Differential Calculus of Several Variables
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Lecture Number 20
Application of IFT: Inverse Function Theorem

Okay! So in the last two lectures of this course, we'll discuss another application, another very important application of 'Implicit Function Theorem', namely, the 'Inverse Function Theorem'. So, as a name suggest, Inverse Function Theorem says, or gives you condition when particular function has a inverse in terms of function. Okay let us start with very simple one. Suppose I have a function f, from Rn to Rn, and I take the most simplest continuous function, linear. So it's a differentiable function, it's a, it's a continuous function, differentiable of any order, and all of us know, that in that case, there exist actually linear function, so if I fix basis of Rn, with respect to fix basis I can write N cross N matrix.

There exist an N cross N matrix such that for each x, f(x) is actually given by Ax, correct? This is such as, these are all linear functions from Rn to Rn. Now suppose, A is an invertible matrix, that is, determinant of A is non zero. Then, so then there exist A inverse such that, A A inverse equal to A inverse A, yeah, A inverse A equal to N cross N identity matrix. Now see if I define g, another function from Rn to Rn, given by g of y equal to A inverse y, then, this g is actually the functional inverse of A, in the sense that, so g is inverse of f, inverse function of f; in the sense, that if I have f applied on g(y), that will give me what? g(y) is A inverse y, f of g inverse y will be A A inverse y, so that will be y, and, g f(x) will be equal to x, for any x.

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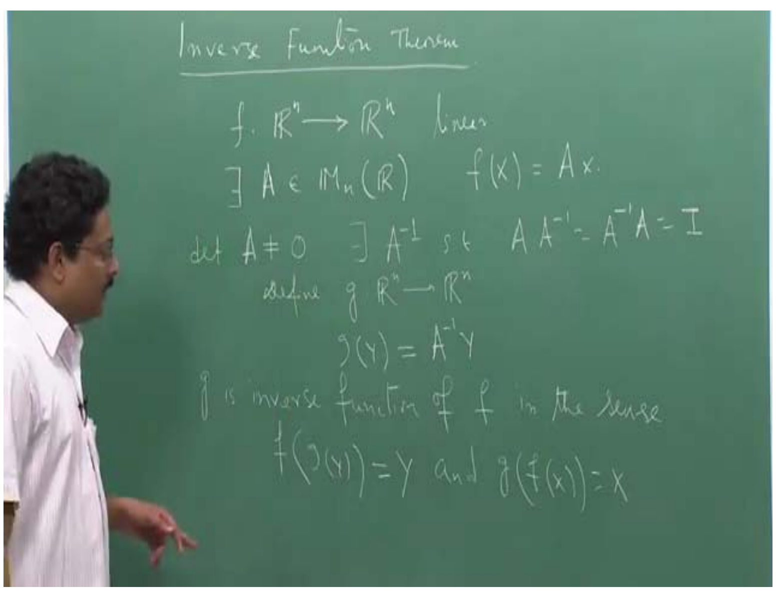

So if I have a linear function, which has, which is determined by a, invertible matrix, then it has a inverse. Now if I go back, if we go back to a differentiable function, suppose now f from Rn to Rn is differentiable at, say x naught, then what I know? We have been always writing, that this means that such a formula holds where, right? By the way I forgot to mention something, here. So come to this board. That we started with a linear function, which is given by f(x) equal to Ax, A is invertible, then I can straightforward define g(y) equal to A inverse y.

On the other hand, this converse is also true, that converse is also true, here, in the sense that, suppose I have this f(x), linear, and there exist a g, such that this thing happens, that f of g(y) equal to y and g of f(x) equal to g(x), then g has to be given by A inverse y. It's easy to see, but while you prove you have to careful. So try proving it, correctly. I mean it's easy to see, it looks like it's obvious, but it's not that obvious. It needs some fact, that, when you apply this fellow, so you have to apply chain rule here, that will give you, the derivative of g at f(x), composed with derivative of f at x equal to identity.

But derivative of f(x) is A, so derivative of g at f(x), that will be inverse of f, but, sorry inverse of derivative of f, that is, inverse of A. But A is linear. So if a linear operator has an inverse, any inverse, it has to be linear again. So that fact has to be used, so be careful while proving it. Anyway, let's come back to general setup, so this is about the differential function, this is the definition. Now what it says, we have used it very, mini, minim, many time, that if is small, well, small in the sense that, this goes to zero as H goes to zero, then, f of x naught plus H minus f of x naught is approximated by D x naught, D f x naught at H. Correct?

So, suppose, without loss of generality, I start with f x naught, differentiable at x naught, and f x naught is zero. I start with such an x naught, then what we'll have? f of H, for any H is approximated by, sorry a small by it's derivative. Because of f of x naught is zero. Okay? Sorry f of x naught of H, oh sorry. So now if I have this fellow, this D f ox, f of x naught, is invertible, that is, what is the matrix? Matrix of D f x naught, we usually write at Jacobian.

So if determinant of J f x naught, which is the matrix of the derivative D f x naught is non zero, that is this is invertible, as a linear operator, then, we may expect that around x naught, f has, f is invertible or f has an inverse, in the sense that this thing happens. Okay? It happens for linear function, for differentiable function, it is approximated the difference, so if it, if it was not without loss of denial taken, put zero. f of x naught plus H is approximated by this, so around x naught, I may find an inverse. Everything depends on this, x naught, because I have x naught plus H is approximated by D f x naught, so, I can, at the most expect things happening in an intri um, in an neighbourhood around x naught.

So, inverse functions theorem says, that yes, we can do it. The statement of 'Inverse Function Theorem' says, theorem implies, yes. As shall most, al, it is true. But you have to assume something more, because we are just doing first order approximation, but we have motivated you see, from f(x) of, f(x) equal to Ax, which is actually, whose derivative is A, so it is differentiable of, upto any order. So here 'Inverse Function Theorem', we'll put some condition that f is c one, that is, we'll need to prove that not only this, we need to have f to have 'continuous partial derivatives', of first order. That we'll do.

Okay! So as in the case of 'Implicit Function theorem', we have seen. So I will state the 'Inverse Function Theorem', in full generality. But before proving, as we did earlier, we saw for 'Implicit Function Theorem', if we prove it for a function from R two to R, then we're done. Rest of the step is mere form, steps are mere formalities. And this, here, in the proof of 'Inverse Function Theorem' is more easier, if we can prove it for function from R to R.

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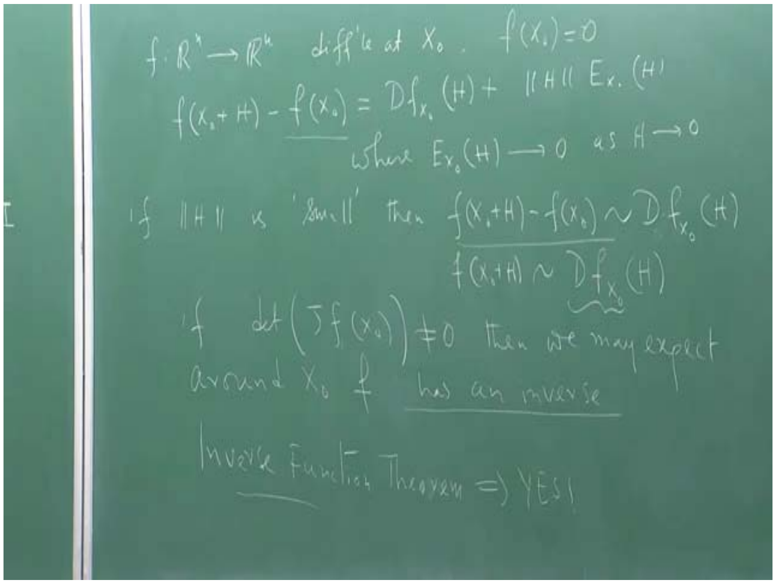

Then the proof for Rn to Rn is done. So all I need to gather, all I need to do, is to see what happens to a function, nice functions, differentiable, continuously differentiable function, if it has non zero Jacobian determinant at a point, and that too, only for R to R, so that means, a function from some interval, may be two hour, it's not either interior point, what happens, if f prime x naught not equal to zero? Before stating anything else, so let us try to see, let me put it in this way, that observation on functions f from say interval U, a um sorry um, opens at U in Rn to R. x naught in You, and, determinant of J f x naught not equal to zero.

So we start with n equal to one, so now I have f from (a,b) to R, a simple point x naught, may be, and, f of x naught, f prime x naught not equal to zero. And I'm taking f to be differentiable, so f is, so this is differentiable, on the entire interval. Okay we'll do it in a way that we can, we'll make the observation one two three four, in such a way, that it straightforward generalizes to n to n, so you must be careful that we should not be use something very special for function of single variable. For example, one can see if I claim such a thing, that f is one one near x naught, that is, in an around around in an interval around x naught, f is one one, why?

Okay for a function of one variable, you see, if I take x minus f of x naught, this is f prime of x, f prime of xi, into x minus x naught, and, if f prime of x naught not equal to zero, and I put this condition as I said we need this that f prime is also conse continuous, at x naught, then suppose it is non zero, then this will be f prime xi, so xi will be in the interval x naught to x. So I can choose an interval where f prime xi is also non zero, because f prime is continuous, then, this right hand side is never zero, so left hand side is not zero, say it will be one one.

So this is simple em, application of MBT will tell you that if f prime is continuous at x naught, f is one one. Okay? But, I cannot use it for Rn, because in Rn, we do do know that this version of MBT is not true. So you have to do it in some other way. And in doing so, we'll observe something more. Okay. So here is our assumption. Again, f from a to b R differentia, inter differentiable on the inter interval, x naught (a,b), f prime x naught non zero, f prime is also continuous at x naught. Okay.

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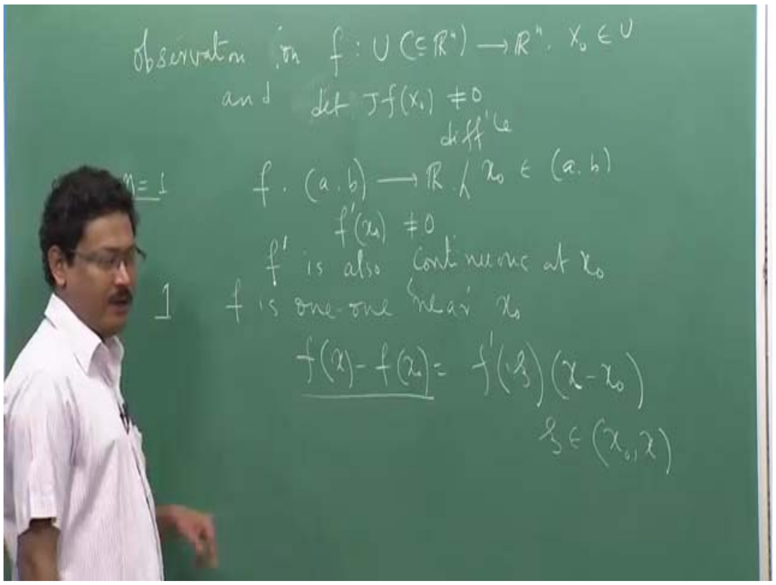

So let us assume f prime is continuous in the, throughout the interval. There's no harm in it. Fine. I can of course choose a smaller interval a naught, b naught. Of course x naught is continuing in a naught, b naught, where. This is by continuity of f prime. If it is non zero at some point, the continuous function non zero I can choose on interval around it, where it is non zero. Okay. And, f of a naught is not equal to f of x naught, and f of b naught is also not equal to f of x naught.

Why I can do that? Because if for every interval around x naught, f of a naught equal to f of x naught, and f of b naught equal to f of x naught, then around x naught f is a constant function, and in that case f prime at x naught will be zero. But we have assumed non zero, so you can always find an interval, where, f of a naught not equal to f x naught, f of b naught not equal to f x naught. Otherwise, around x naught, f will be constant. And in that case, f prime will be zero. Okay. Very good.

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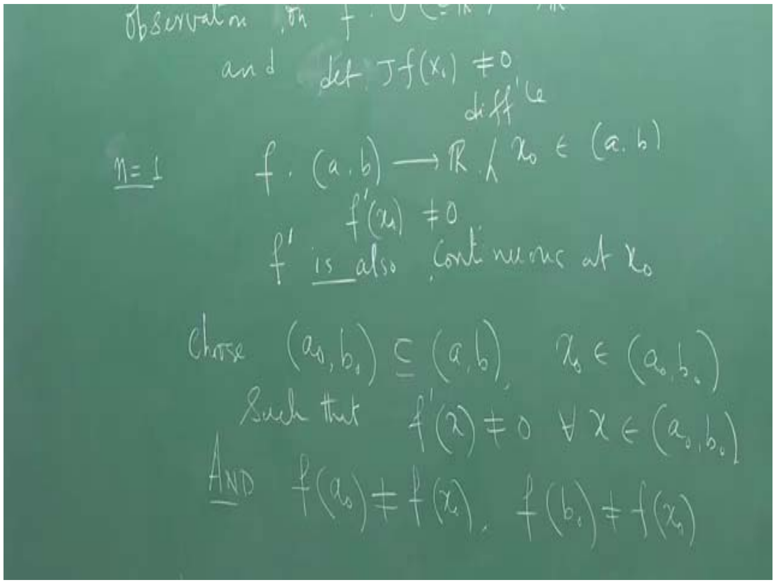

Now, let m, now this will be greater than zero. And I assume, this is less than, this also I can do, without loss of generality. So think for a moment, that I can also choose the interval such a way, that I get f of a naught minus f x naught minus, is strictly less than f of b naught minus x naught. I mean one of them. Either f of b naught minus x naught great than, less than this thing, f of a naught minus x naught, or this way, because if these two are equal, you can find out a contradiction to f prime x naught is zero.

Okay? So now, what I do, I've chosen y, pick y in this intervals. So even small had done, this m, m by two. And consider, pick any y, and consider, the function h(x) equal to f(x) minus y. So I pick and fix an f(x) f of h of x equal to f(x) minus y. So this is a continuous function right? And continuous function on the closed interval a naught, b naught, because f is continuous in a bigger interval, so I can choose it it a h is contin for, um, f is continuous function on this closed interval a naught, b naught.

I can choose a naught, b naught in such a way, that closer of a naught, b naught is also inside this. So may be I should have written here this, and, this, I can always do that. Now a continuous function on a naught, b naught, so that implies there exist a z naught in the closed interval a naught, b naught, such that, h of z naught is minimum of z in a naught, b naught of h(z). A continuous function in a closed interval at x is minimum. Okay I make some, so upto this is okay. I make an observation.

Can z naught be equal to a naught? See, f of z naught minus y. Why it is y in between f x naught minus m by two f x naught plus m by two. So f and h of f(x) is f ek, f(x) minus y. So f of z naught minus y, this value is less than m by two. Correct? Which is strictly less than h of a naught, which is f of a naught minus y, um, f of a naught minus y, and y is here, so it has to be less than f of b naught minus y, which is h of b naught.

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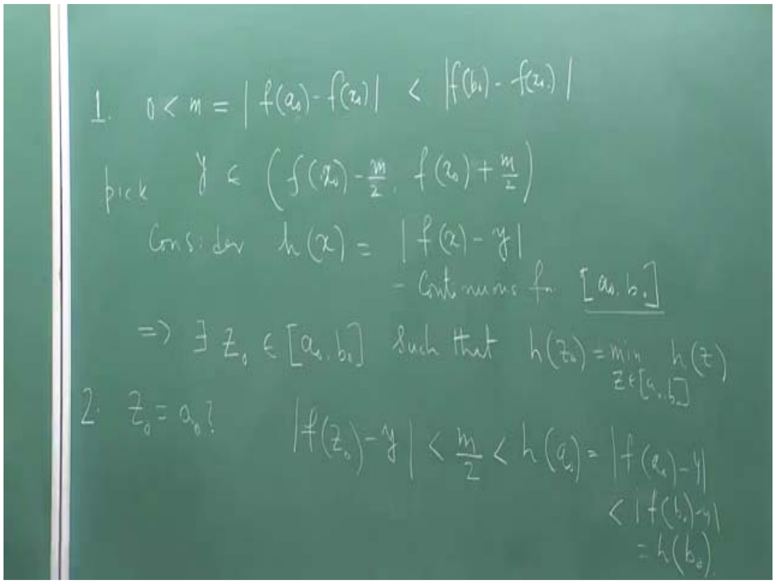

This is easy observation, okay? So z naught cannot be a naught. So z, uh, or z naught cannot be b naught either, so this too cannot happen. So, but z naught is inside this closed interval, so this will implies z naught is in the open interval a naught, b naught. Why I need that? Because I am going to put that z naught is minimum, for h(x), for g(x), which is h of x square, because h is a positive function, which is, f of x square minus two f(x,y) plus y square.

This implies, since z naught is in the interval, open interval, g prime of z naught equal to zero. But what is g prime at z naught? You see, g prime of x is two f prime x f(x) minus two f prime x y square. So g prime of z naught is two f prime z naught f of z naught minus two f prime z naught, oh, sorry, y, y is, y. This is equal to zero. This implies f prime z naught, f z naught equal to f prime z naught y. What happened? I have chosen the interval such a way, that f pr note, f prime z naught.

I've chosen the interval in such a way that this is not equal to zero. So this implies, f of z naught equal to y. So what got, dood I did I get? I get for any such y, there exist a z naught, where? So there exist, so any such y, there exist a z naught in the interval, open interval a naught, b naught around x. So as that f of z naught equal to y. And this, from this, also implies, that, f of a naught, b naught, this entire interval, contains an open interval. Decide how. It's there already in the proof.

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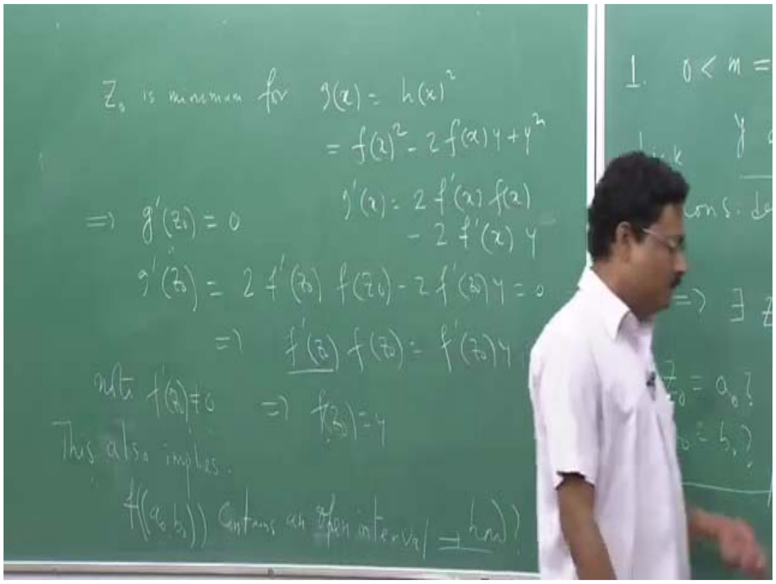

In the proof, while I showed this fact that f of z naught equal to y also shows that f of a naught, b naught contains an open interval. Decide how. And next time, you will see how to get Inverse Function Theorem from only these observations.