Probability and Random Variables/Processes for Wireless Communication
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Module No. 2
Lecture 8
Maximum Aposteriori Probability (MAP) Receiver

Hello, welcome to another module in this massive open online course on probability and random variables for wireless communications. So in the previous module, we have looked at Bayes Theorem. Let us now look at an application of Bayes Theorem in the context of digital communication systems and wireless communications. As we said, this is a course on applied probability theory, so we have looked at Bayes Theorem. Let us look at an important application of this in the context of wireless communications.

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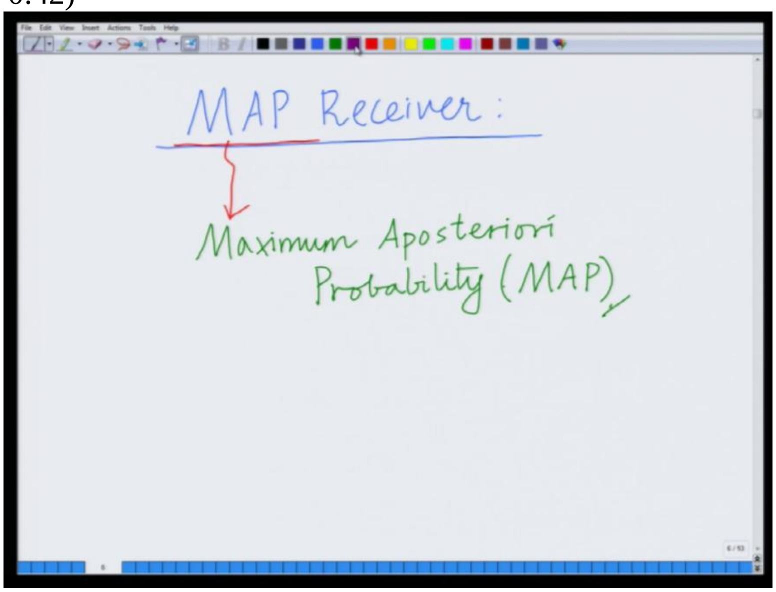

So what we are going to look at today is known as the MAP receiver and this map: this acronym MAP stands for Maximum Aposteriori Probability or basically the MAP receiver. This is a very important receiver in the context of wireless communication or even a traditional digital communication. That is digital information symbol based communication system. This MAP receiver which computes the Aposteriori probabilities that we had illustrated previously in the context of Bayes Theorem. So let us now consider a digital communication system with binary information symbols in which the sample space is S consisting of the binary information symbols, 0 and 1.

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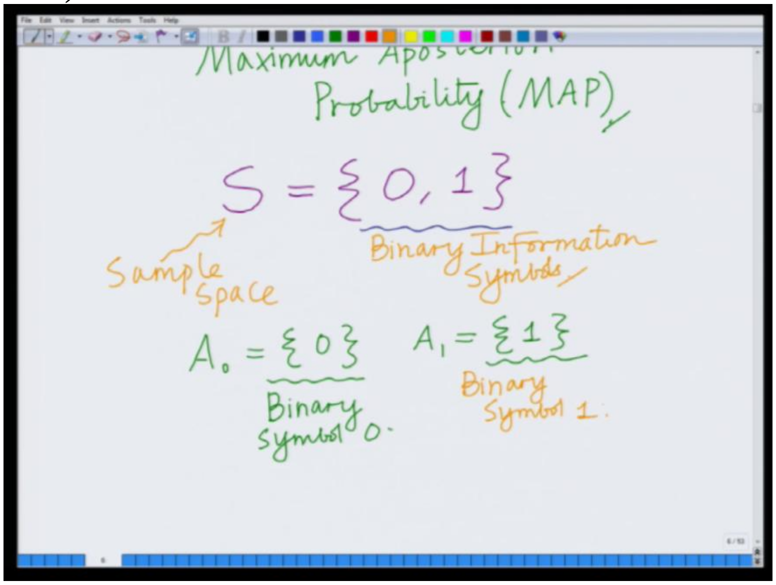

So let us consider a digital communication link based wireless communication system in which the sample space consists of the binary information symbols. So this is my sample space

$$S = \{0,1\}$$

and these are my binary information symbols. Therefore, now, so binary information symbols are 0 and 1 and the sample space contains these 2 binary information symbols, 0 and 1. Now let us consider the events

$$A_0 = \{0\},$$

$$A_1 = \{1\}$$

So there are 2 binary symbols, 0 and 1. A0 corresponds to the event of the transmitted binary symbol being 0. A1 corresponds to the transmitted binary symbol being 1.

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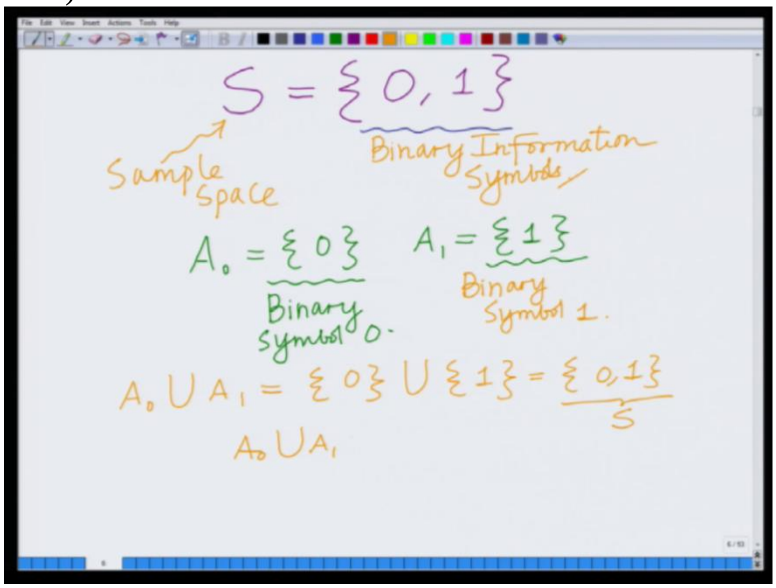

And therefore now, you can clearly see that

as,

$$A_0 \cap A_1 = \emptyset \text{ and}$$
$$A_0 \cup A_1 = S$$

Thus, A_0 and A_1 are mutually exclusive and exhaustive.

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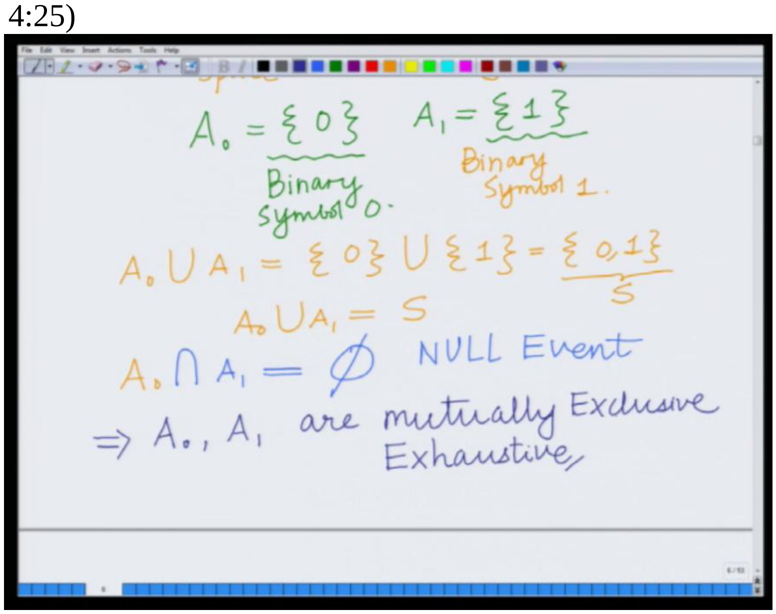

So, A0 and A1 are therefore for this binary information symbol based digital or wireless communication system, mutually exclusive and exhaustive. Therefore this satisfies one of the prerequisite conditions for our Bayes Theorem. Remember, for Bayes Theorem, we said whatever are the sets, A_0 and A_1, they have to be mutually exclusive, that is their intersection has to be the null event and they have to be exhaustive that is the union of these has to be the entire sample space, S. So this satisfies the prerequisite for the Bayes Theorem.

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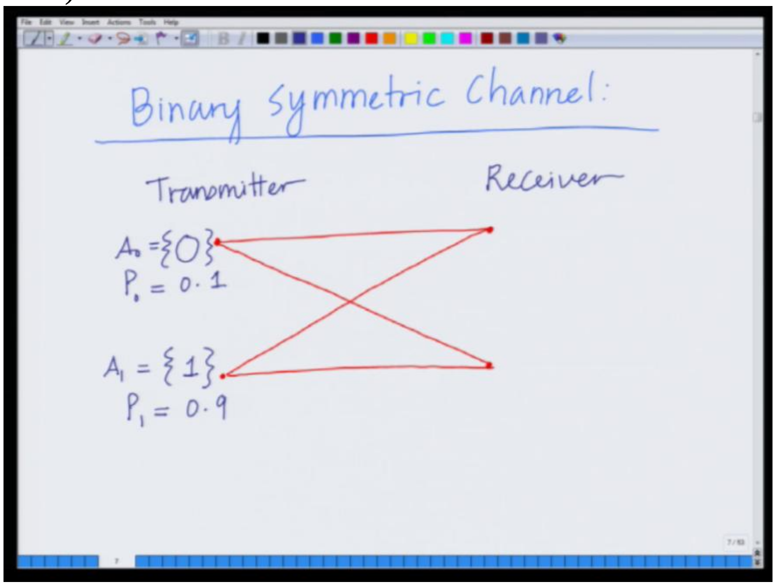

Now let us look at the application of this in the context of communication. We consider what is known as a binary symmetric channel. Let us consider a standard channel in communication. This is a binary symmetric channel and this can be described as follows. I am showing a schematic representation of a binary symmetric channel where I can have the left side is the transmitter and the right side is the receiver. I can transmit either a 0 or 1. That is A_0 as yet seen before, A_0 corresponds to the event is 0, A_1 corresponds to the event that the transmitted symbol is 1. Let us say that the

$$p(A_0) = P_0 = 0.1$$

and

$$p(A_1) = P_1 = 0.9$$

So we have a binary symmetric channel in which I have the transmitter side and I have the receiver side and at the transmitter side, I am saying that we can transmit a symbol that is A_0 that is a symbol 0, that is A_0 corresponds to the transmission of the symbol 0. Event A_1 corresponds with the transmission of symbol 1. And we are also saying that the probability of A_0 is 0.1. That is, the probability of the symbol, 0 is transmitted is 0.1. The probability that the symbol, 1 is transmitted that is the probability of A_1 which is equal to P_1 is equal to 0.9.

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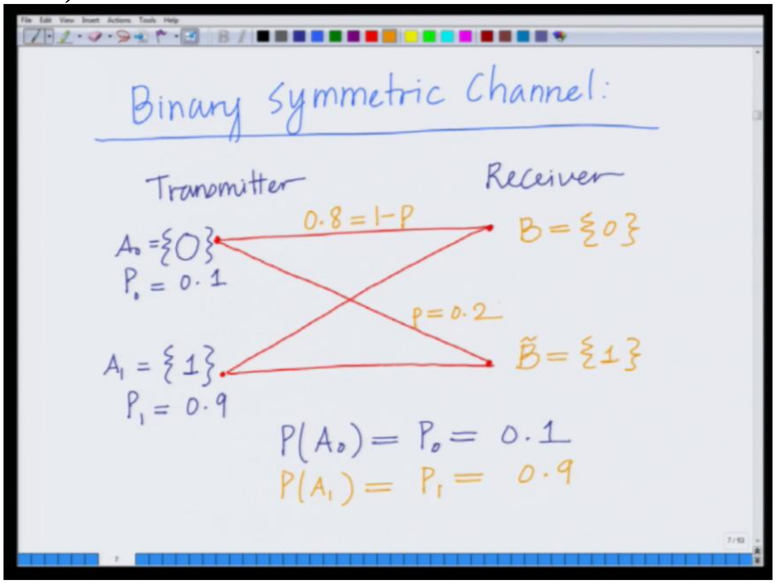

Further, at the receiver also, we can have 0 that is the event B which corresponds to receiving 0 i.e.

$$B = \{0\}$$

$$\text{and } \tilde{B} = \{1\}$$

or the event $\tilde{B}$ which corresponds to receiving 1.

Further, a transmitted 0 can either be received as a 0, the probability of which is equal to 0.8 which is equal to 1 minus P or the probability of 0 can get flipped to 1, the probability of which is P equal to 0.2.

So therefore we are saying that this is an interesting scenario where a transmitted 0 can either be received as a 0 with a probability 0.8 or the transmitted 0 can be flipped to 1. That is, there is an error which occurs on the channel due to which the 0 that is transmitted gets flipped to a 1 and the probability of this happening is 0.2.

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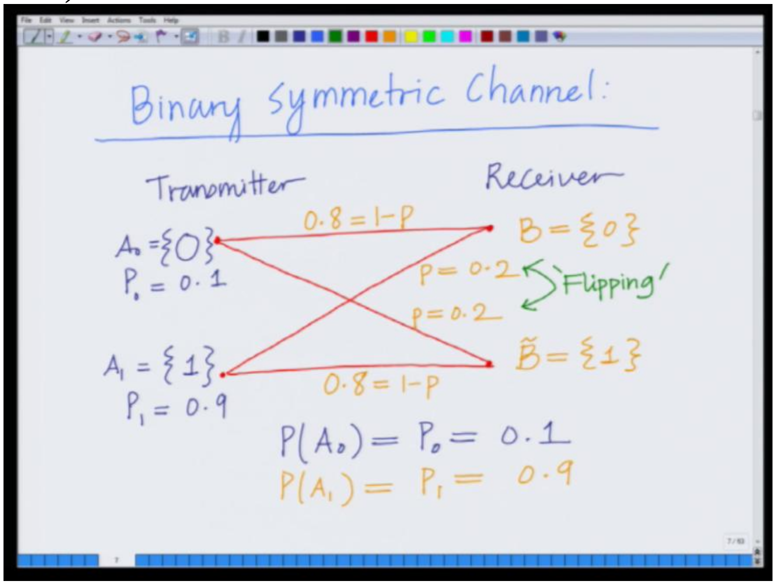

Similarly, the 1 that is transmitted, can be either received as 1, the probability corresponding to which is 0.8 which is a call to (1-p). Or it can get flipped to a 0 the probability of which happening is P = 0.2 .

So these 2 links, these 2 things that we show here, these correspond to the flipping. What do we mean by flipping? That is, a 1 gets flipped to a 0 or a 0 gets flipped to a 1 because of errors on the channel and this channel is known as a binary symmetric channel basically because it represents binary information symbols, 0 and 1 which are transmitted as 0 and 1 and can be received as either 0 or 1, and symmetric because these different probabilities are symmetric between 0 and 1.

The probability that a 0 can get flipped to a 1 and the probability that a 1 can get flipped to a 0, both these probabilities are symmetric and these probabilities are equal to 0.2. Therefore, this is known as a binary symmetric channel. This is a very important channel which is used to model the transmission of digital information symbols across both, a digital communication channel and also, a wireless communication channel. A wireless communication channel which uses digital symbols of course is also based, is a special case of a digital communication system. So this is a binary symmetric channel which is used to model the transmission of binary information symbols over a communication channel.

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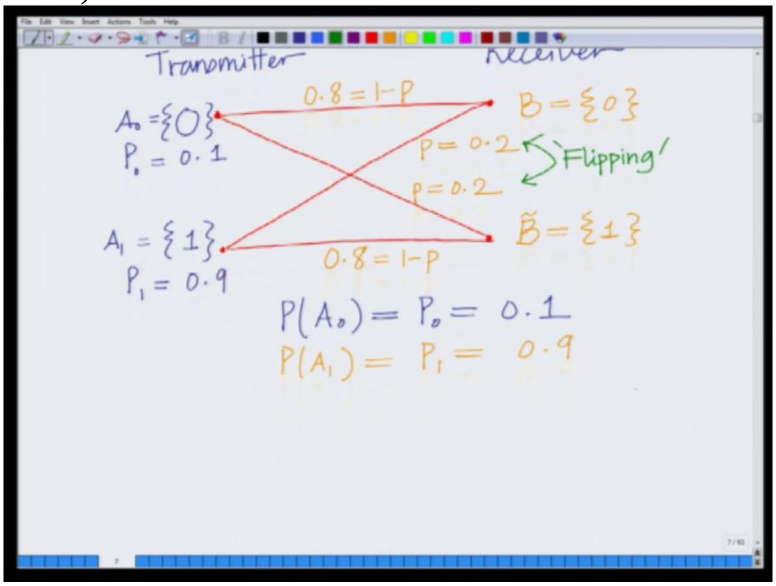

And let us now look at this. What we have given? We have given that the probability of A_0 that is the probability that the transmitted symbol is 0 is 0.1. The probability of A_1 which is equal to the P_1 which is equal to 0.9 is basically the probability that the transmitted symbol is 1.

$$P(A_0) = 0.1$$
$$P(A_1) = 0.9$$

And as we had said before, these are the prior probabilities.

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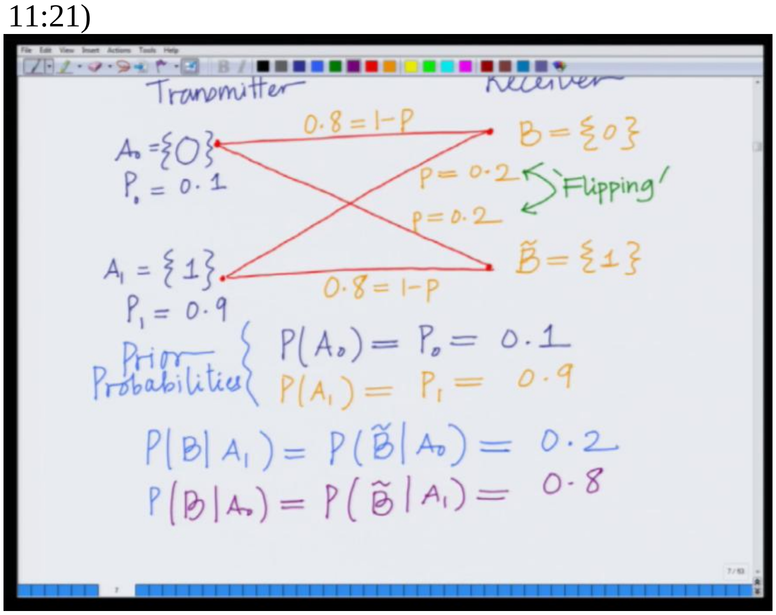

Now also observed, the probability of flipping that is

$$P(B|\ A_1) = P(\widetilde{B}\ |\ A_0) = 0.2$$

these are the flipping probabilities

$$P(B|\ A_0) = P(\widetilde{B}\ |\ A_1) = 0.8$$

these are the non flipping probabilities

These conditional probabilities together are known as the likelihoods, for this case.

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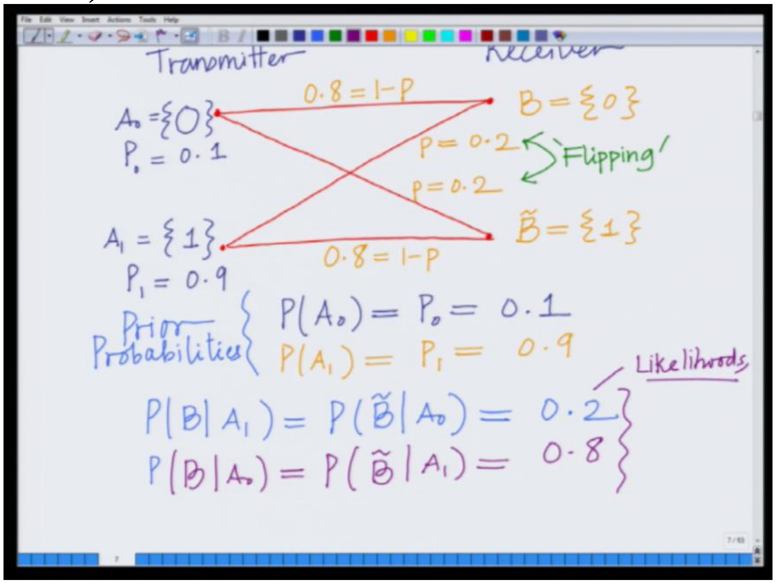

So we are given the prior probabilities, we are given the likelihoods. So now we would like to employ the Bayes Theorem to compute the Aposteriori probabilities. What is the Aposteriori probability?

We would like to compute, what is the $p(A_0|\ B)$? Remember, this is the Aposteriori probability. That corresponds to the probability corresponding to the transmission of 0 and B is the event corresponding to the reception of 0. Therefore, $p(A_0|\ B)$ corresponds to the probability that a 0 has been transported given that a 0 has been received.

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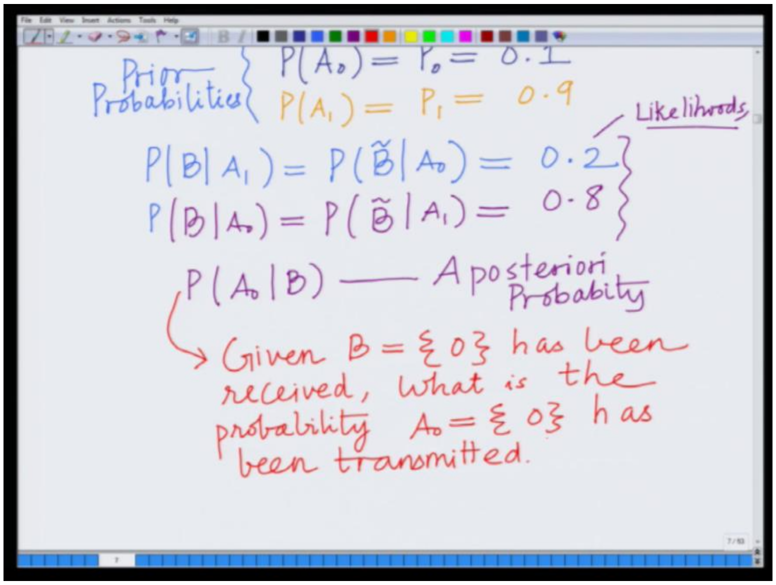

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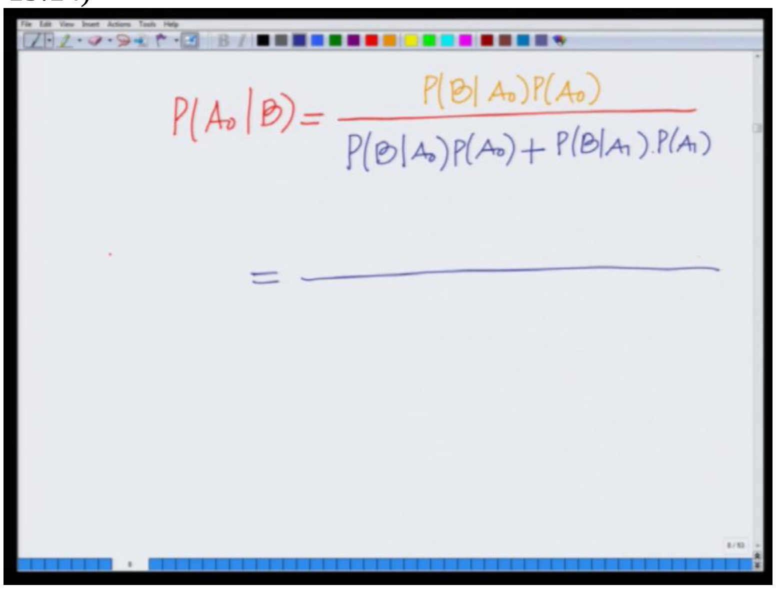

And now, let us compute this. We know that the Aposteriori probability of $p(A_0| B)$ from the Bayes theorem that this is equal to

$$P(A_0|B) = \frac{P(B|A_0).P(A_0)}{P(B|A_0).P(A_0)+P(B|A_1).P(A_1)}$$

And now let us go back and take a glimpse at this channel.

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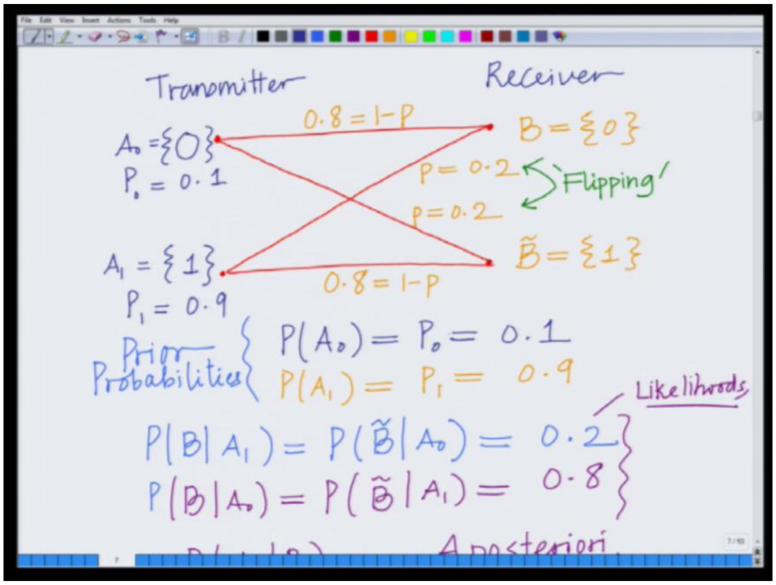

$$P(B|A_0) = 0.8$$

$$P(A_0) = 0.1$$

$$P(A_1) = 0.9$$

$$P(B|A_1) = 0.2$$

Now, all we have to do is substitute these values in the Bayes Theorem and this can be calculated as

$$P(A_0|B) = \frac{(0.8)(0.1)}{(0.8)(0.1)+(0.2)(0.9)}$$

$$= \frac{8}{26}$$

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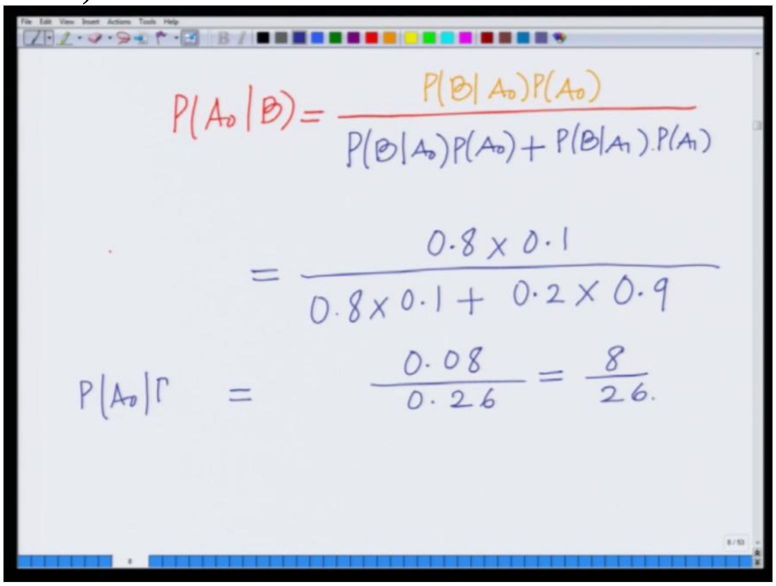

So the probability, the Aposteriori probability of A0 given B is equal to 8 divided by 26.

So what we have seen? Using the Bayes Theorem, we have calculated the Aposteriori probability of (A_0 | B). That is what is the probability that a 0 has been transported given that a 0 has been received at the receiver. Now let us compute the other Aposteriori probability. What is the probability that a 1 has been transmitted given that a 0 has been received?

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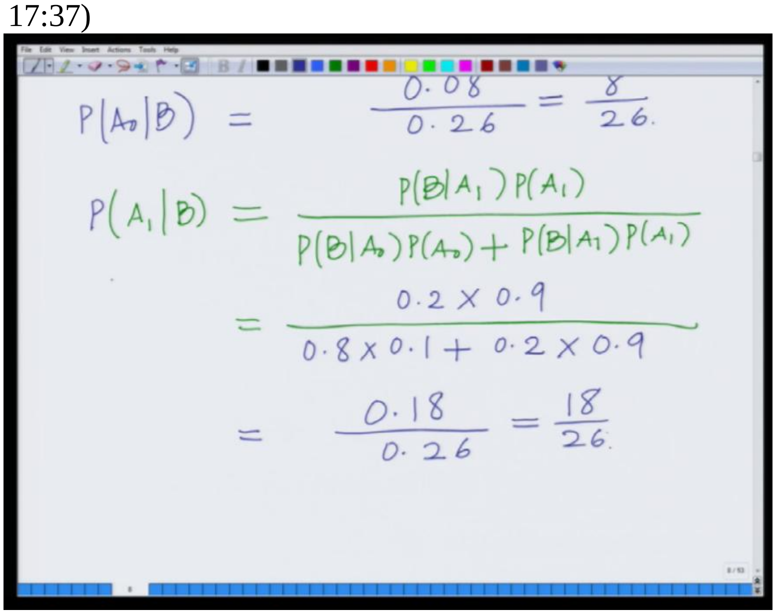

And this can be calculated as the $P(A_1|B)$ which is equal to

$$P(A_1|B) = \frac{P(B|A_1).P(A_1)}{P(B|A_0).P(A_0)+P(B|A_1).P(A_1)}$$

$$P(A_1|B) = \frac{(0.2)(0.9)}{(0.8)(0.1)+(0.2)(0.9)}$$

$$= \frac{18}{26}$$

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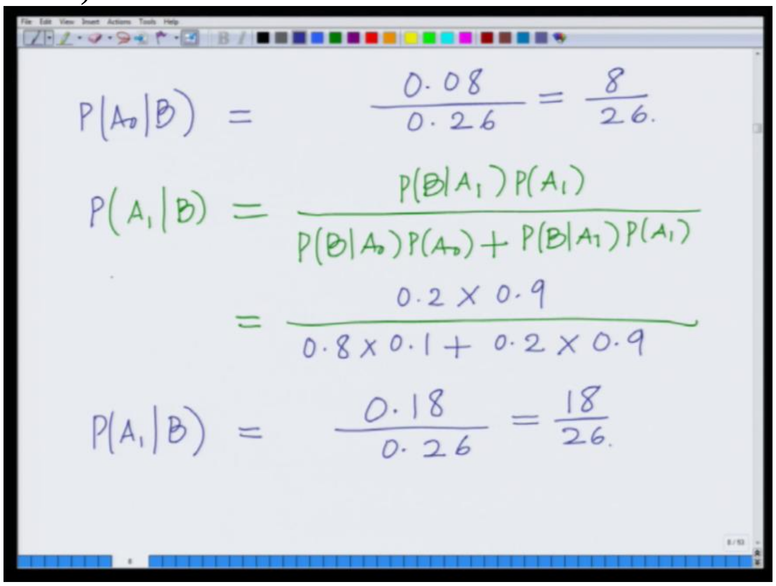

And now you can observe, if you look at these 2 quantities, the $P(A_1|B)$ and $P(A_0|B)$, that is the probability corresponding to transmission of 0 given a 0 has been observed and the probability of A1 given B that is the probability corresponding to a transmission of 1 given a 0 has been observed. Very interestingly, you observe that the probability of A1 given B which is 18/ 26, which is higher than the $P(A_0|B)$ which is 8/ 26. So what you observe is something that is very fascinating.

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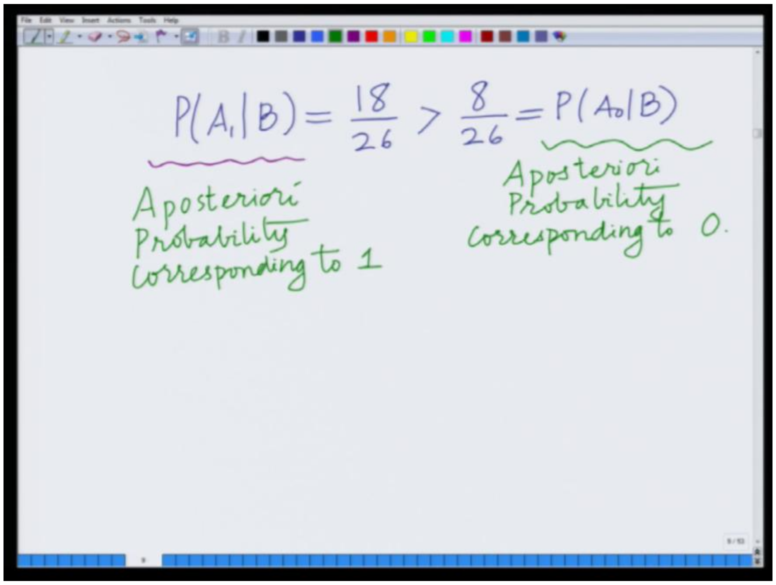

What you observe is that probability or the posterior probability $P(A_1|B)$ which is equal to 18/26 is greater than 8/26 which is the posterior probability $P(A_0|B)$. Right?

what we are saying is, the Aposteriori probability even though a 0 has been received, the Aposteriori probability corresponding to the transmission of 1 is higher than the Aposteriori probability corresponding to the transmission of 0.

Therefore, based on this, at the receiver even though we are observing 0, that is the event B which corresponds to the reception of 0, we conclude that the actual transmitted symbol must have been 1 with a higher probability than 0.

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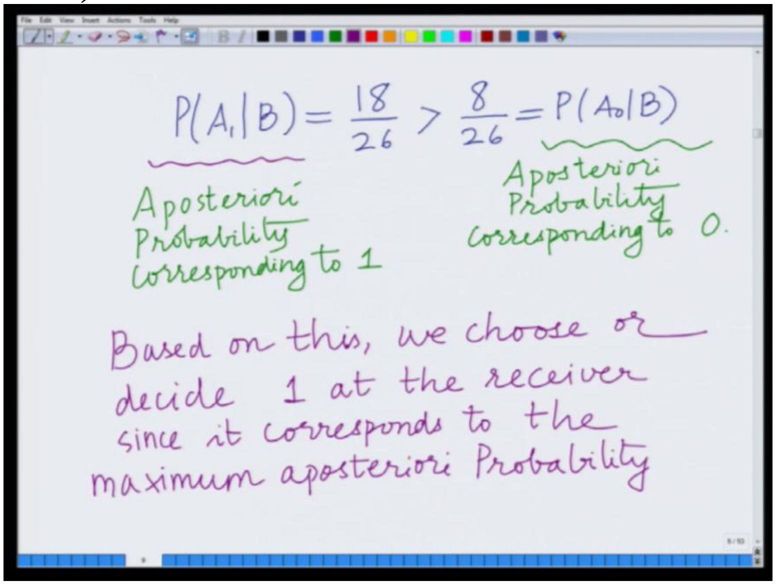

So based on Maximum Aposteriori, so based on this computation, we choose or we basically decide or detect 1 at the receiver since it corresponds to the Maximum Aposteriori probability. And this receiver is therefore known as the Maximum Aposteriori Probability receiver. That is, corresponding to the observation, we compute what are the Aposteriori probabilities of the various transmitted symbols and we choose that transmitted symbol which has the Maximum Aposteriori Probability and this is known as the Maximum Aposteriori Probability receiver.

That is, even though we observe a symbol 0, we compute the Aposteriori probability corresponding to the transmission of 0, we compute the Aposteriori probability corresponding to transmission of 1 and we conclude that transmission of 1 has a greater Aposteriori probability. Therefore we decide that as a 1 has been transmitted by the transmitter. This is known as the Maximum Aposteriori Probability receiver or the MAP receiver. And this minimizes the probability of error at the receiver. This is the optimal receiver which minimizes the probability of error at the receiver. So this is known as the MAP receiver. This principle is known as the MAP principle.

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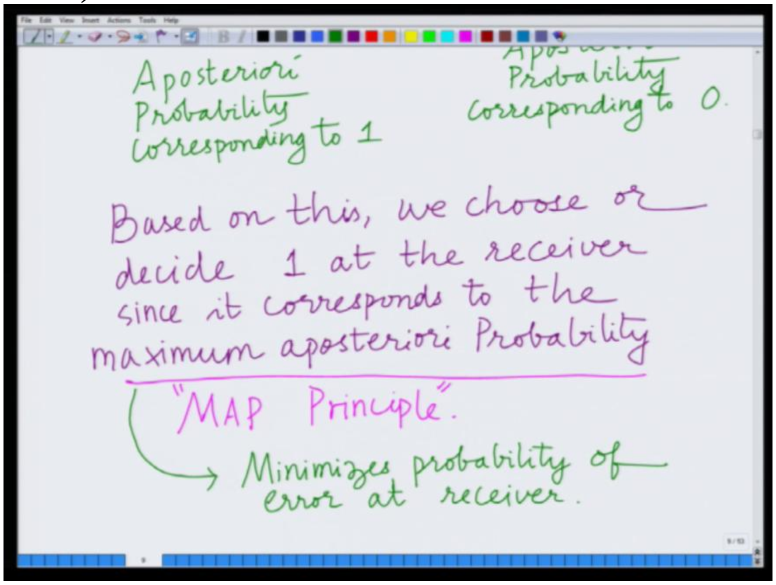

This is a very important principle in the context of communication systems. This is known as the MAP or the Maximum Aposteriori Probability and this leads to minimization of the probability of error at the receiver.

Thus the MAP receiver minimizes the probability of error at the receiver. Therefore, this MAP receiver which is built on the principle of the Bayes Theorem which computes Aposteriori probabilities of the various transmitted symbols and chooses the transmitted symbol which has the Maximum Aposteriori Probability is known as the MAP receiver.

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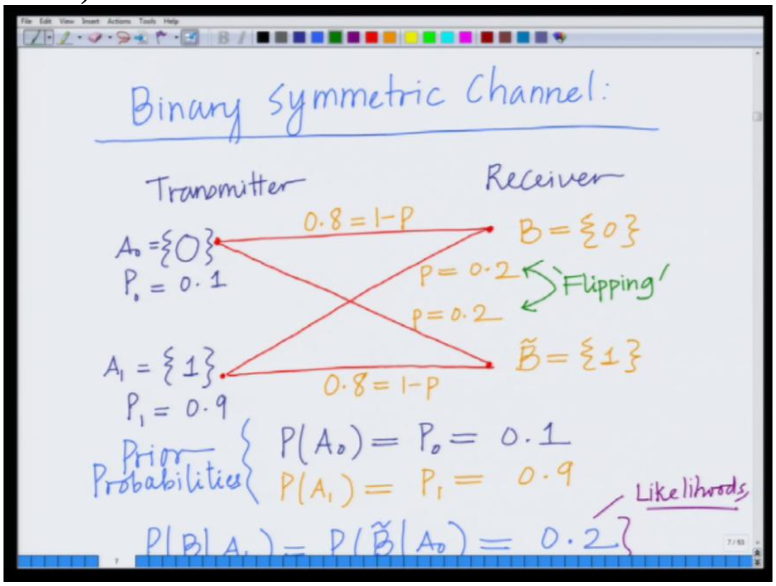

On the other hand, if you look at a simple maximum likelihood receiver, now if you look at the likelihoods, what are the different likelihoods? If you look at the likelihoods, the likelihood of this B given A0 is 0.8 and likelihood of B given A1 is 0.2.

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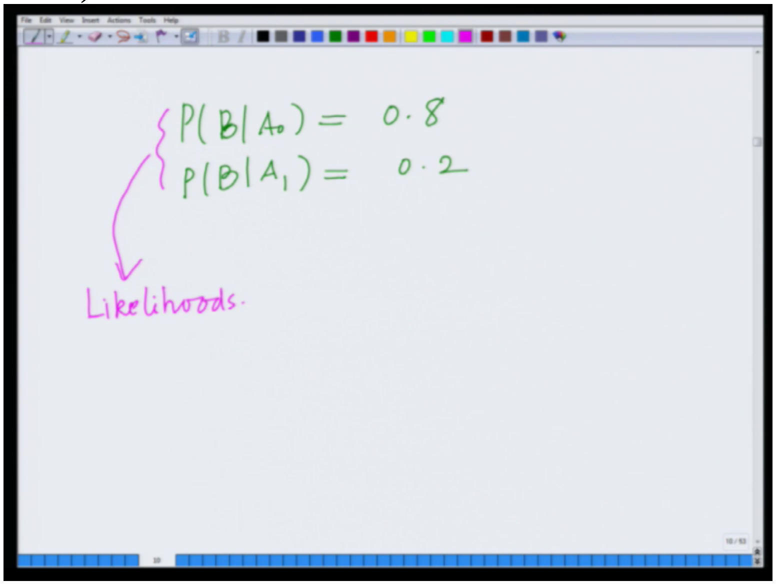

So now if you look at the likelihood. Simply if you look at the likelihoods, probability of A0 given that is the probability of B that is of the received observation 0 corresponding to the transmission of 0 equals 0.8, Probability of B given A1 equals 0.2, This is the flipping probability and these are the likelihoods, and based simply on the likelihood, we can see that A0 has L. Because the probability of B given A0 is higher than the probability of B given A1,

$$P(B|A_0) > P(B|A_1)$$

we can see that transmission of 0 has a higher likelihood. Therefore, if we decide based on this that the transmitted symbol is 0, this is known as the Maximum likelihood receiver.

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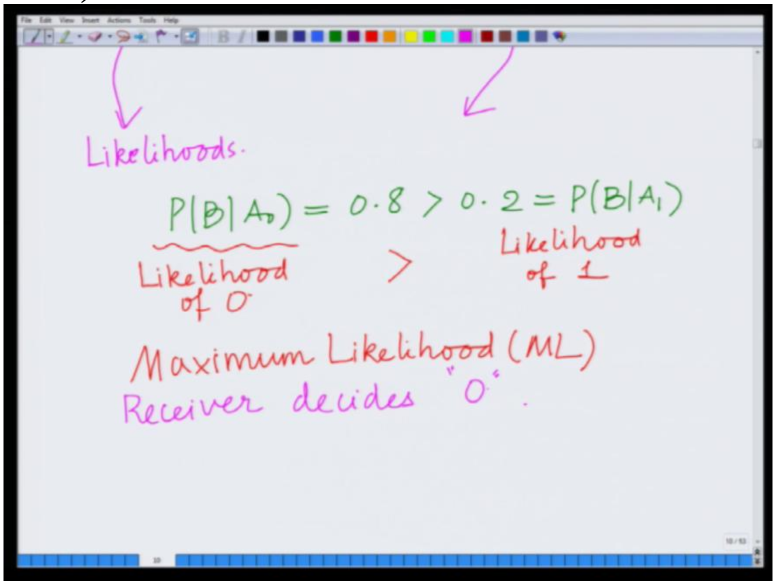

So, In Maximum likelihood receiver, we are saying that

$$[P(B|A_0)= 0.8] > [P(B|A_1)= 0.2]$$

Therefore, likelihood of 0, not the Aposteriori probability, likelihood of 0 is greater than the likelihood of the binary information symbol 1. Therefore, Maximum likelihood receiver or ML receiver decides 0. So, Maximum likelihood receiver which purely looks at the likelihood decides the 0 as against 1 because the likelihood of 0 is greater than the likelihood of 1. However, the MAP receiver looks at the Aposteriori probability and decides a 1 rather than 0.

And remember, the Maximum Aposteriori Probability minimizes the probability of error, not the maximum likelihood receiver. So the maximum likelihood receiver, although it is an easy receiver to look at, it is appealing, it does not minimize the probability of error. So the optimal receiver is the MAP receiver, not the Maximum likelihood receiver. So this is, so we have looked at 2 different kind of receivers, one is the MAP receiver and the Maximum likelihood receiver…

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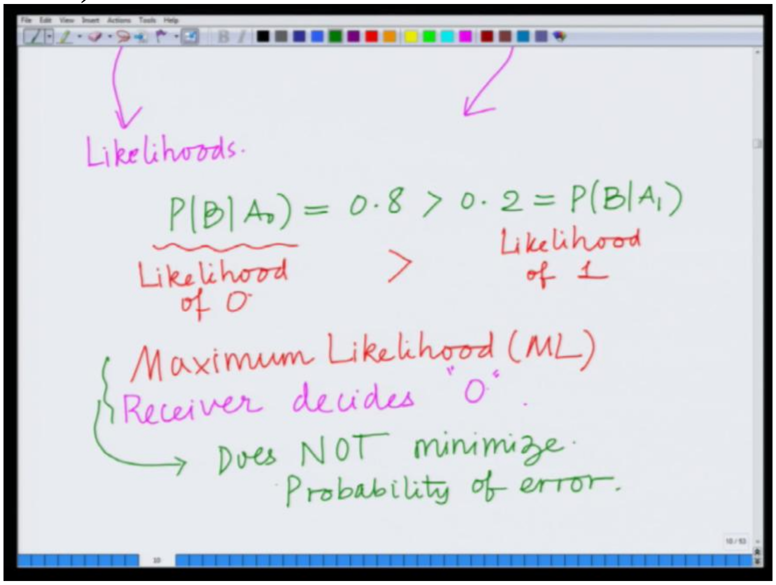

…but the Maximum likelihood receiver, this does not minimize the probability of error. So while it is appealing to use the Maximum likelihood receiver, it is suboptimal as to the MAP received which computes the, which chose the symbol with the Maximum Aposteriori Probability is the optimal receiver and that uses the Bayes Theorem. So the Bayes Theorem is very important. In fact, I have to say, it is one of the key principles used in digital communication or wireless communication system. Therefore, it is very important to understand the Bayes Theorem and as well as the application of this Bayes Theorem in the context of modern wireless and digital communication symbols because that leads to the optimal receiver which minimizes the probability of error at the receiver.

So this module, while the previous module has explained the theoretical or abstract concept of Bayes Theorem, this module has justified, has illustrated a very important application of this Bayes Theorem for computing this Aposteriori probabilities and then choosing the symbol with the Maximum Aposteriori Probability at the receiver in the context of a wireless communication system or the MAP receiver. So we will conclude this module here. We will explore other aspects of probability and random variables in subsequent modules. Thank you very much.