Estimation for Wireless Communications-MIMO/OFDM Cellular and Sensor Networks.
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Lecture -17.
Properties of Least-Squares Estimate – Mean, Covariance and Distribution.

Hello, welcome to another module in this massive open online course on estimation for wireless system. So, we are considering currently the maximum like destination of a vector parameter H bar and we have shown that in the maximum likelihood estimate, which is also the solution or that corresponds to the minimum of the least-squares cost function, this maximum likelihood estimate or the least-squares estimate H hat is given as,

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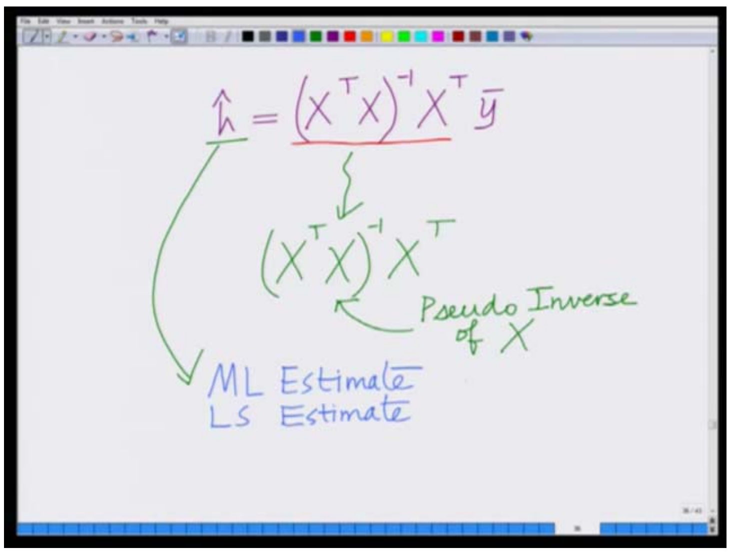

your H hat equals X transpose X inverse X transpose Y bar, okay. And this matrix X transpose, we have called this matrix as a pseudo-inverse of X, remember this is called the pseudo-inverse.

Okay, and this we have also called the ML estimate for the vector parameter, this is your ML estimate or LS, that is least-squares estimate.

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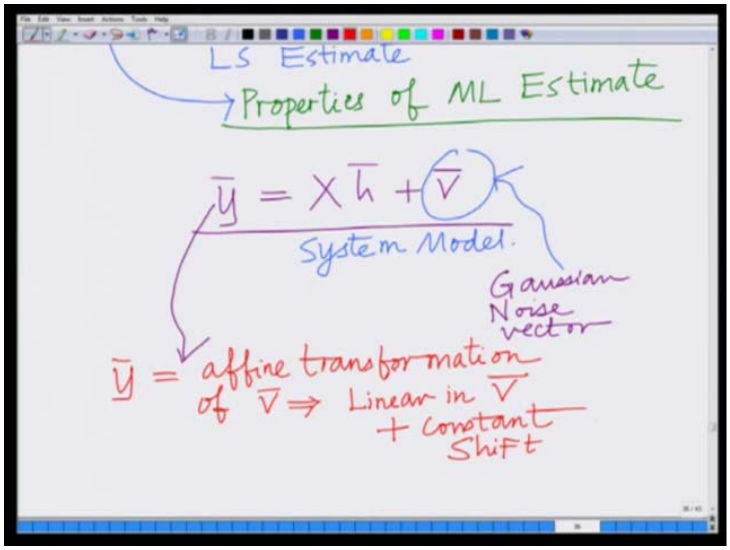

Now, let us explore the properties of this maximum likelihood estimate, let us now explore what we want to do is we want to find, explore the properties of this, we would like to explore the properties of this maximum likelihood estimate which is basically H hat equals X transpose X inverse times X transpose Y bar, where Y bar is of course the vector of observations and the matrix X is the pilot matrix for this multi-antenna system.

Correct, now recall, they let us go back again to our system model, recall that our system model is Y bar equals H X bar + V bar, this is basically your system model where V bar is the Gaussian noise vector, remember, this vector contains Gaussian noise samples, so this is a Gaussian noise vector. Now, Y bar, which is equal to H X bar + V bar, it is related in affined fashion to V bar, that is it is a linear transformation of V bar, basically V bar itself, + a constant, shifted by a constant. That is H X bar. Therefore this is known as affined transformation, right. So, Y bar is basically related in affined fashion or Y bar equals to an affined transformation of V bar, which basically implies linear in V bar, that this linear formation of V bar + a constant shift.

Now, affined transformation, now Gaussian variables remain Gaussian under an affined transformation

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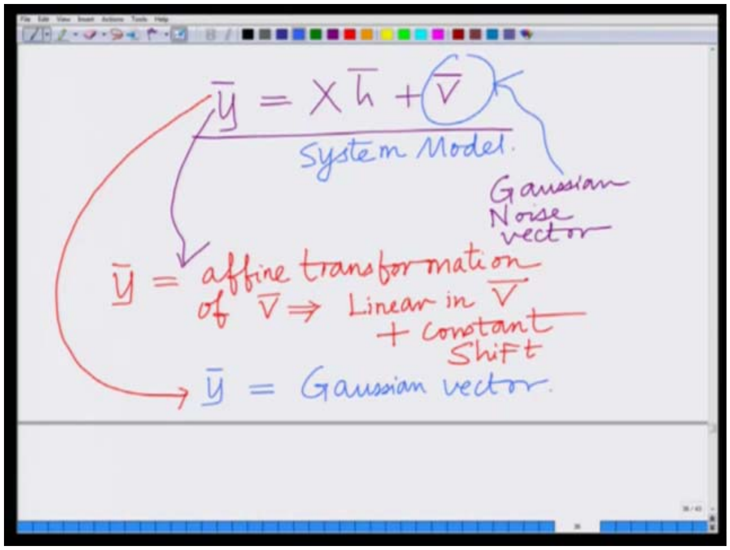

implies Y bar is basically equals a Gaussian vector. So, what we are saying is basically the following thing. If I take a Gaussian random variable, if I scale it and if I shift it by another constant, right that is I consider affined transformation of this Gaussian random variable, I basically get another Gaussian random variable. Right, so the Gaussianness remains invariant under affined transformation. Now, I have Y bar equals X H bar + V bar, V bar is Gaussian, so I am taking V bar, shifting it by X H bar, therefore Y bar is the Gaussian vector.

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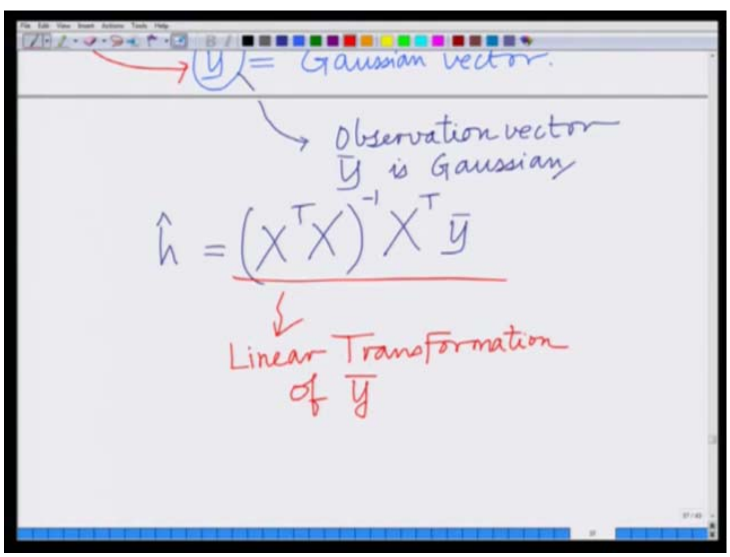

So, that is, let us start with that was, that is the observation vector is Gaussian. So, Y bar which is your observation vector, so let me repeat that observation vector Y bar is Gaussian, now if you look at X H hat, H hat equals X transpose X inverse X transpose Y bar. This is basically a linear transformation of Y bar. What is this, this is a linear transformation, of the vector, this is a linear transformation of the vector Y bar

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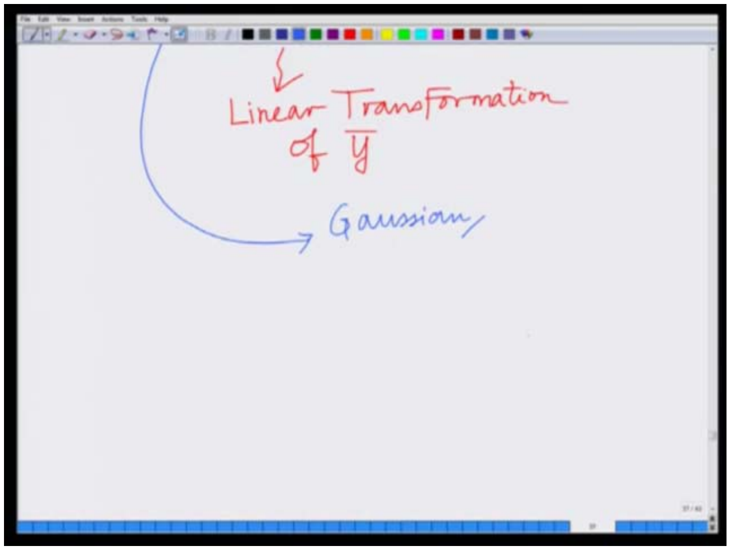

and hence Y Y bar is Gaussian, a transformation of a Gaussian vector results in another Gaussian vector H hat is Gaussian in nature.

Since a transformation of a Gaussian vector results in another Gaussian vector, the estimate H hat is a Gaussian vector. So, that is the 1st thing that we have established. That is H hat is the estimate of the vector parameter H, that is H hat is the Gaussian vector, that is the 1st thing.

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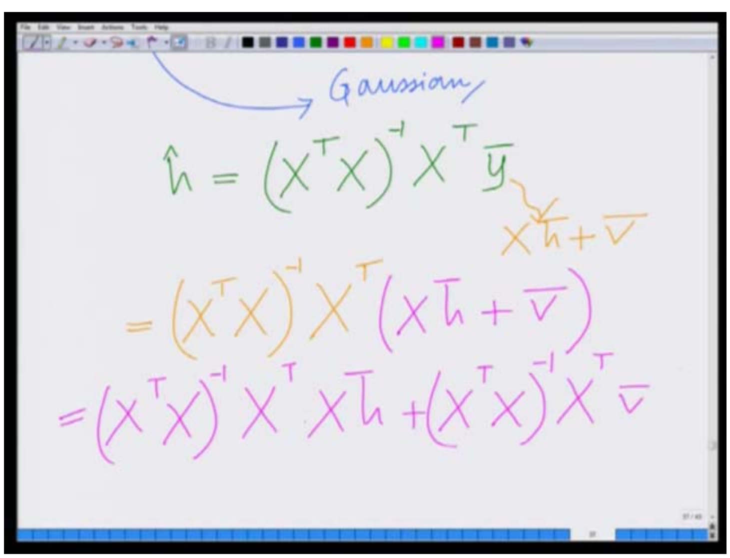

Now, 2nd, let us look at H hat equals X transpose X inverse X transpose Y bar but recall your Y bar equals X H bar + V bar, so I substitute that over here now, so I have X transpose X inverse X transpose times the Y bar, I will substitute XH bar + V bar which is basically Y bar, and now expanding this, I have X transpose X inverse, X transpose times XH bar + X transpose X inverse X transpose V bar.

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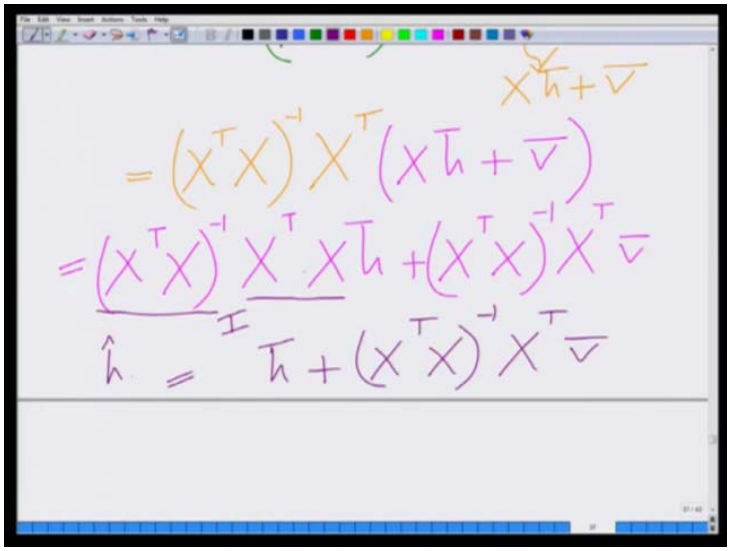

Now if you look at this quantity over here X transpose X inverse times X transpose X, this is identity, therefore I simply have identity times H bar which is H bar + X transpose X inverse times X transpose V bar and this is your H hat. So, the least-squares estimater H hat can be expressed as H hat equals H bar, the original parameter, unknown parameter vector + something that depends on noise, that is X transpose X inverse times X transpose V bar.

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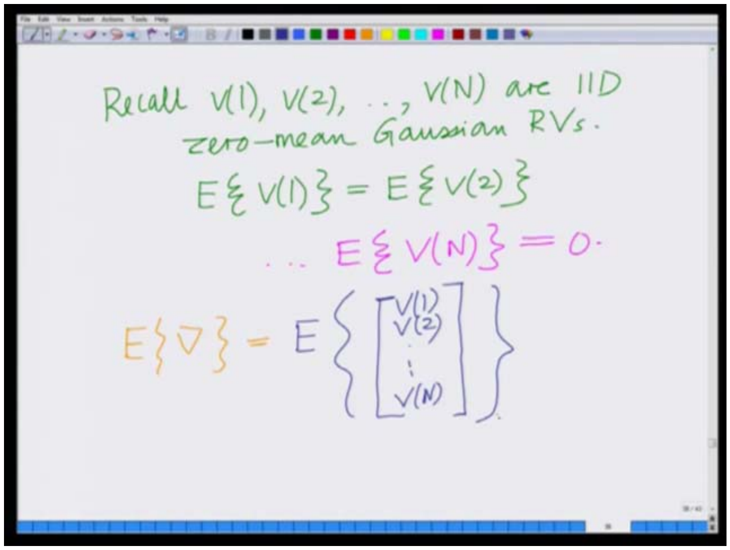

Now, again recall V1, V2… Up to VN are IID 0 mean Gaussian random variables. Implies expected value, since they are 0 mean, expected value of V1 equals expected value of V2, so on and so forth that is expected value of each random variable equals 0.

And therefore, now if I look at the expected value of this vector as a whole, that is if I look at the expected value of this Gaussian vector V bar, that is the noise vector V bar, that is equal to basically your expected value of your vector comprising of components V1, V2… Up to VN

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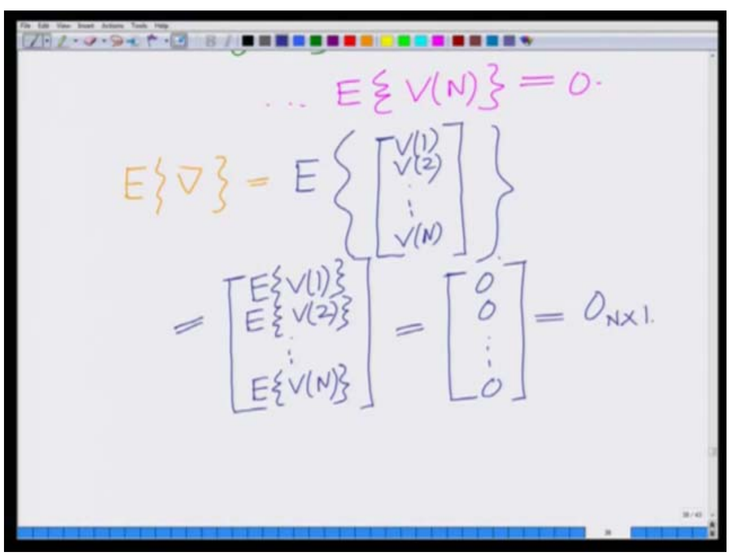

which is basically, now if I take the expected value inside, that is nothing but the expected value of V1, expected value of V2, so on expected value of VN and each of these is basically 0, so this is the N dimensional 0 vector. And this is fairly natural again, that is what we are saying is if you look at the noise vector, it is the noise components V1, V2… Up to VN, each of these Gaussian noise components has 0 mean, therefore the expected value of the vector V bar is 0. That is it is the 0 vector.

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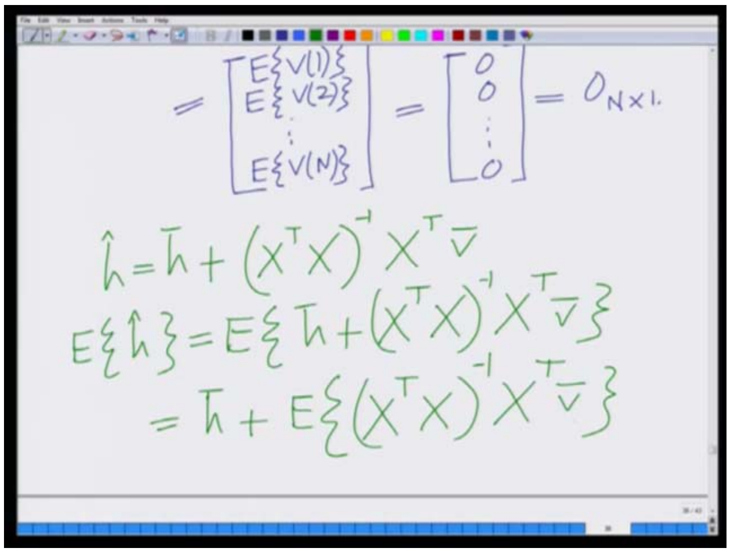

Now, I can use this to simplify the expected value of our estimate H hat, recall that H hat equals X transpose X inverse X transpose, that is equals your H hat equals H + X transpose X inverse X transpose V bar which means expected value of H hat equals expected value of H + X transpose X inverse X transpose V bar. Which is basically nothing but, now H bar is a constant, so I take it outside, H bar + expected value of X transpose X inverse X transpose V bar and now again, the pilot matrix X is constant, so I can move the expectation inside.

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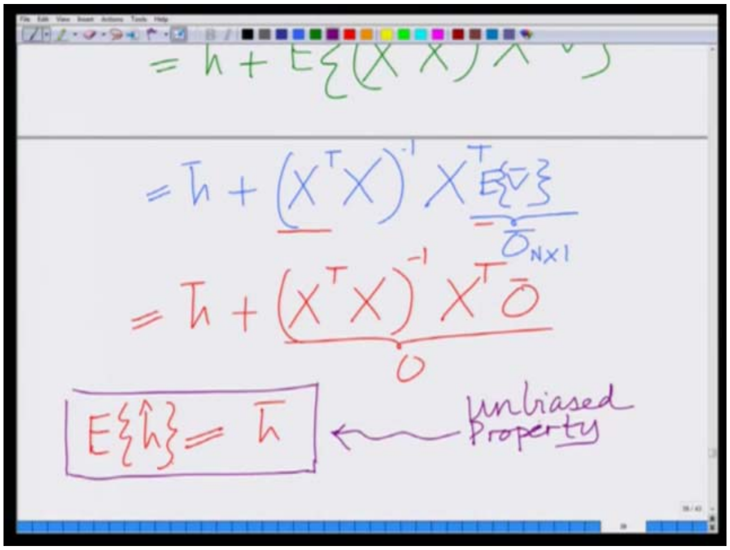

I have X transpose X inverse X transpose expected value of V bar but the expected value of V bar is the 0 vector, that is N x 1 zero vector, hence as a result of that basically, this quantity, which is X transpose, so let me just write that down, this is H bar + X transpose X inverse X transpose 0 bar, this quantity is 0, therefore what we have, again similar to what we have seen many times before that is your expected value of the vector estimate H hat equals H and this is basically your unbiased property. That is what we are saying is something very interesting and which we already said before in the context of a scalar estimate that is H hat is the estimate and it is a random quantity, remember, it is not deterministic quantity, remember we have already said that H hat is a Gaussian, is Gaussian in nature, it is a Gaussian vector.

However, if I look at the average value of H hat, that is expected value of H hat, that is equal to H bar, that is expected value is basically equal to the true value of the underlying unknown parameter. Therefore such an estimator is known as an unbiased estimator and the estimate is known as the unbiased estimate. So, this maximum likelihood estimate or the least-squares estimate is unbiased in nature.

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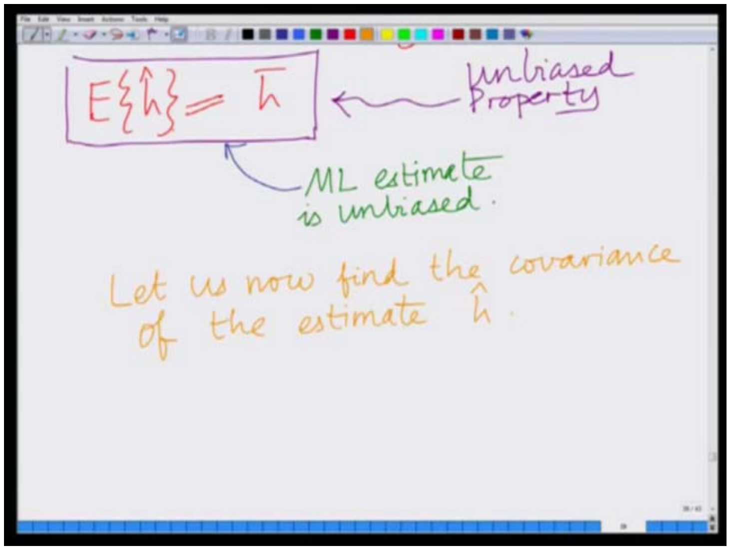

That is the 1st thing that we have explained established. So your ML or basically least-squares estimates, the ML or least-squares estimate is unbiased, that is the 1st thing. Now what we want to do, the 2nd thing that we have to do is find the **various** variance or the covariance, since this is a vector, right for a scalar random variable, we find the various, scalar estimate we find the variance.

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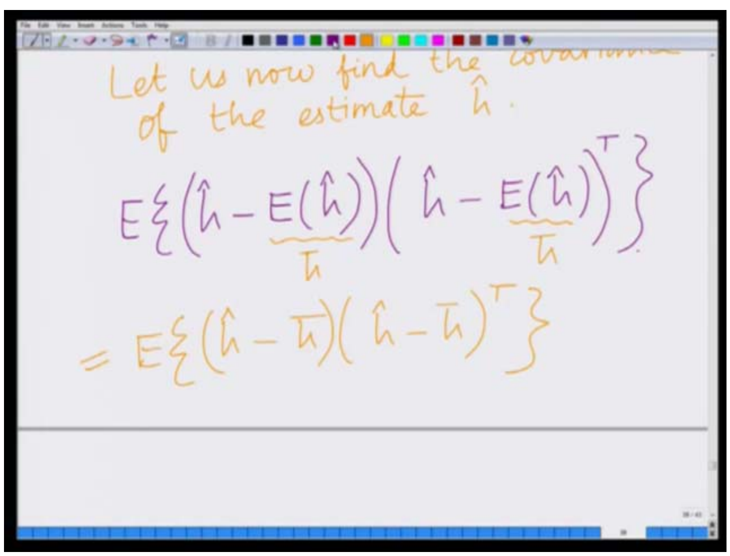

Since this is a vector, let us find the covariance of the estimate. So, let us now find the covariance of the estimate. Let us now find the covariance of the estimate H hat that is the covariance of the estimate is defined as, remember H hat - the expected value of H hat times its transpose, H hat - expected value of H hat slightly H hat - expected value of H hat transpose, I now realise that expected value of H hat is basically your vector H bar, because this vector is, the estimate is unbiased, so basically that is basically nothing but your expected value of H hat - H bar times H hat - H bar transpose. Now we have already seen that H hat, look at this, H hat equals H bar + X transpose X inverse X transpose V bar, so what I have here is basically,

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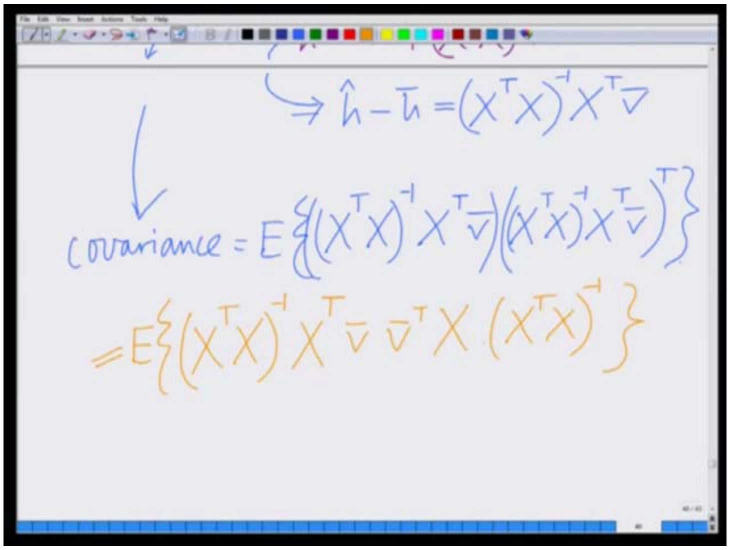

I will use the property H hat or let me use the property H hat equals H bar + X transpose X inverse X transpose V bar which basically implies that, now if you look at this, this basically implies H hat - H bar equals X transpose X inverse X transpose V bar.

And now if I substitute that in this expression over here, I will get the covariance is equal to expected value of H hat - H, that is X transpose X inverse X transpose V bar times its transpose, that is X transpose X inverse X transpose V bar transpose of this quantity. And now I can simplify this quantity, this quantity can be simplified as follows, this is the expected value of, well, the 1st quantity I can take it as it is, it is X transpose X inverse X transpose V bar. The 2nd quantity, transpose of that is V bar V bar transpose X transpose transpose which means X, X transpose X inverse transpose, that is again this is symmetric matrix, the inverse is also symmetric, therefore the transpose is itself.

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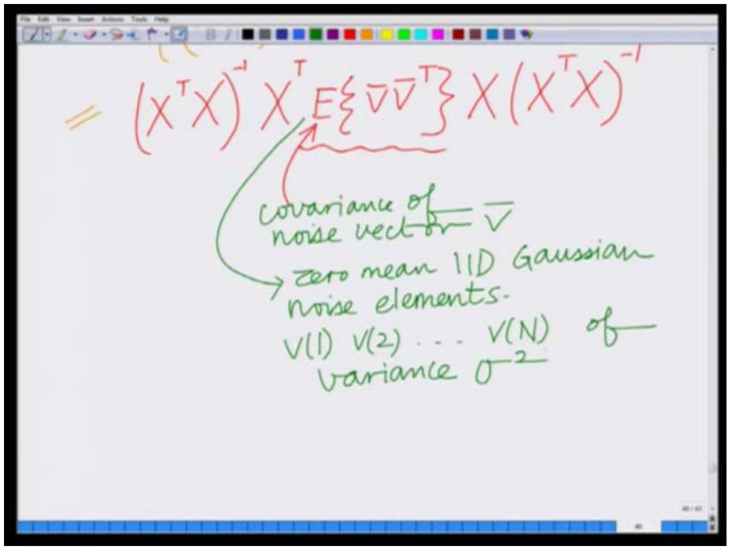

Now I can move the expectation operator inside, when I move the expectation operator inside, what I have is basically, I have your X transpose X inverse expected V V bar times X transpose X inverse, and now this expected V V bar transpose, this is nothing but covariance of the noise. Remember, expected V bar V bar transpose is the covariance of the noise vector V bar and remember again we are considering 0 mean IID Gaussian noise elements V1, V2… Up to VN, which means the covariance matrix is Sigma square times identity. That is also something that we have seen before, remember this is we are considering 0 mean IID Gaussian noise aliments V1, V2… Up to VN of variance that is each noise sample has mean 0, variance Sigma square and these noise samples are independent, implies,

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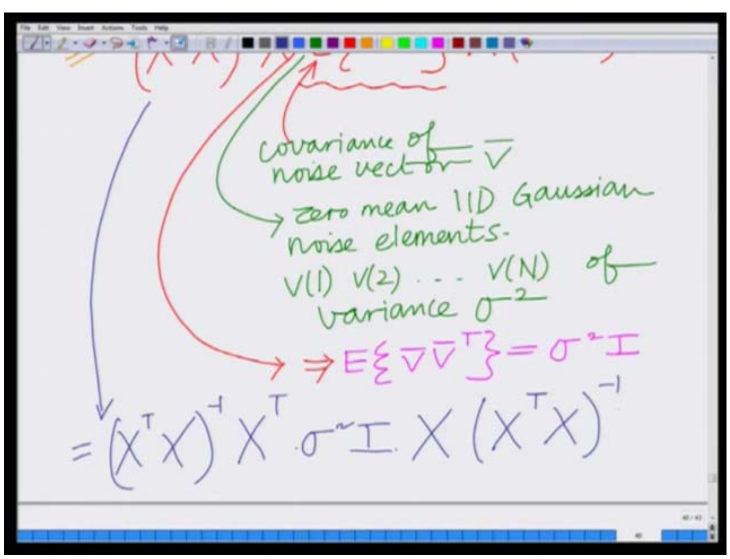

which implies the covariance expected V bar V bar transpose is equal to Sigma square times identity.

That is the noise covariance expected V bar V bar transpose considering 0 mean IID noise samples of mean 0 and variance Sigma square is basically Sigma square times identity. And now I substitute that in this expression for the covariance of the estimate H hat to simplify it further. And now once you substitute that things, I see it has a nice form, this is basically reduces to equal to X transpose X inverse X transpose times the covariance which is Sigma square times identity times X times X transpose X inverse

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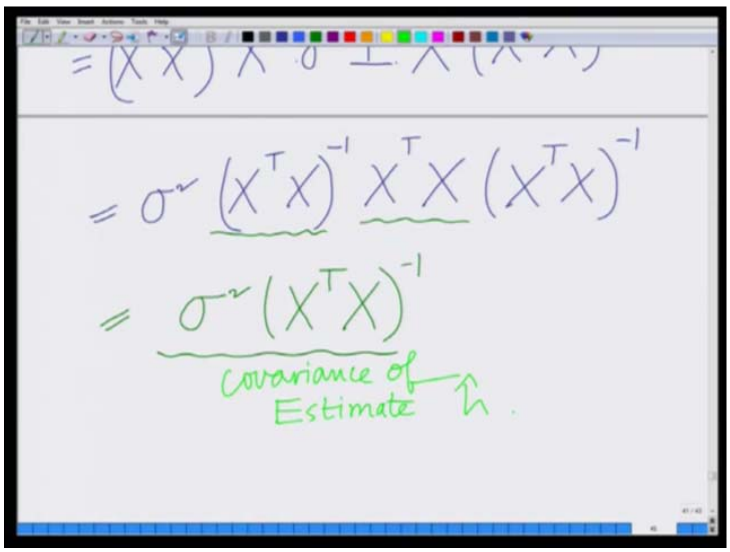

and now you can clearly see I can bring the Sigma square outside, that is a scalar quantity and identity times any matrix is the matrix itself.

So, this reduces to X transpose X X X transpose X inverse, now you can see that this I have an X transpose X inverse, I have an X transpose X, so basically this is identity, so this is Sigma square times what I am left with is X transpose X inverse and this is the covariance of the estimate. What is this, this is basically your covariance of the estimate H hat of the vector parameter. And finally to summarise, we already said 3 things, H hat is a Gaussian parameter, H hat has mean, this is an unbiased estimate, that is expected value of H hat is parameter vector H bar and its covariance is Sigma square X transpose X inverse, therefore now I can neatly summarise this as, is basically it is Gaussian, it is a constant vector.

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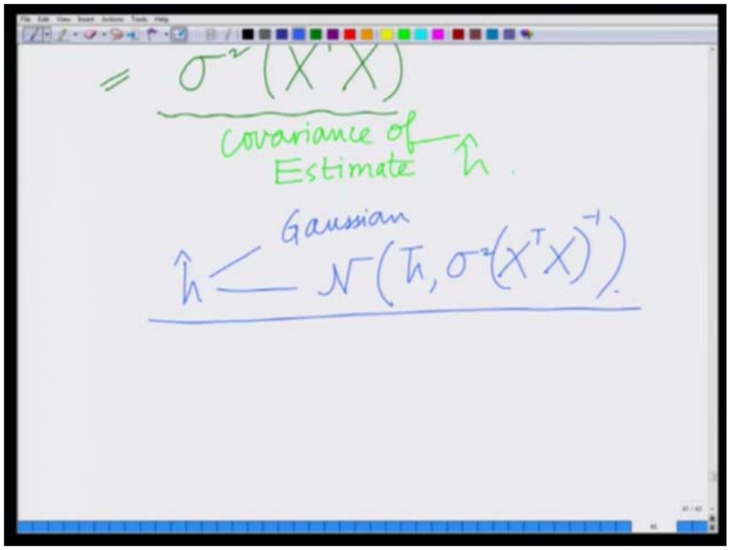

We denote it by N with mean H bar covariance X transpose X inverse. Alright, so this summarises the behaviour of the estimator which is Gaussian with mean H bar and covariance Sigma square X transpose X inverse. And now also realise something interesting that diagonal elements of the covariance matrix correspond to the variances of the components of the vector estimate.

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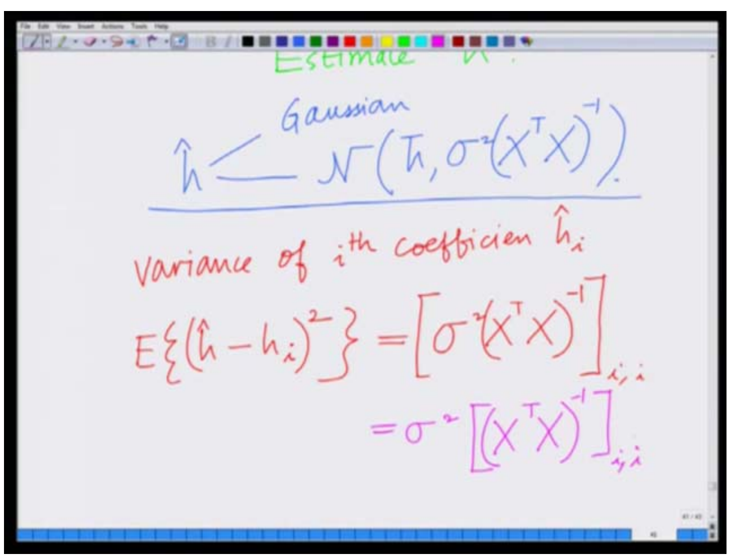

That is, if you look at the Ith estimate, variance of the Ith element, variance of the Ith coefficient H I, that is, rather variance or the estimate of the Ith coefficient, that is expected value of H I - H I square, this is that I element of the Ith diagonal element of the covariance matrix, this is Sigma square X transpose X inverse, that is the I, Ith element. That is Ith diagonal element or the I, Ith element of this matrix which is basically nothing but, Sigma square is a constant, so you can bring that out, that is I take the Ith diagonal element of this matrix.

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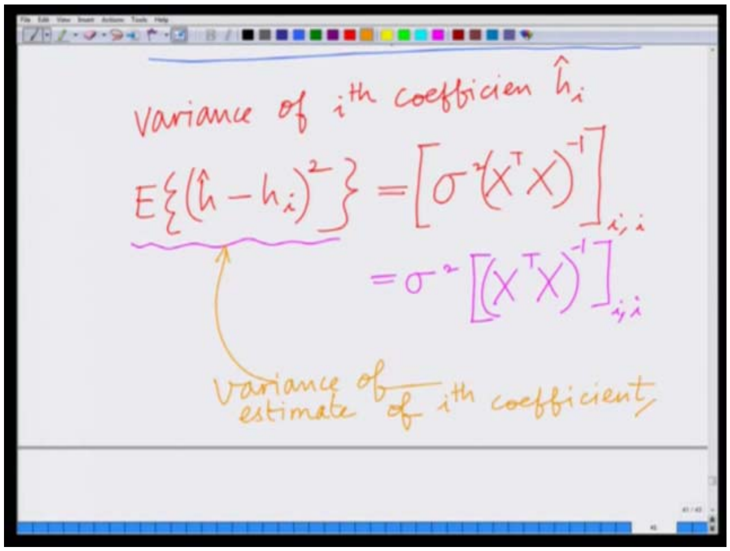

This is, what is this, this is the variance of the estimate of the Ith coefficient. This is the variance of the estimate of the Ith coefficient in the vector parameter H hat. Alright, so what we have seen in this module is basically we have looked at the least-squares or the maximum likelihood estimate H hat which is equal to X transpose X inverse times X transpose Y bar and we have simplified, we have explored the properties of this vector estimate and similar to before, we have said that 1st this vector estimate, it is a Gaussian vector, remember, unlike the scalar estimates that we have considered previously, now we are considering a estimate, a vector estimate or an estimate of a vector parameter. So, what we have is a Gaussian vector which is unbiased, which means the average value of this estimate a chat is basically the underlying parameter H bar.

At the covariance is basically given by Sigma square X transpose X inverse, where X remember we said is the pilot matrix. Alright, and finally we have also said the variance of the Ith, the variance in the estimate of the Ith coefficient is given by the Ith diagonal element or I, Ith element of this covariance matrix. Alright. So, that basically succinctly summarises the properties of this maximum likelihood estimate and we will explore other aspects in the subsequent modules, thank you very much.