Estimation for Wireless Communication –MIMO/OFDM on Cellular and Sensor Networks
Professor A K Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Lecture Number 23
Example of ISI Channel and Least Squares Based Zero Forcing Channel Equalizer

Hello, welcome to another module in this massive open online course on estimation for wireless communications. So currently we are looking at Internet Symbol Interference in wireless communication, right. And equalisation to remove the Inter Symbol Interference, we have looked at least-squares principle for equaliser design, right.

So today let us look at a simple example for equaliser design in a um in a wireless communication channel with Internet Symbol Interference. So we are looking currently we are looking at, we are looking at channel equalisation and channel equalisation as you know is done for ISI removal that is, to remove the Inter Symbol, where ISI stands for the Inter Symbol Interference, correct.

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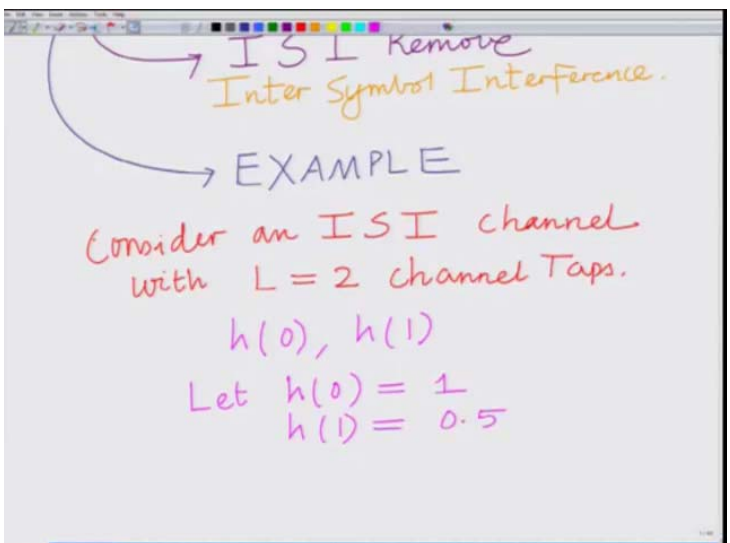

So today let us look at a simple example for channel equalisation. So today, in this module we want to look at a simple example for channel equalisation. So consider an Inter Symbol Interference with 2 channel taps. Consider an ISI Channel with L equal to 2 channel taps, correct alright. So we said in the equalisation model that is in the model with Inter Symbol Interference.

We are denoting the number of channel taps by capital L and these channel taps are denoted by the actual channel coefficients or channel taps are denoted by h 0, h 1 up to h L - 1. So we have L equal to 2 channel taps that is, h 0 and h 1, okay. So we have 2 channel taps that is h 0, h 1 further, let h 0 equals 1 and h 1 equal to 0.5, right. And therefore, now our system model our received symbol y k equals.

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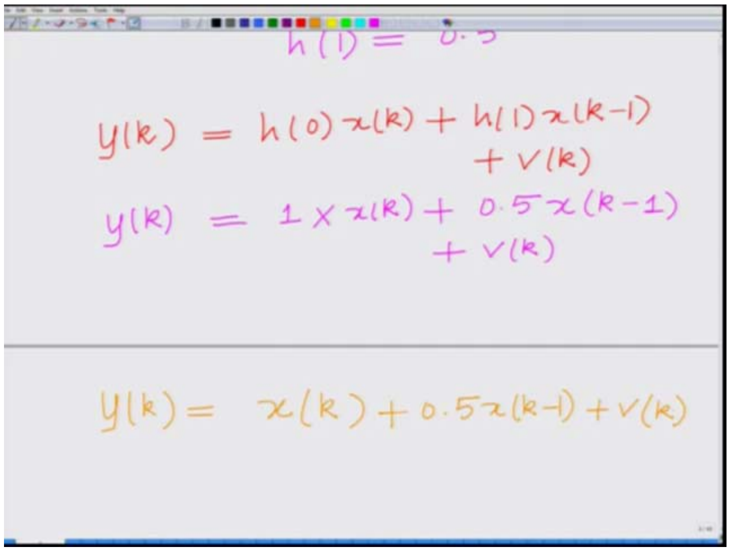

Remember, h 0 times x k + h 1 times x k - 1 + v k and now let us substitute the values for h 0 and h 1 and this gives me y k equals 1 times x k, since h 0 equals 1 + 0.5 times x k - 1 + v k, which is basically we have y k equals. Therefore, x k + 0.5 times x k - 1 + v k.

This is the model for the received symbol y k in which remember y k is the received symbol at time k, x k is the transmitted symbol at current time instant k, x k - 1 is the previous transmitted symbol that is the transmitted symbol at the previous time instant k -1, which is interfering with the current symbol x k that is the current transmitted symbol x k and v k is the noise in the channel, okay.

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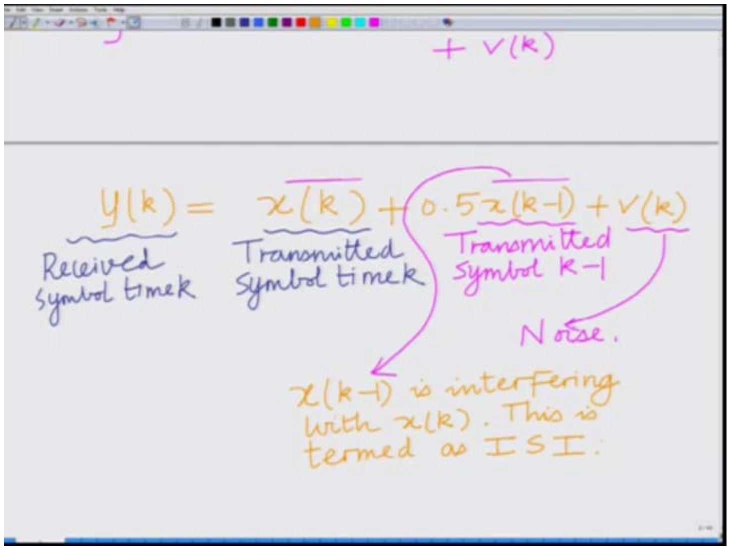

So let me just rewrite those that is, y k is the received symbol, this is the transmitted symbol, x k - 1 this is the transmitted symbol at k - 1 at time k - 1 and v k is the noise. Therefore, x k - 1 is interfering, what we have basically we have x k - 1 is interfering with x k. That is, x k - 1 is interfering with the detection of x k, x k - 1 is interfering.

And this is termed as ISI or Inter Symbol Interference. That is this wireless channel in which y k in y k that is we have x k as well as x k - 1 which is interfering with um previous symbol x k - 1 which is interfering with the current symbol x k, this is termed as Inter Symbol Interference and the removal of this Inter Symbol, the process of removal of this Inter Symbol Interference is termed as equalisation.

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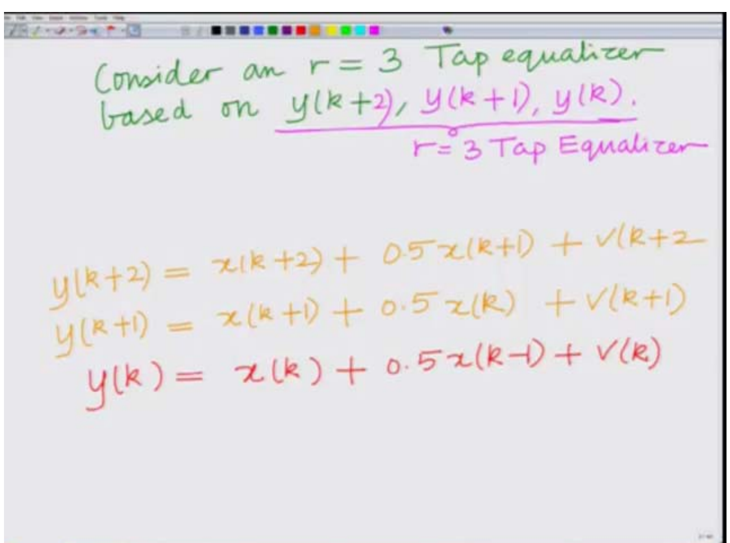

So we perform equalisation at the receiver to remove the effect of ISI, okay alright. Now consider an r equal to 3 tap equaliser again r equal to 3 tap equaliser based on the received samples y k + 1 or based on the received sample y k + 2, y k + 1 and y k. So we have 3 samples and basically using this, we want to design an r equal to 3 tap equaliser, correct. So now let us write the equation for these symbols y k + 2, y k + 1 and y k.

I have y k equals x k + 0.5 times the previous symbol x k - 1 + v k, right. Similarly, y k + 1 equals the current symbol x k + 1 + 0.5 times the previous symbol to x k - 1 that is x k at time instant k + v k + 1. And y k + 2 equals x k + 2 + 0.5 times x k + 1 + v k + 2, correct. Now let us write this as a matrix, now if I write this as a matrix I have, so writing as a vector.

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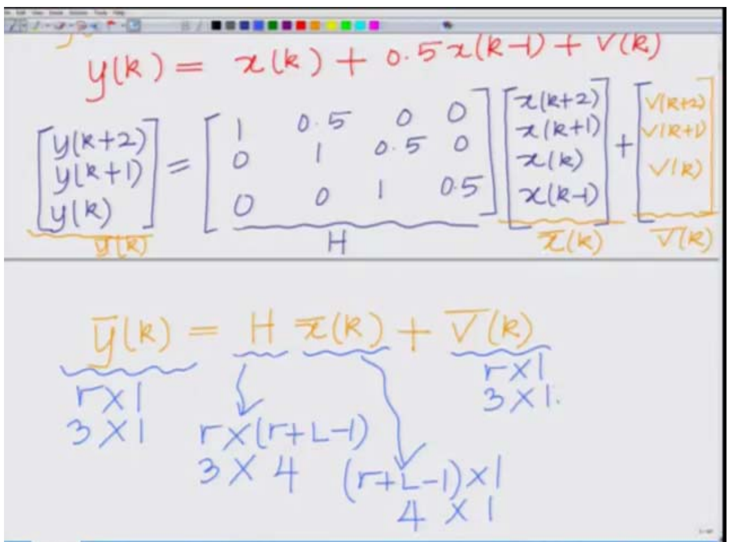

I have y k + 2, y k + 1, y k, which is equal to 1 0.5, 0, 0, 0, 1, 0.5, 0, 0, 0, 1 and 0.5, this is same as the matrix H that we have written in the previous module except now I am substituting for the channel coefficients h 0 and h 1. This is the matrix H times x k + 2, x k + 1, x k - 1 + we have the noise vector that is v k + 2, v k + 1 and v k. And we said this is this is the received the vector y bar of k.

This is the transmitted vector this is x bar of k, this is the noise vector v bar of k. And basically using this, now we can basically write the vector model for this Inter Symbol Interference as y bar k equals H times x bar k + v bar k, where v bar k is the noise vector, correct. So therefore, I can write this as y bar k equals H times x bar k + v bar k, v bar k is the noise vector.

Remember, this is an r cross 1 vector; r equals 3, so this is basically a 3 course 1 matrix. H is an r cross r + L - 1 matrix, r equals 3, L equals 2, so this is a 3 cross matrix. This is x bar k, which is r + L - 1 cross 1, this is 4 cross 1 and this is an r cross 1 vector which is a 3 cross 1 noise vector. So basically what we have done is we have developed this model that is, y bar k equals H times x bar k + v bar k for this ISI the Inter Symbol Interference.

To basically capture the effect of this Inter Symbol Interference in this wireless channel and now let us design over r equal to 3 tap equaliser, okay. So we are going to design the 3 tap equaliser to remove the effect of Inter Symbol Interference in this wireless channel, correct. And let us denote the 3 taps by the coefficients c 0, c 1, c 2. So basically now so let the coefficients or taps of equaliser.

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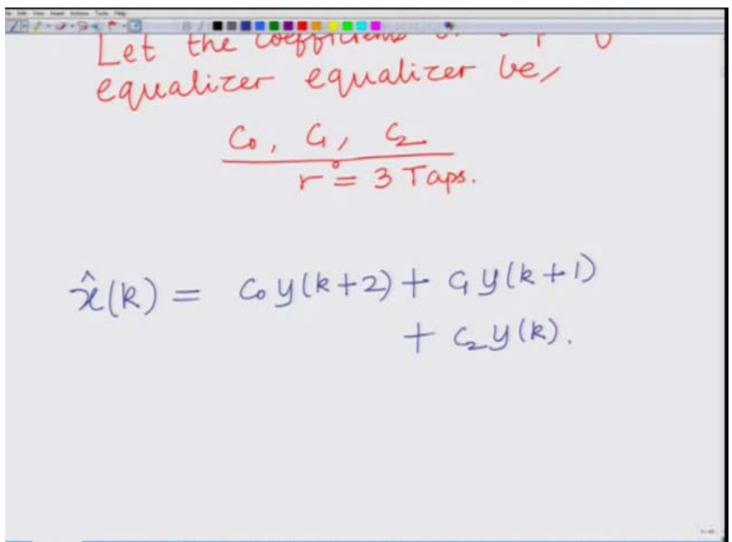

These are the 3 taps of the equaliser, so we have r equal to, we have r equal to 3 taps and therefore, now I can write the equaliser that is, equalised symbol x k c 0 times y k + 2 + c 1 times y k + 1 + c 2 times y k that is, I am estimating the symbol x k; x hat k is the estimate of the symbol x k after removing the interference from the other symbol that is x k + 2, x k + 1 and x k - 1.

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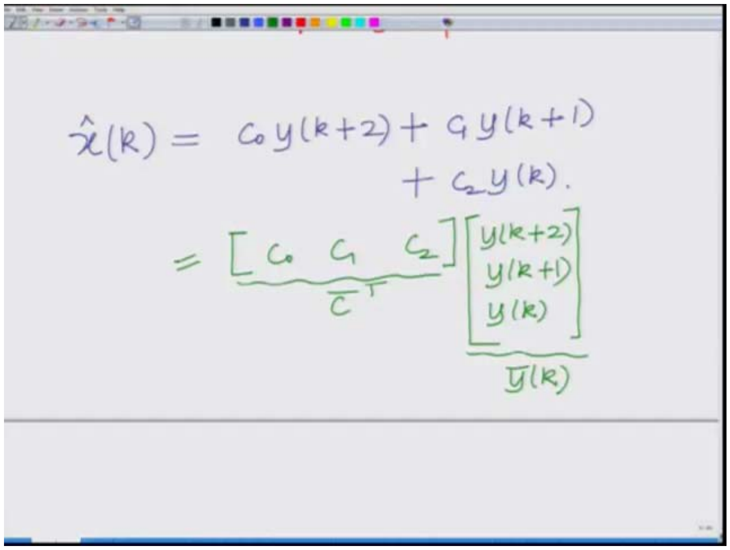

And that estimate is given by the linear combination of output symbols y k + 2, y k + 1 and y k as c 0 times y k + 2 + c 1 times y k + 1 + c 2 times y k which now I can write as basically the row vectors c 0, c 1, c 2 times the column vector y k + 2, y k + 1, y k, this is the transpose of the equaliser vector c bar, this is c bar transpose, this is y bar k, so the equalised symbol x hat k equals c bar transpose y bar k.

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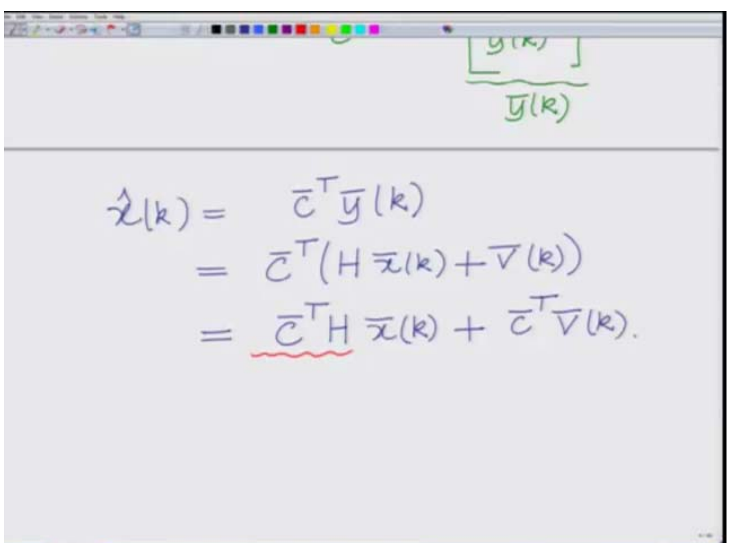

And now substituting the expression for y bar k, which is equal to H times x bar k + v bar k, which can be now by removing the brackets, I have c bar transpose h times x bar k + c bar transpose v bar k. And now said to achieve equalisation, this vector c bar transpose H, this vector for perfect equalisation, ideally this should be the vector 0, 0, 1, 0, alright.

If it is not exactly 0, 0, 1, 0, we would like to at least approximate the vector 0, 0, 1, 0, as closely as possible so that we it picks, the vector picks x k while it nulls the other symbols x k + 2, x k + 1 and x k - 1 and that is the basic idea in this equaliser design. This is therefore known as also the 0 forcing equalise, 0 forcing criterion for equaliser design because you are forcing the interference from other symbols x k + 2, x k + 1 and x k - 1 to 0.

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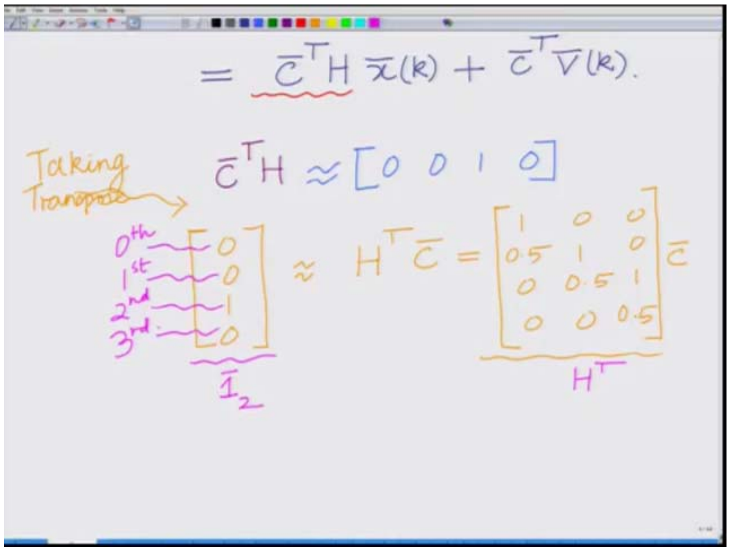

And therefore, this cannot be expressed as, I have to have c bar transpose H, this vector should basically be as close and approximation as possible to 0, which means if I now take the transpose.

So taking transpose, I have this should be equal to H transpose times c bar or basically should be as close as possible to catch transpose times c bar. And H transpose is remember the matrix, it is the transpose of H which is 1, 0.5, 0, 0, 0, 1, 0.5, 0, 0, 0, 1 0.5 times c bar and this basically we said is our matrix h transpose. This is the matrix 1 bar 2 with 0s everywhere and 1 in the second position.

So this has 0 in the 0th position, in the first position, 1 in the 2nd position and again 0 everywhere. So 12 is basically the vector which has 1 in the second position, start counting from 0, 1 in the second position and 0s everywhere. And therefore, now the least squares criterion for equaliser design can be expressed as. In fact, the least squares criterion for 0 forcing equaliser design.

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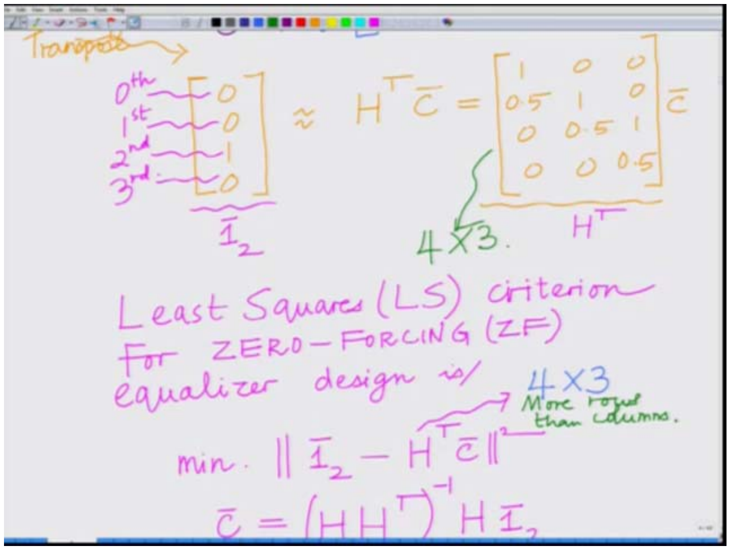

Let us, for 0 forcing or Zee F equaliser design, is minimise norm 1 bar 2 - H transpose c bar square and the equaliser c bar, the least squares basically solution for the above this thing. The equaliser c bar is basically given as H H this something that we have seen before H H transpose inverse times H into the vector 1 bar 2.

And remember, that is because this is basically because H transpose the um matrix H transpose is a 4 cross matrix and therefore, it has more rows than columns. If you look at the matrix H transpose, it is a 4 cross 3 matrix. So the matrix H transpose is a 4 cross 3. Therefore, if you look at the system of equations, 1 bar to equals 2 H transpose times c bar, that is an over determined system, alright.

So that is an over determined system, so one can solve it such as one cannot solve it exactly. One can solve it only to minimise the error between this vector 1 bar um 2 H transpose c bar that is the least squares solution c bar. In fact, this is similar remember, yesterday we have said, this is similar to the multiple antenna downlink channel estimation, there we have y bar equals x times H bar.

And x is the pilot matrix, which has more rows than columns and colloquially this is also known as a tall matrix since basically it has a structure that appears tall that is, it has more rows than columns. If it is an N cross M matrix, where N is the number of observations is more than M, the number of antennas in the context of channel estimation that is what we have seen previously.

Anyway; coming back now to our equalisation problem, so now the equaliser can be designed by substituting these matrixes H and 1 bar 2. So now let us look at our matrix, let us look at this product H H transpose, this product H H transpose, now let us substitute the expression for H.

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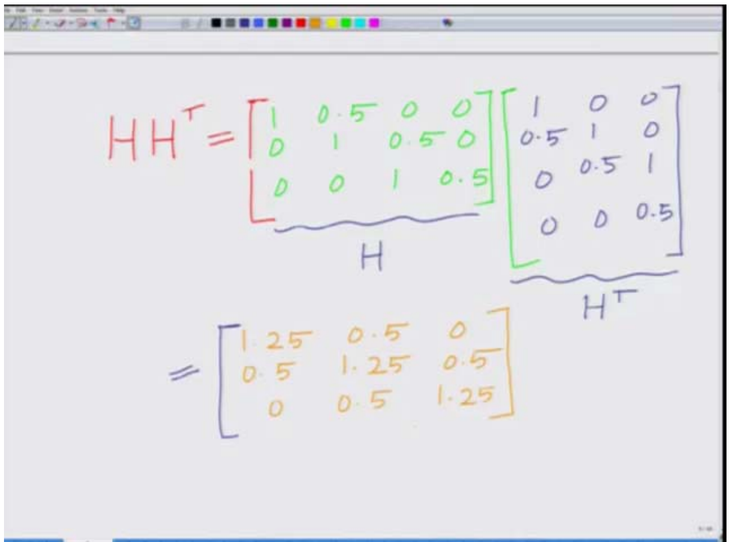

This expression for H is given as 1, 0.5, 0, 0, 0, 1, 0.5, 0, 0, 1, 0.5 times the matrix H transpose and the matrix H transpose is 1, 0.5, 0, 0, 0, 1, 0.5, 0, 0, 0, 1, 0.5. This is my matrix H, this is my matrix H transpose and you can see, you can compute this, you can evaluate this and you can see H H transpose is given as 1.25, 0.5, 0, 0.5, 1.25, 0.5 and again 0, 0.5. 1.25.

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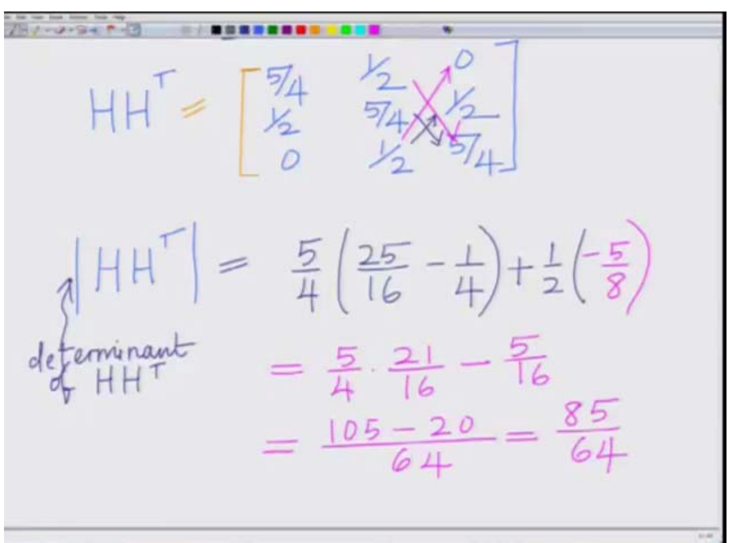

And basically just writing this as fraction for convenience I have well 5 by 4, this is half, 0.5, 0, this is half, 5 by 4, this is half, 0, half, 5 by 4, this is my matrix H H transpose. So basically substituting the values, so substituting the substituting the matrix H, I have basically evaluated H H transpose. Now we have to evaluate H H transpose inverse.

And to evaluate H H transpose inverse, first we have to evaluate the determinant of H H transpose, okay. So the determinant of H H transpose, let me write it this way that is simply denoting this by the determinant, this is basically the determinant of H H transpose and you can calculate this, you can evaluate this, you can see this is 5 by 4 times 25 by 16, this product - this product 25 by 16 - 1 by 4 + half into.

Well, this product - this product that is - 5 by 8 that is basically 5 by 4 into 21 by 16 - 5 by 16, which is equal to that is basically105 - 20 divided by 64, which is basically you can again check this, this is 85 divided by 64 that is the determinant, we have evaluated the determinant of this matrix H H transpose.

Now to evaluate um evaluate the inverse, again you should be familiar with um the the evolution of the inverse of a simple 3 cause 3 matrix. Basically, I have to I have to replace each element by the cofactor of the transpose, yeah. So in case you are not familiar, you can look to the procedure, it is a very simple procedure where I am where I am replacing each element by the cofactor of the element corresponding element in the transpose.

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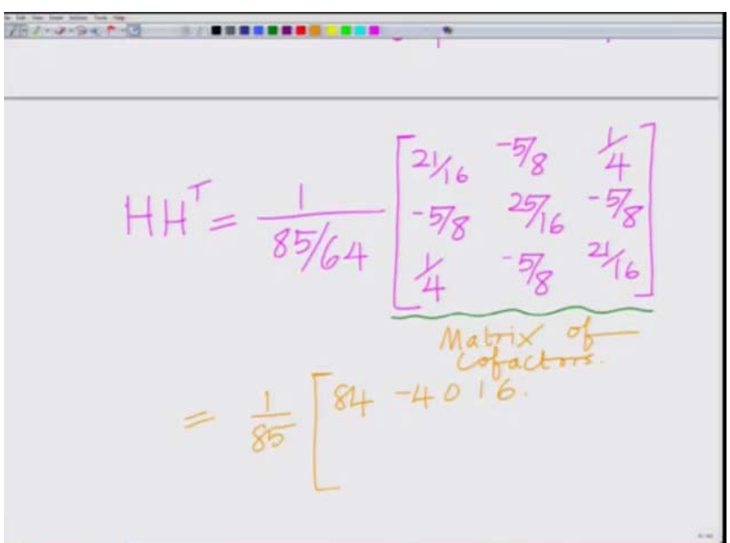

Yeah, so um therefore H H transpose can now be computed as H H transpose is 1 over the determinant that is 1 over the determinant times this matrix of cofactors, which is given as 21 by 16, - 5 by 8, 1 by 4, - 5 by 8, 25 by 16, - 5 by 8, 1 by 4, - 5 by 8, 21 by 16 and basically if you look at this, the cofactor is very simple. For instance, the cofactor of this element, we have already evaluated the cofactor of 5 by 4.

The cofactor of this element is basically 5 by 4 times 5 by 4 - that is 5 by 4 times 5 by 4 - 1 by 4 equals 21 by 16 and so on. So this is basically, you can look it up, this is the matrix of cofactors can be found in any standard um textbook on “Linear Algebra”.

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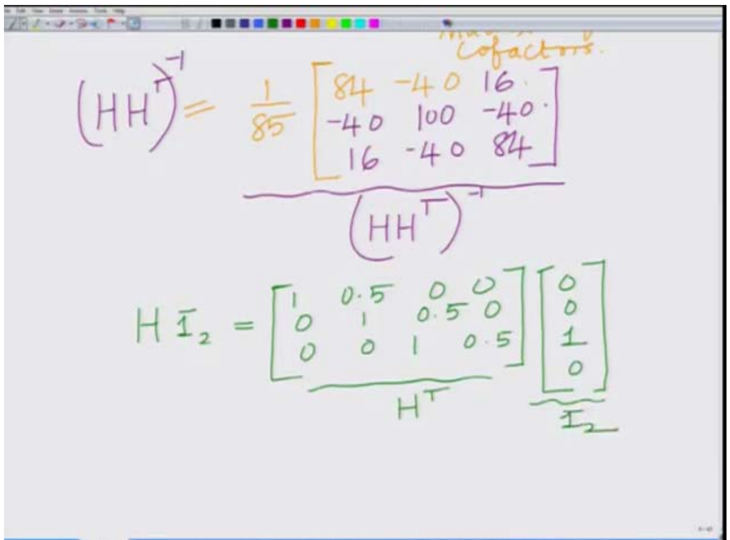

And now taking the factor of 64 inside this, this is basically you can further simplify this as 1 over 85 times 84, - 40 and 16, - 40, 100, - 40, 16, - 40 and 84. And now therefore, this is what is this? This is basically H H transpose inverse. So this is basically H H transpose, so we have evaluated H H transpose inverse. Remember, the equaliser is H H transpose times H into 1 bar 2.

So I have to evaluate this other component that is, H into 1 bar 2 and that is very simple. That can be evaluated as H into this vector 1 bar 2 is basically your matrix 1, 0.5, 0, 0, 0, 1, 0.5, 0, 0, 0, 1, 0.5 times this is the vector 1 bar 2 that is 0, 0, 1, 0, so this is basically your H transpose, this is basically your vector 1 bar 2, so this is equal to…

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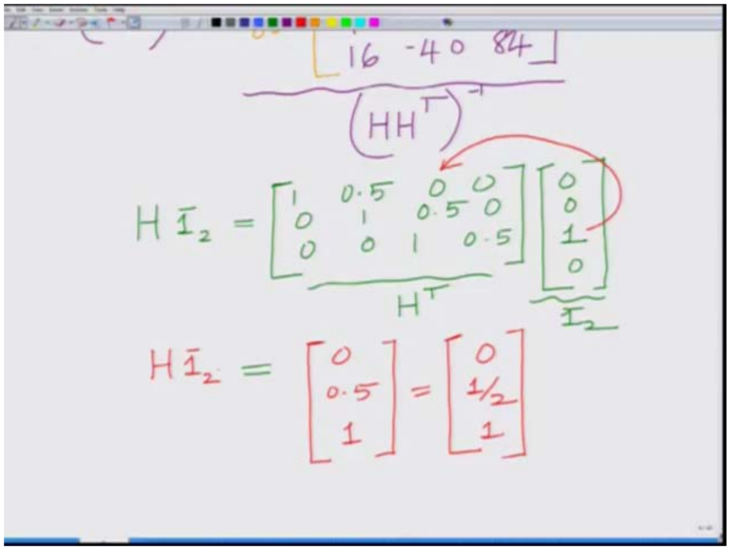

And look at this, this 1 bar 2, this simply takes the the second column that is start counting from 0, this simply picks this column and you can multiply this that is simply your 0, 0.5, 1 or in other words that is basically your 0, half, 1. Therefore, the equaliser c bar is the equaliser c bar, the equaliser c bar is now c bar equals H H transpose inverse times H into this matrix 1 bar 2.

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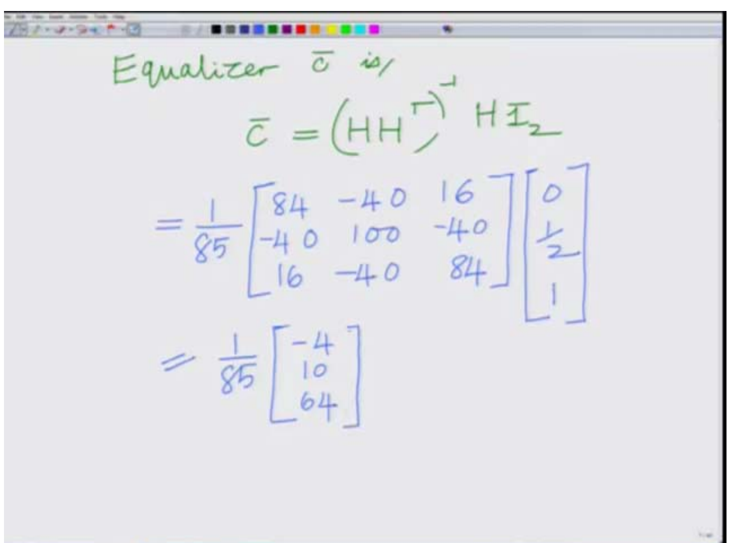

We have computed both these quantities above H H transpose inverse is 1 over 85 times 84, - 40, 16, - 40, 100, -40, 16, - 40, 84 times H 1 bar 2 is 0, half, 1 and this when you multiply you can check this, what you will get is 1 over 85 times - 4, 10, 64 and finally taking this 1 over

85 inside, what I have here is basically C bar, which is equal to - 4 divided by 85, 10 divided by 85, 64 divided by 85.

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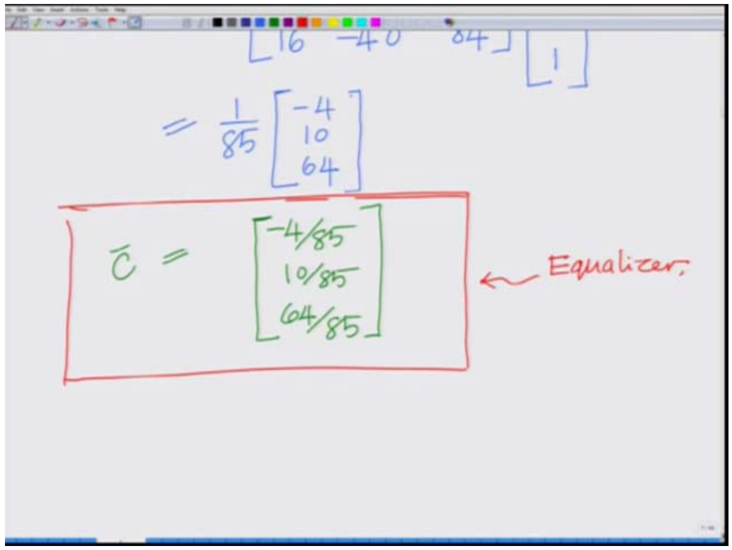

And this is basically, the equaliser. So what we have done is basically we have evaluated this quantity H H transpose inverse times H into this vector 1 bar 2 and that is given as this equaliser C bar. And now what we have to do is basically the symbol x hat k that is the equalised symbol x um x hat k can be evaluated as C bar transpose the vector y bar k. That is the last step that is that is the implementation of the equaliser.

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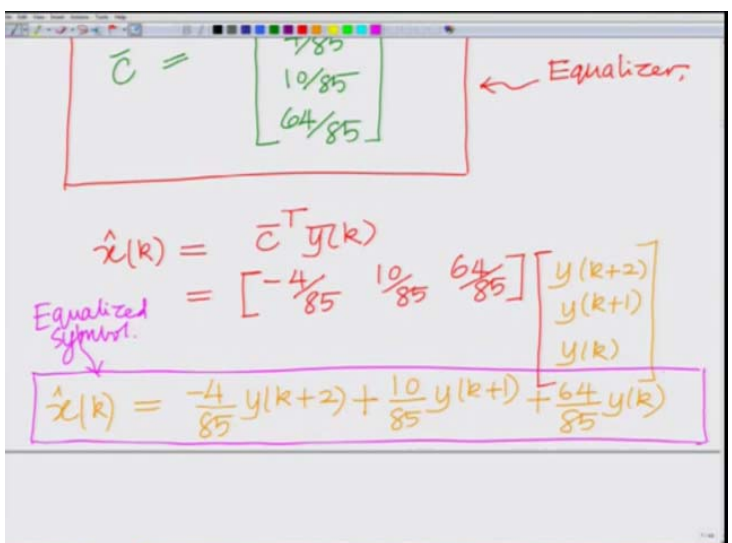

So x hat k equals C bar transpose y bar k, which is basically equal to your um the equaliser which we have evaluated, which is - 4 divided by 85, 10 divided by 85, 64 divided by 85 times y of k + 2, y of k + 1 and y of k. And that is basically x hat k and that shows that x hat k is basically equal to - 4 by 85 times y of k + 2 + 10 by 85 times y of k + 1 + 64 by 85 times y of k.

And finally therefore, this is basically your equalised symbol that is, x hat k equals - 4 by 85 times y of k + 2 + 10 by 85 times y of k + 1 + 64 by 85 times y k. So what the equaliser is doing is basically performing a linear combination of the symbols y k, y k+ 1, y k + 2 to eliminate the Inter Symbol Interference from x k. So what we have seen in this basically, we have seen a simple example based on L equal to 2 taps.

That is, um a wireless channel model which is Inter Symbol Interference from one previous symbol that is x k with Inter Symbol Interference from x k - 1, we have seen a simple example of this 2 tap Inter Symbol Interference channel and then subsequently we have demonstrated the procedure to derive an r equal to 3 tap equaliser, for this channel we have evaluated the 3 tap equaliser.

And finally also demonstrated how the equalisation can be done using this 3 tap equaliser for this wireless channel with Inter Symbol Interference, all right. So with this we will stop this module and we will explore other aspects in the subsequent modules. Thank you very much