Probability and Random Variables/Processes for Wireless Communication
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Module No. 2
Lecture 9
Random Variables, Probability Density Function (PDF)

Hello, welcome to another module in this massive open online course on probability and random variables for wireless communications. Today, we are going to start with a new topic that is the concept of a continuous random variables.

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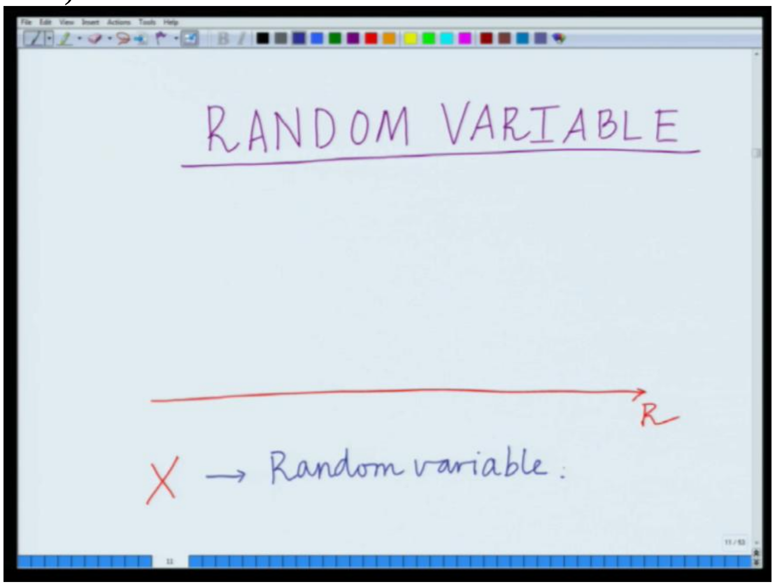

So we are going to start dealing with the notion of a random variable which is an important concept. So we will start dealing with a random variable and a random variable can take basically as the name implies, can take values randomly. So a random variable, X. If X is a random variable, then X can take values randomly from either the entire set of real numbers or a subset of real numbers.

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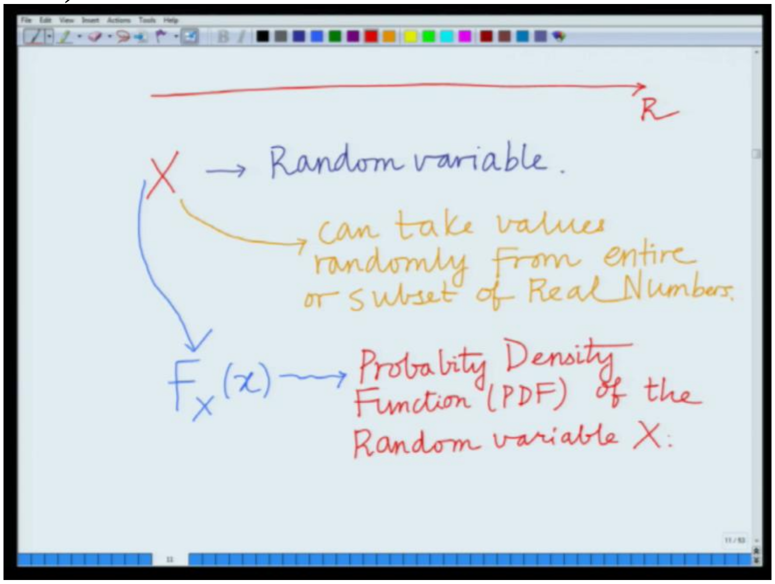

So X which is a random variable can take values randomly from entire or subset of real numbers. So X is a continuous random variable. It can take values randomly from the set of real numbers. And this random variable X is characterized by a very important function. This is known as the probability density function. That is, this is characterized by $F_x(x)$, the subscript X denotes the random variable. The small X denotes the value of this probability density function at a particular point X. So this is the probability density function also denoted by PDF.

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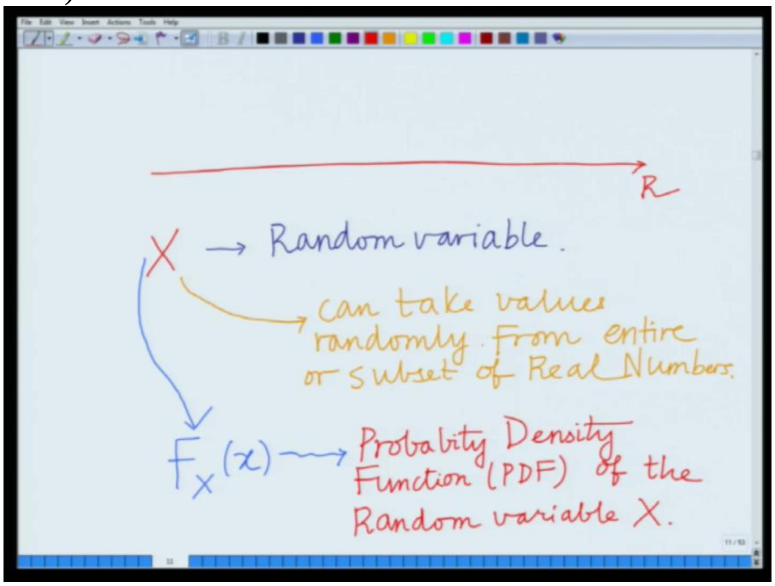

And what does this PDF, probability density function of the random variable X, what does it to denote?

Let us consider a small interval of width dX around X. That is let us consider a small interval between X and X + dx.

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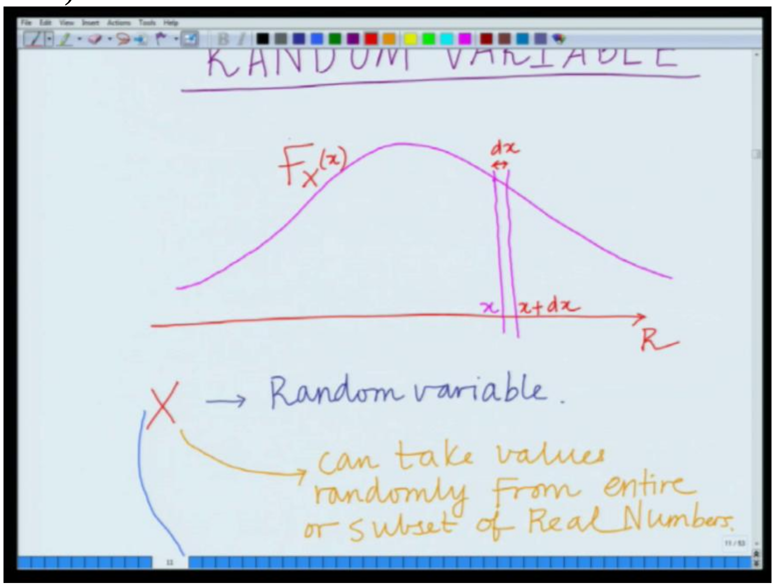

So let us say this is the PDF. Let us consider a small interval, infinitesimally small interval between X and X + dX. We are considering an infinitesimally small interval of width dX. This is my probability density function $F_x(x)$.

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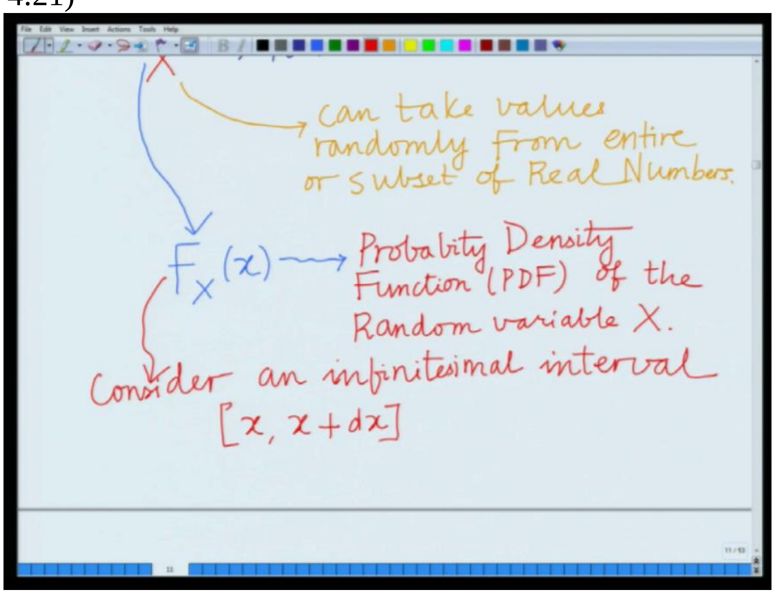

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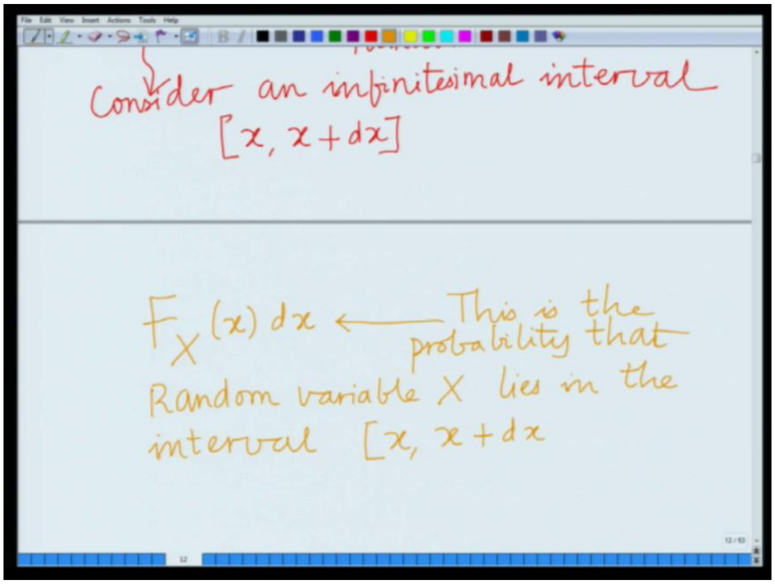

Then the quantity, $F_x(x)$times dX, represents, the probability that random variable X lies in the interval X ~~and~~ to X + dX.

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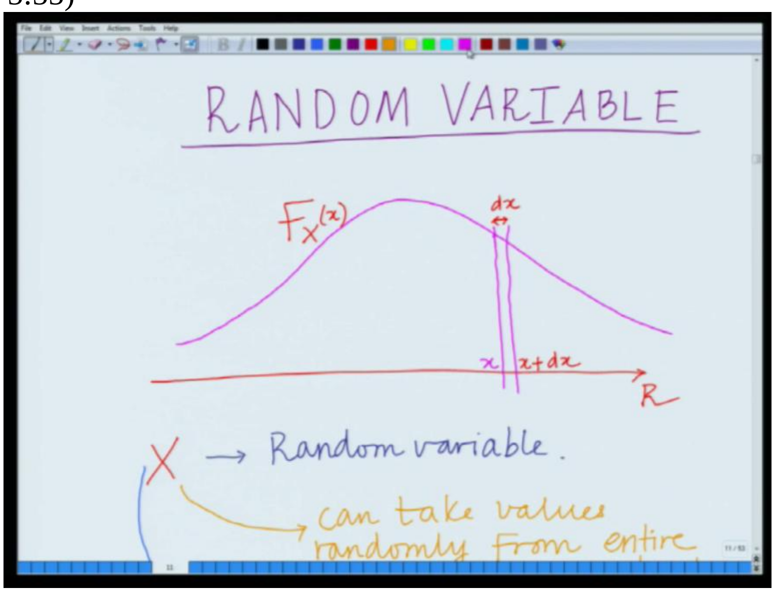

So if we look at this infinitesimal interval between X and X + dX, the quantity $F_x(x)$dX basically at a point X denotes the probability that the random variable, capital X takes value in this infinitesimal interval of width dX. That is, in this infinitesimal interval around X which is the infinitesimal interval X to X + dX.

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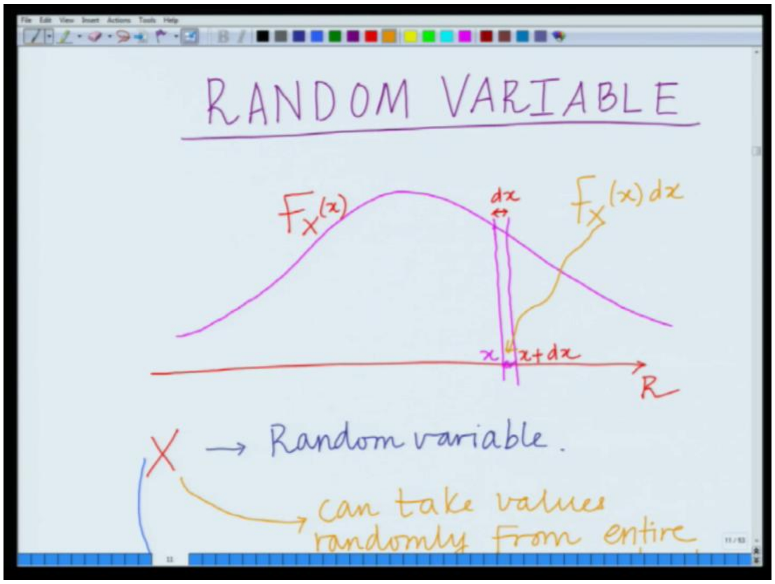

so $F_x(x)$ times dX represents the probability that the random variable takes a value in this infinitesimal interval. So that is the significance of this probability density function.

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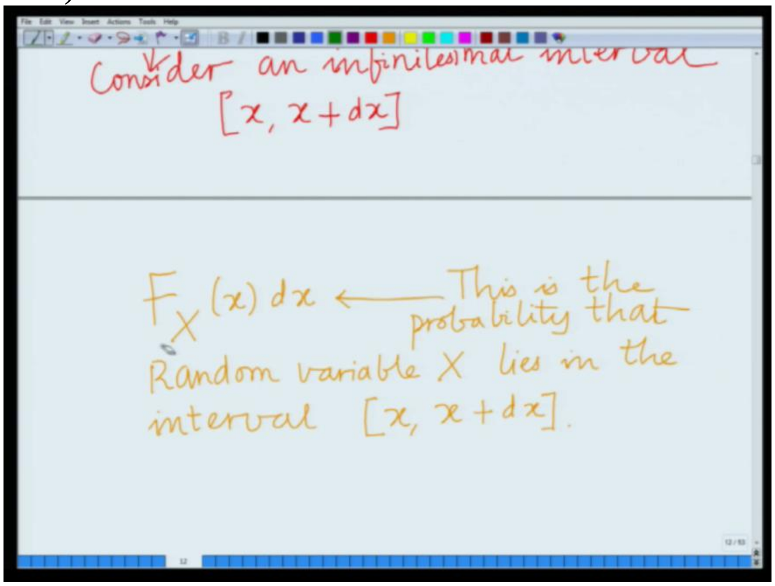

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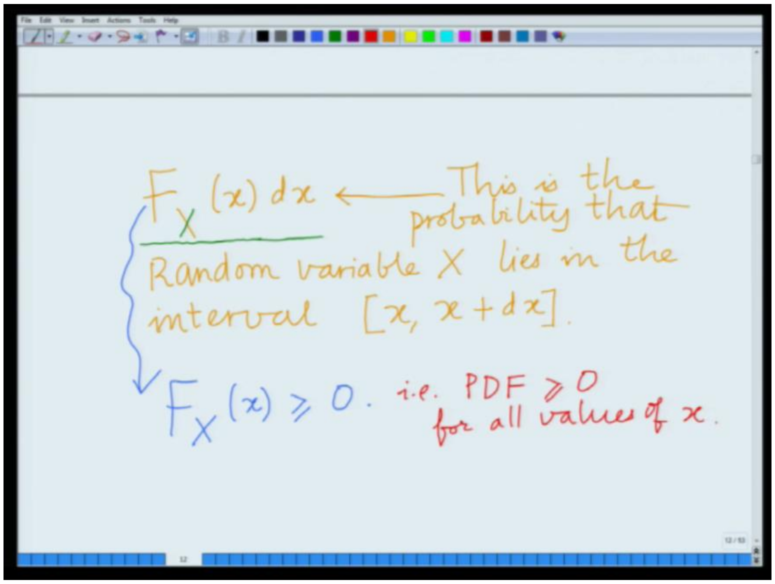

And naturally, since $F_x(x)$ represents the PDF, represents the probability, it must be the case that since probability cannot be 0, $F_x(x)$ has to be greater than or equal to 0. That is, the PDF

$$F_x(x) \geq 0, \quad \forall x$$

It cannot happen that the probability density function is negative for a certain value of X. Okay? So the probability density function represents the probability density of the random variable X which when multiplied by dX gives the probability that the random variable X lies in the infinitesimal interval around small X that is lies in the infinitesimal interval between X and X + dX.

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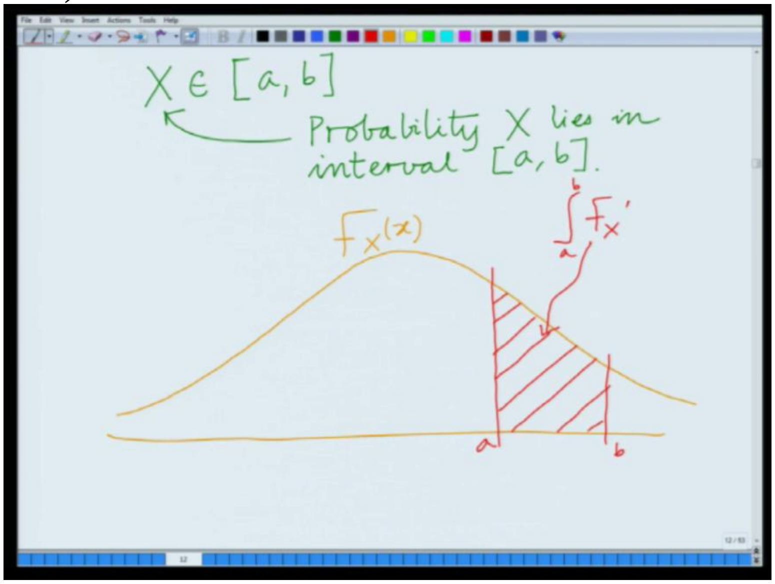

Also, this naturally means that, therefore now the probability that X lies in this interval, if we consider any interval AB, probability that X lies in this interval AB, is basically now if you look at the probability density function again, this is the probability density function, $F_x(x)$. And now let us say, I am considering this interval between 2 points, A and B. The probability that this random variable lies in the interval A to B is the area under the probability density function between the points A and B that is equal to-

$$\int_a^b f_x(x)dx$$

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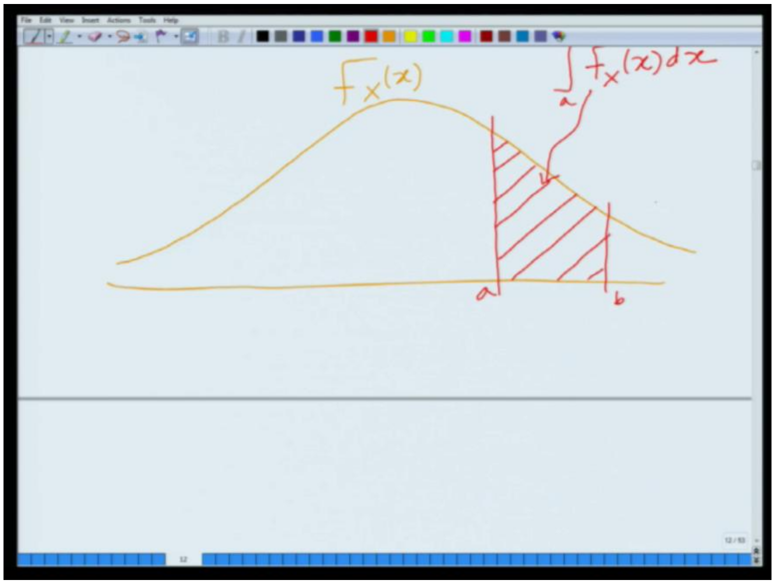

Therefore what we are saying is that the probability

$$P(X \in [a, b]) = \text{area under pdf } F_x(x) \text{ between points A and B} = \int_{x=a}^{x=b} f_x(x)\,dx$$

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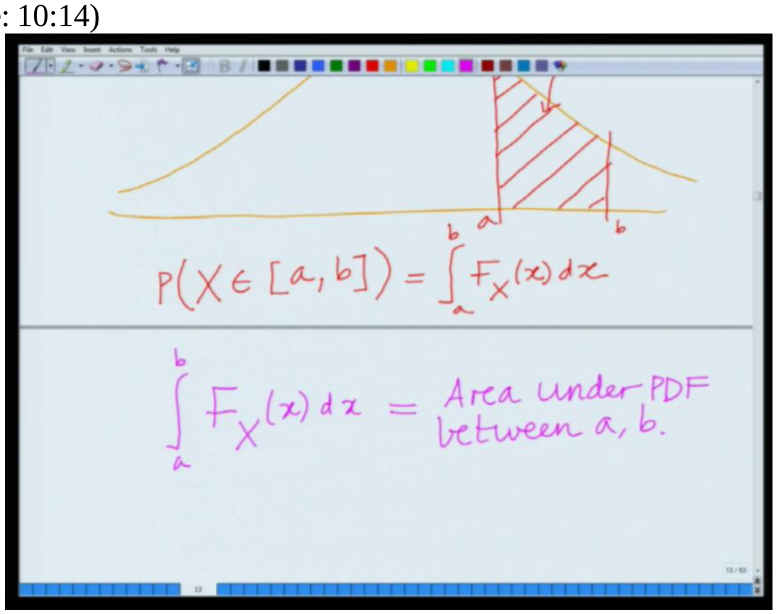

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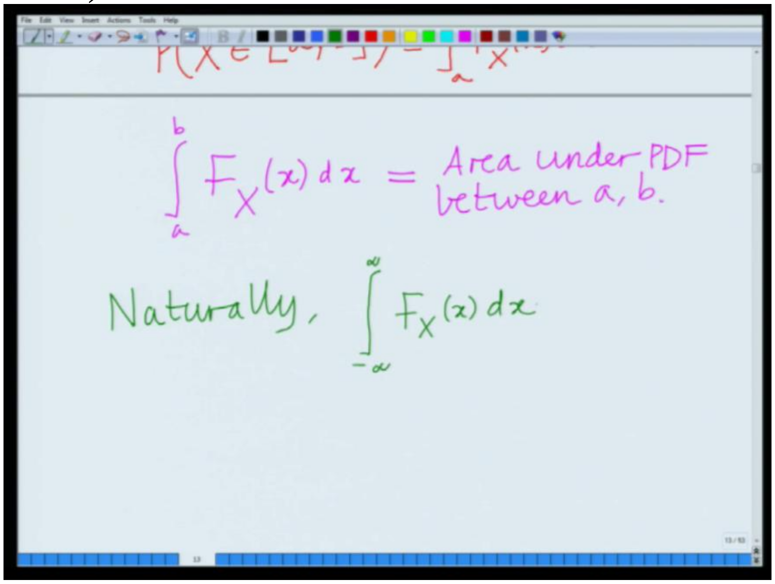

Naturally therefore, it follows that I am saying naturally because it follows from the above result, if we consider integral,

$$\int_{-\infty}^{\infty} f_x(x)dx = 1$$

If I consider the integral between the limits $-\infty$ to ∞ of $f_x(x)dx$ where $f_x(x)$ is the probability density function. This integral over the entire set of real numbers must be infinity because the entire set of real numbers is the sample space. It is the sample space for this random variable, therefore the probability that it takes any value from the sample space must be 1. The total probability that it takes any value from the Sample space must be equal to 1.

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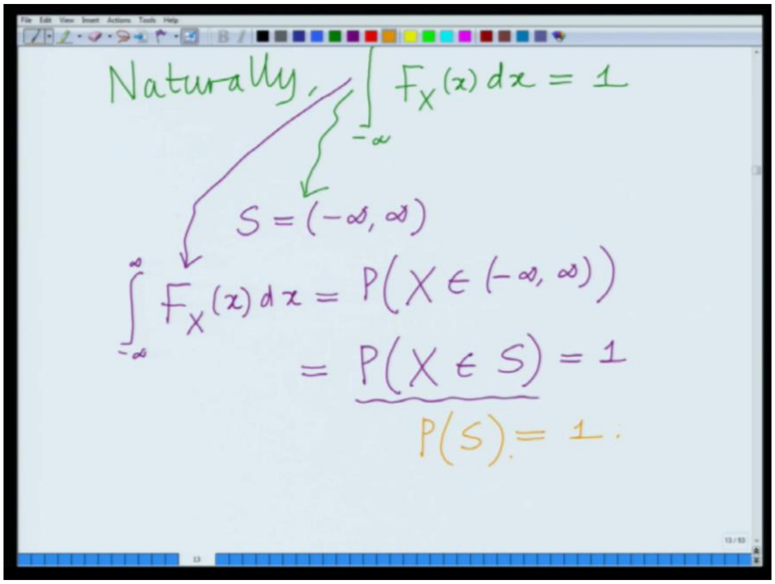

So what we are saying is, for this random variable, X, the sample space S equals

$$S = (-\infty,\infty)$$

therefore, integral over $(-\infty,\infty)$ of $F_x(x)$ dX is basically the probability that the random variable X belongs to $(-\infty,\infty)$, which is the same as the probability that the random variable, X belongs to the sample space which is equal to 1.

Because this is basically, a quantity as we had seen in the axioms of probability before, this is the probability of the sample space. and from the exempts of probability, that is from the 2nd axiom of probability, we have the probability of the sample space equal to 1.

So we have 2 interesting properties, First is that the probability density function $F_x(x)$ is always greater than or equal to 0. And 2nd , the area between $-\infty$ to ∞ under $F_x(x)dX$ is equal to 1, i.e.

$$f_x(x) \geq 0$$

$$\int_{-\infty}^{\infty} f_x(x)dx = 1$$

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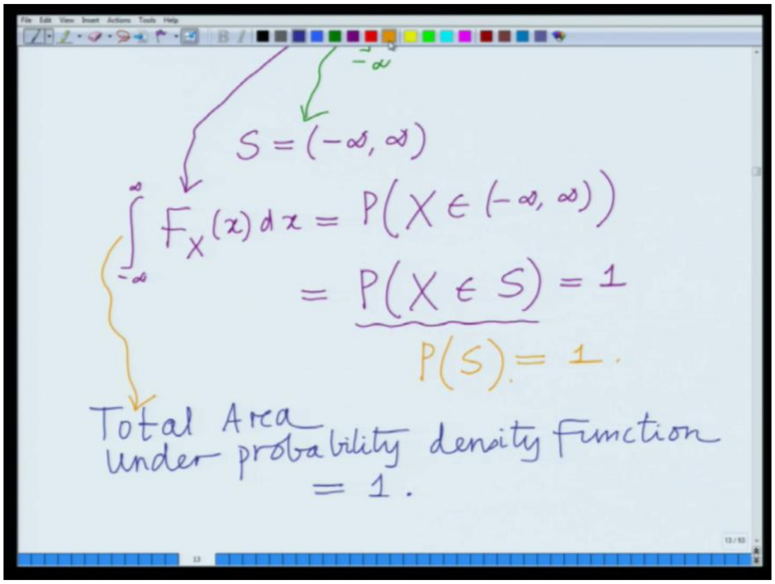

And, $F_x(x)$ times dX represents the probability that the random ~~page table~~ variable X takes a value in the infinitesimal interval of width dX around X. That is it lies in the interval between X to X + dX.

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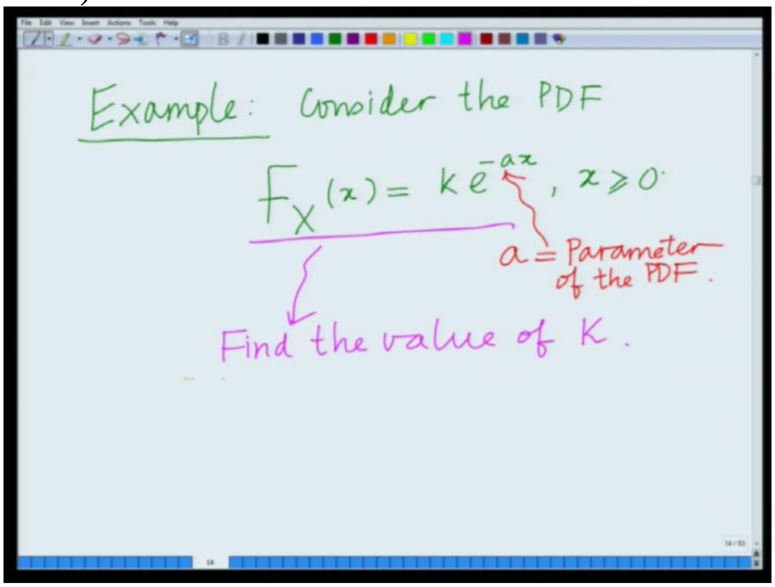

So for example, let us take a look at this simple example now. For example, let us take a look at, consider the PDF,

$$f_x(x) = ke^{-ax}, \qquad x \geq 0$$

Where quantity *a* is termed as the parameter.

and what we are asked to do is we are asked to find the value of the unknown constant K.

Naturally, we can find it as follows. We know that the total probability is 1.

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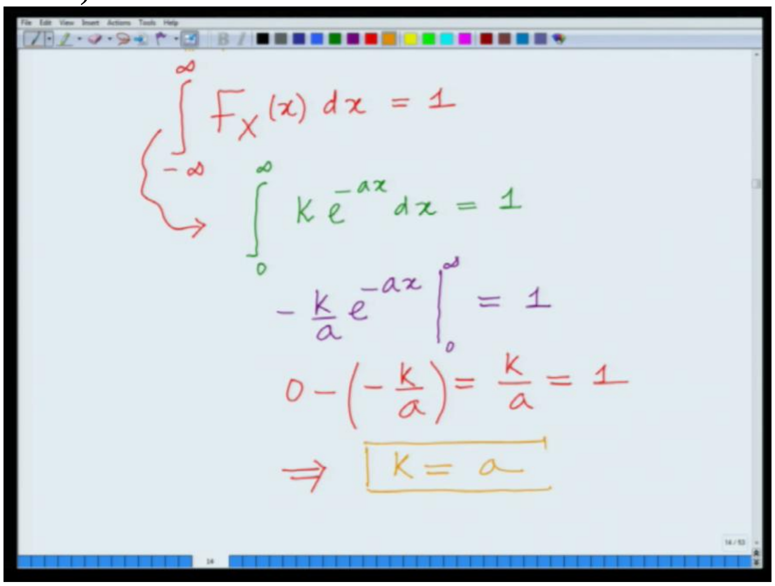

Therefore,

$$\int_{-\infty}^{\infty} f_x(x)dx = 1$$

$$\int_{-\infty}^{\infty} ke^{-ax}dx = 1$$

$\frac{-k}{a}e^{-ax}$ from 0 to ∞

0-($\frac{-k}{a}$)= ($\frac{k}{a}$)= 1

⇒ k =a

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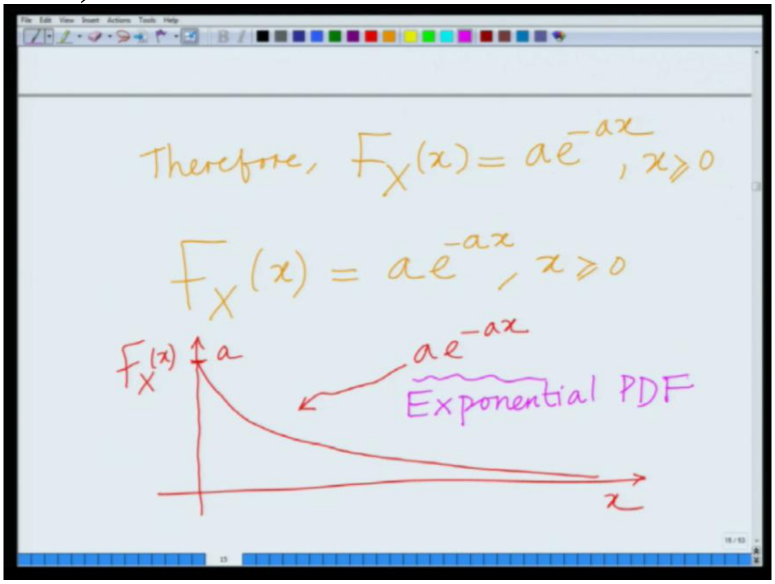

Therefore the PDF is given as

$$f_x(x) = ae^{-ax}$$

greater than or equal to 0 and this PDF is decreasing exponentially. This PDF for a is constant. It is decreasing exponentially. It starts with a at x=0 and it is decreasing exponentially. At x=0, this is basically A e to the power of - 0 which is a and it is decreasing exponentially, this PDF is also known as the exponential PDF. This is one of the standard probability density functions also known as an exponential probability density function.

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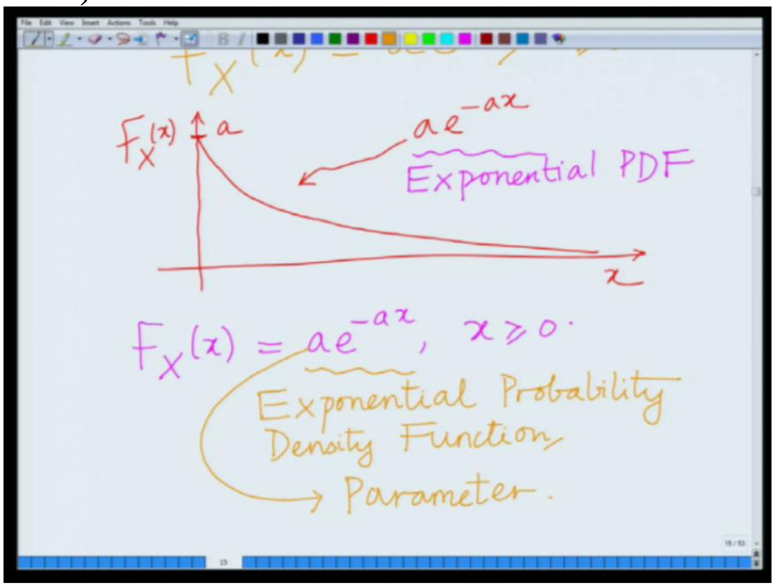

Here *a* is the parameter which characterizes this exponential probability density function. So in this module, we have seen a definition of the probability density function of a random variable, the properties of the probability density what it represents, the probability density function, the properties of this probability density function and also we have done a simple problem involving this exponential probability density function. So let us stop this module here. We will look at other aspects in the subsequent modules. Thank you very much