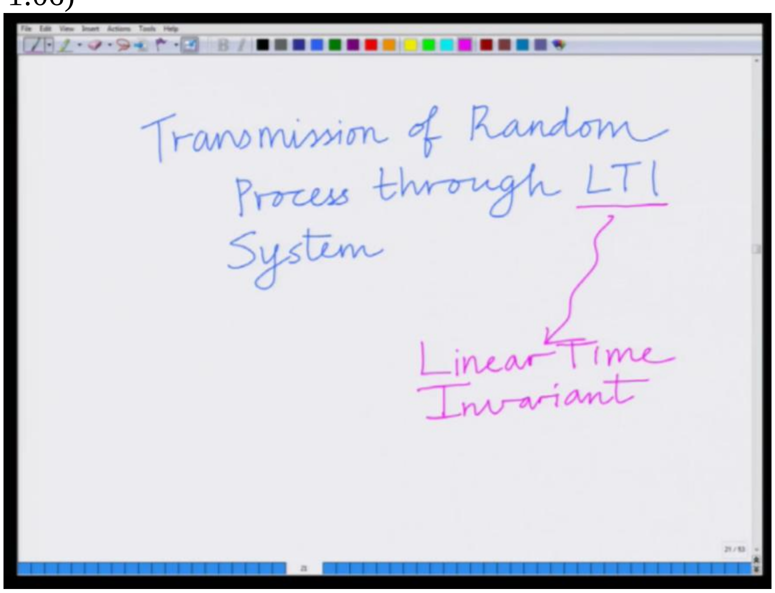
Probability and Random Variables/Processes for Wireless Communication
Professor Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Module No. 4
Lecture 21
Transmission of WSS Random Process Through LTI System

Hello, welcome to another module in this massive open online course on probability and random variables for wireless communications. So we are talking about random processes and in the previous modules, we have seen the properties of a wide sense stationary random process and also more importantly we have seen the definition of power spectral density ~~of the power spectral density~~ of a wide sense stationary random process and an interesting application in the context of the bandwidth required for the transmission of a wireless signal.

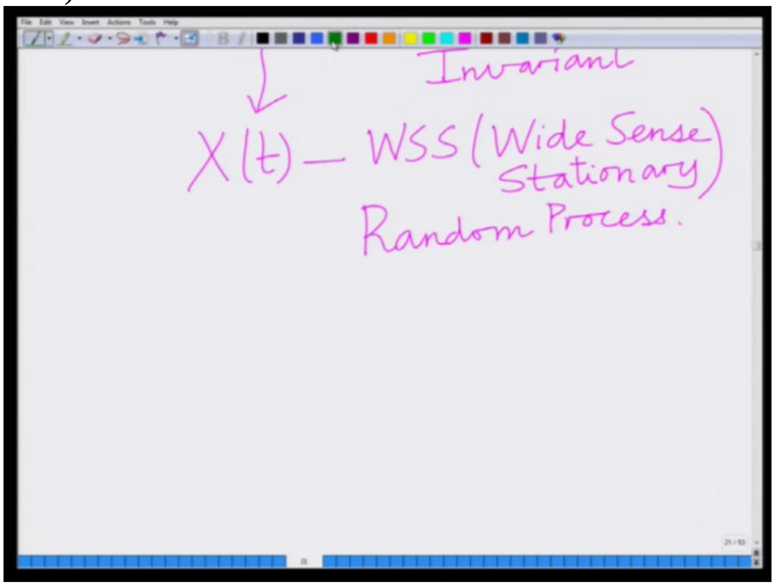
That is the spectrum that is required for transmission of a wireless signal. So let us continue our discussion on random processes and let us today look at the transmission of a random process through an LTI system. That is what happens when a random process is input to an LTI, or the linear time invariant system. Okay?

(Refer Slide Time: 1:06)



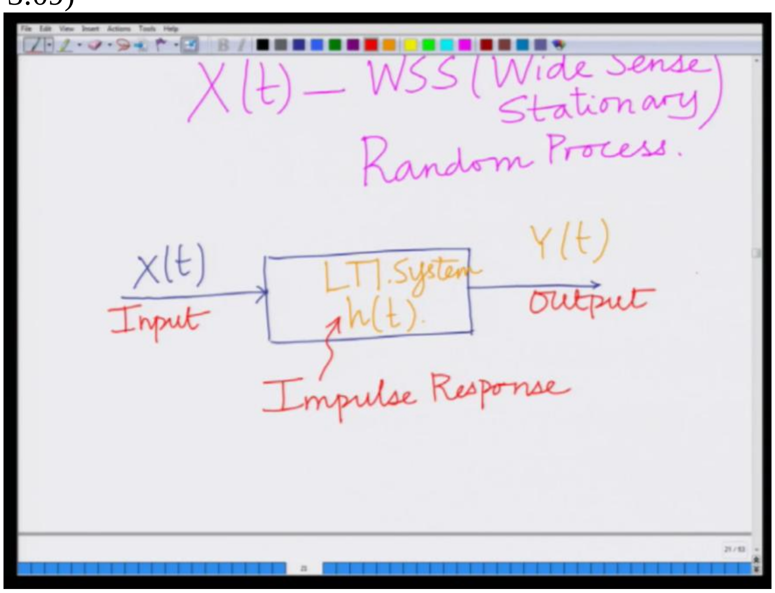
So in today's lecture let us start looking at the transmission, another key property that is the transmission of a random process through an LTI system. where what is the meaning of LTI? LTI denotes obviously most of you would be familiar with this. LTI denotes a linear time invariant system. So what we will be considering? Let us consider a wide sense stationary random process X(t).

(Refer Slide Time: 2:10)



So let us start by considering X(t) which is wide sense stationary random process. So X(t) is a basic wide sense stationary random process. And what we're saying is this random process X(t) is an input to an LTI or a linear time invariant system and remember every linear time invariant system is characterized by impulse response. That is the key aspect of an LTI system. That is every linear time invariant system is characterized by the impulse response. Let us denote this by **h**.

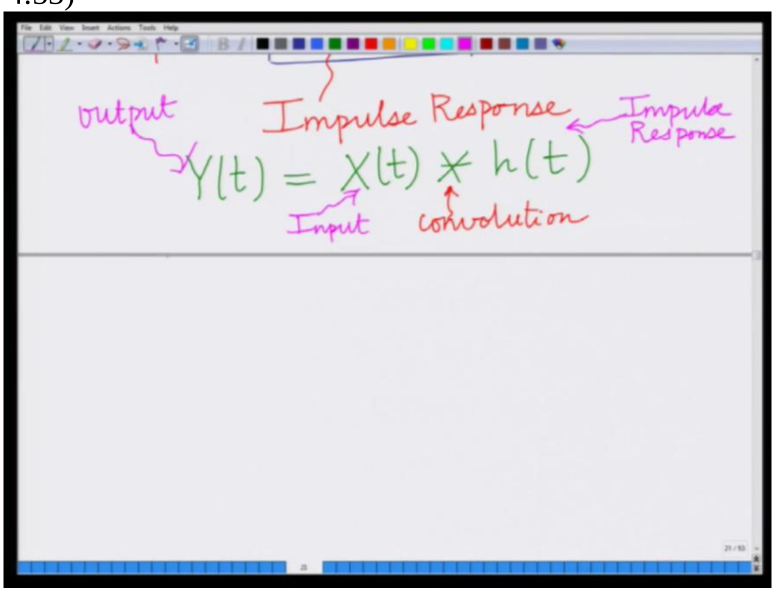
(Refer Slide Time: 3:09)



So we are saying, this wide sense stationary random process X(t) is input to a linear time invariant system with impulse response h(t) and that I want to represent that schematically over here. So I have my LTI system. This is the input to the LTI system. So, X(t) is the input to LTI system, Y(t) is the output of the LTI system. So this is my LTI system which is characterized by the impulse response h(t).

Remember, every LTI system is characterized by h(t). So what is h(t)? h(t) is the impulse ~~impulse~~ response of this system. So this is an LTI system and this is a linear time invariant system X(t). What is X(t)? X(t) is the input, Y(t) is the output. X(t) is the input to the LTI system, Y(t) is the output to the LTI system and the LTI system is characterized by the impulse response h(t). That is, if the impulse δ(t) is applied as an input to this LTI system than the output will be h(t). That is the meaning of the impulse response. That is, impulse response is nothing but a response to the impulse input of this system.

(Refer Slide Time: 4:53)

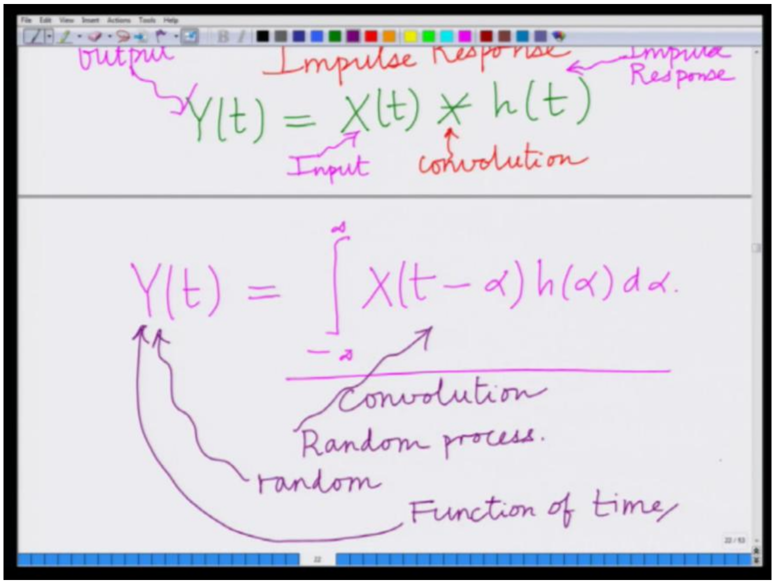


And therefore, remember for an LTI system, the output Y(t) for any signal input signal X(t), the output Y(t) is the signal X(t) convolved with impulse response h(t).

$$Y(t) = X(t) * h(t)$$

This is your basic convolution operator. So the output of this system Y(t) is the input X(t) of the system convolved with the impulse response h(t) of this system. And the convolution operator can be represented in the time domain as follows.

(Refer Slide Time: 5:58)



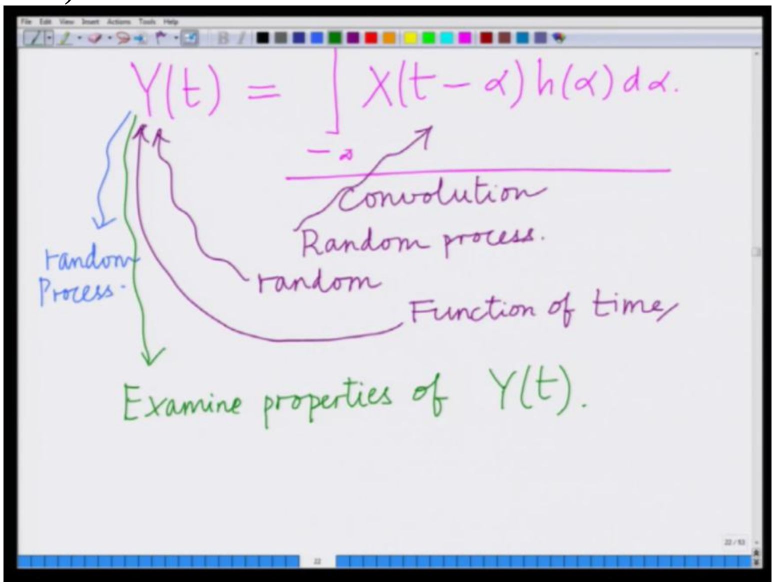
So I have,

$$Y(t) = \int_{-\infty}^{\infty} X(t-\alpha)h(t)d\alpha$$

This is basically the convolution. And also we are given that this X(t), this is a random process. This is random in nature. Further it is a function of time. Now, Y(t) is therefore, it is random. Since X(t) is random, Y(t) is random. Also naturally, Y(t), you can see, this is a function of time.

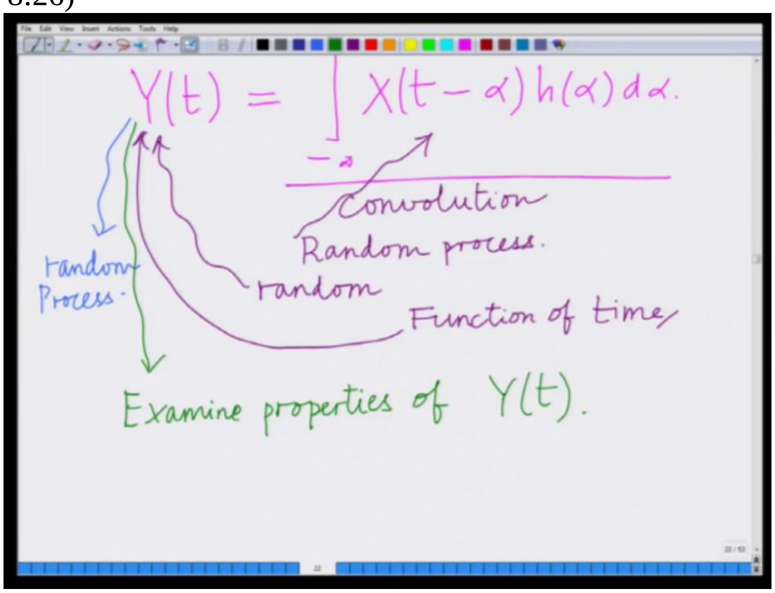
That is, it is random. It is a function of time. Y(t) is random because X(t) is random. That is, the input is random to the LTI system. So naturally, the output Y(t) is random. Further, it is a function of time. That is, it is index pattern. Therefore, Y(t) is also a random process.

(Refer Slide Time: 7:25)



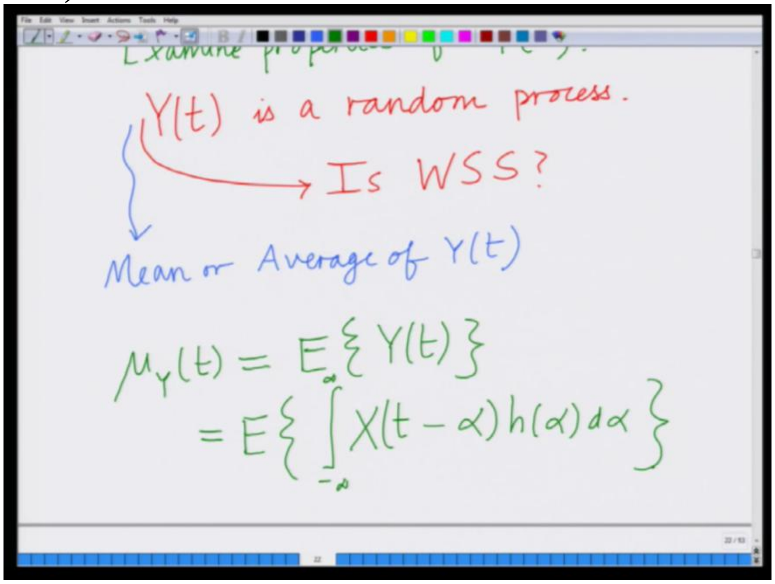
And therefore, now what we would like to do is we would like to examine the properties. We would like to examine the properties of this output random process Y(t). What are the properties of this output random process Y(t)? In particular, we would like to ask the question, is Y(t) wide sense stationary? If Y(t) is wide sense stationary then what is the mean and what is the autocorrelation of Y(t)?

(Refer Slide Time: 8:26)



So we would like to answer the question, is this output random process, Y(t) is this also wide sense stationary? It is not a given that it is going to be wide sense stationarity. We have to demonstrate that is, if it is wide sense stationary which is what we are going to do next.

(Refer Slide Time: 8:47)



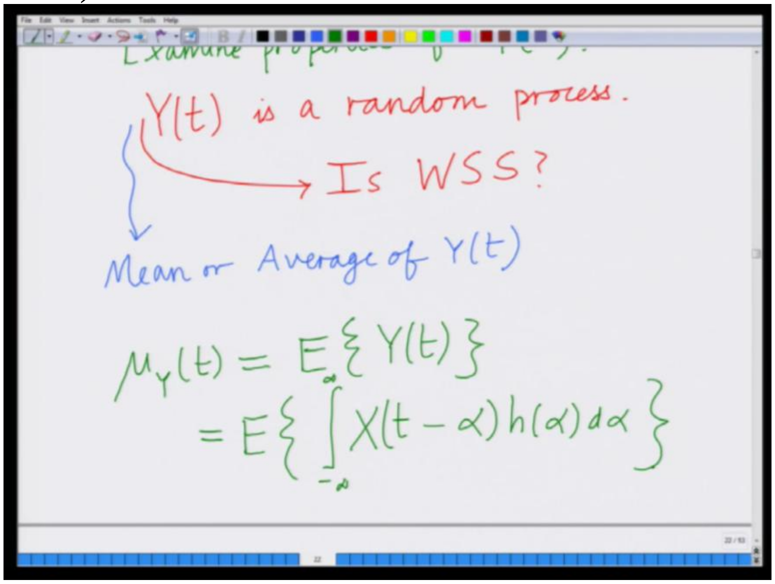
So now, let us examine Y(t). 1st, we have shown that Y(t) is a random process because it is a function of time so Y(t) is a random process. And now, we would like to ask the question, is Y(t) wide sense stationary? Is it WSS or wide sense stationary? So let us start with the mean or average. The average like the time average $\mu_Y(t)$, which is the expected value of this random process Y(t) is defined as follows. That is,

$$\mu_Y(t) = E\{Y(t)\}$$

$$= E\{ \int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha \}$$

So remember, Y(t) is the convolution of X(t) and h(t). And now we're taking the expected value of this whole thing. We're taking the expected value of this convolution between the random process X(t) and the impulse response, h(t).

(Refer Slide Time: 10:34)



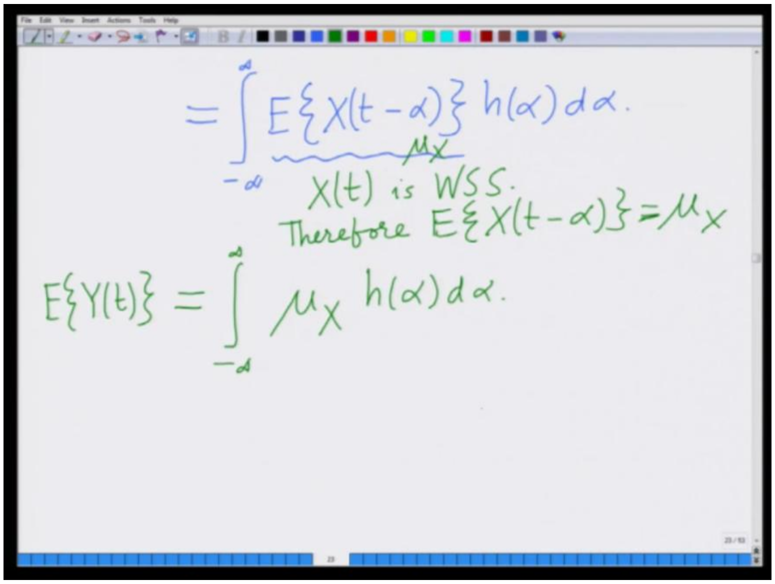
And therefore now remember since the expectation operator is linear, I can move the expectation operator inside this integral. So this becomes,

$$= \int_{-\infty}^{\infty} E\{X(t-\alpha)\}h(\alpha)d\alpha$$

Now remember that X(t) is wide sense stationary. Therefore it is stationary in the mean which means at any time $X(t-\alpha)$, the mean of this is constant. Therefore this quantity is equal to μ_X. It does not depend on time.

$$= \int_{-\infty}^{\infty} \mu_X h(\alpha)d\alpha$$

(Refer Slide Time: 11:28)



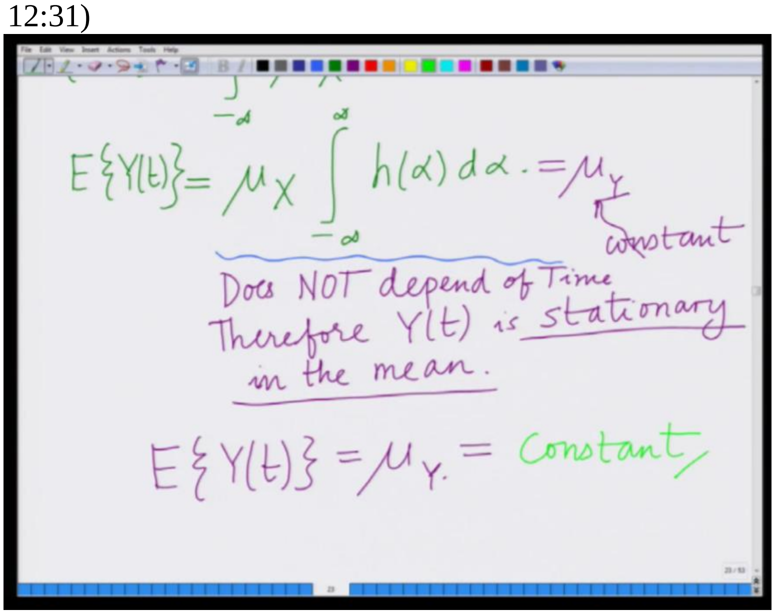
And now, since μ_X is a constant, I can bring this outside.

$$E\{Y(t)\} = \mu_X \int_{-\infty}^{\infty} h(\alpha) d\alpha$$

That is expected value of this random process Y(t). This does not depend upon time. There is no time, t. It is not a function of time.

Therefore, the mean of this random process Y(t) which is the output of the LTI system is constant. Therefore, this is stationary in the mean.

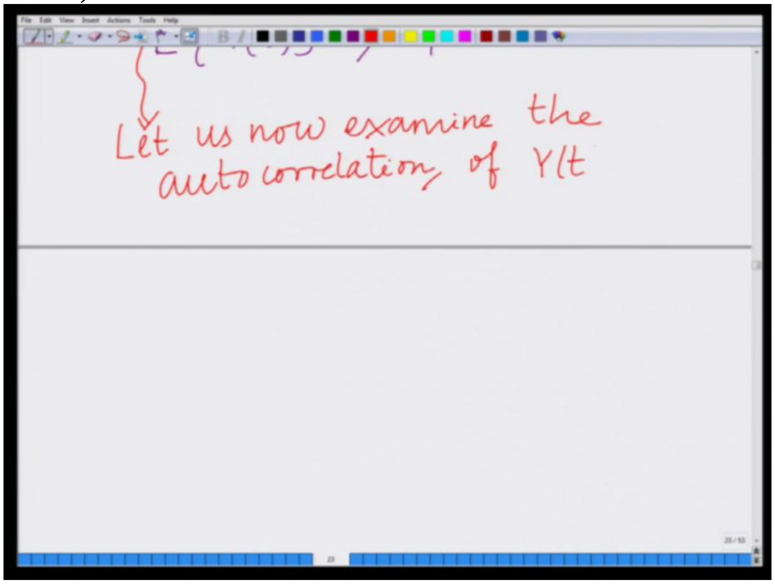
(Refer Slide Time: 12:31)



Therefore, this is equal to a constant, that is μ_Y. Therefore, output random process, Y(t) is stationary in the mean.

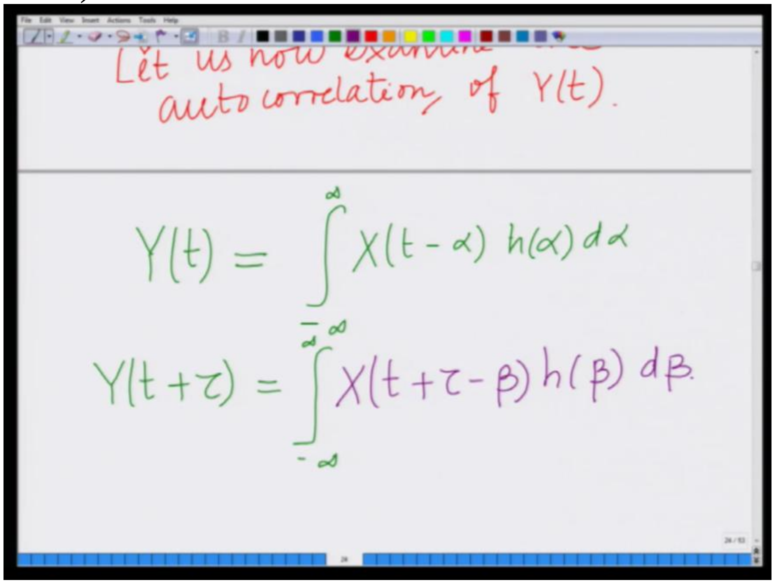
And what do we have? We have the $E\{Y(t)\} = \mu_Y$ which is basically a constant. Expected value of $Y(t)$ this basically demonstrated that for this output random process, the expected value of $Y(t)$ is basically μ_Y which is a constant. It does not depend on time. Therefore the output random process $Y(t)$ is stationary in the mean. Let us now therefore look at the autocorrelation of this random process $Y(t)$.

(Refer Slide Time: 14:03)



So you are demonstrated that it is stationary in the mean. Let us now examine the autocorrelation of Y(t).

(Refer Slide Time: 14:37)



We have,

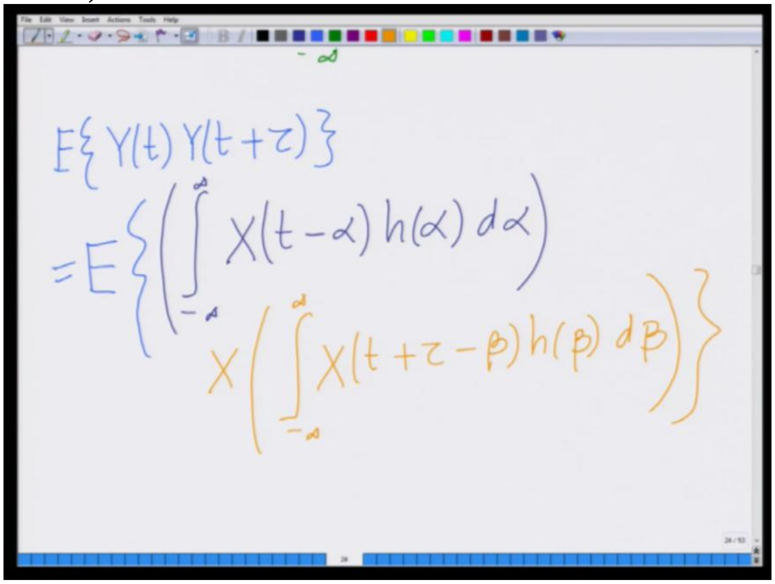
$$Y(t) = \int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha$$

$$Y(t+\tau) = \int_{-\infty}^{\infty} X(t+\tau-\beta)h(\beta)d\beta$$

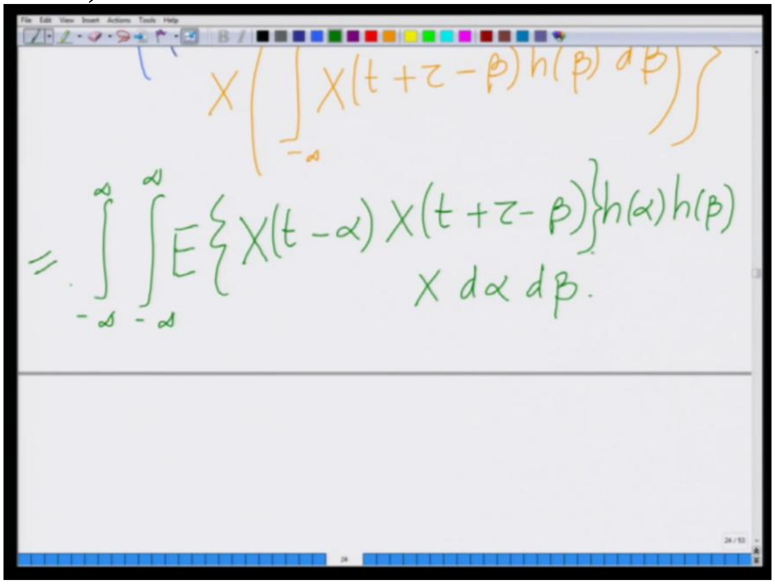
Now I have to look at what is the autocorrelation? Autocorrelation is,

$$E\{Y(t)\,Y(t+\tau)\} = E\left\{\left(\int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha\right) X\left(\int_{-\infty}^{\infty} X(t+\tau-\beta)h(\beta)d\beta\right)\right\}$$

(Refer Slide Time: 15:45)



(Refer Slide Time: 16:47)

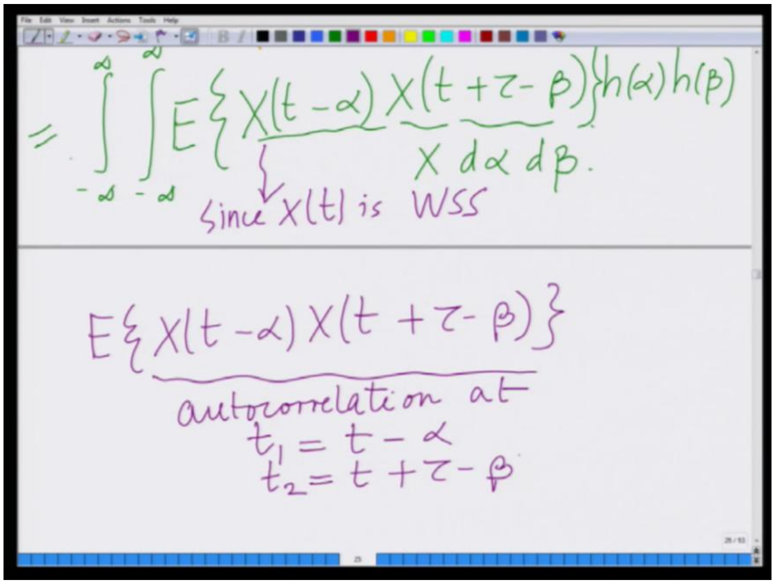


And,

$$= \iint_{-\infty}^{\infty} E\{X(t-\alpha)X(t+\tau-\beta)\}\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$

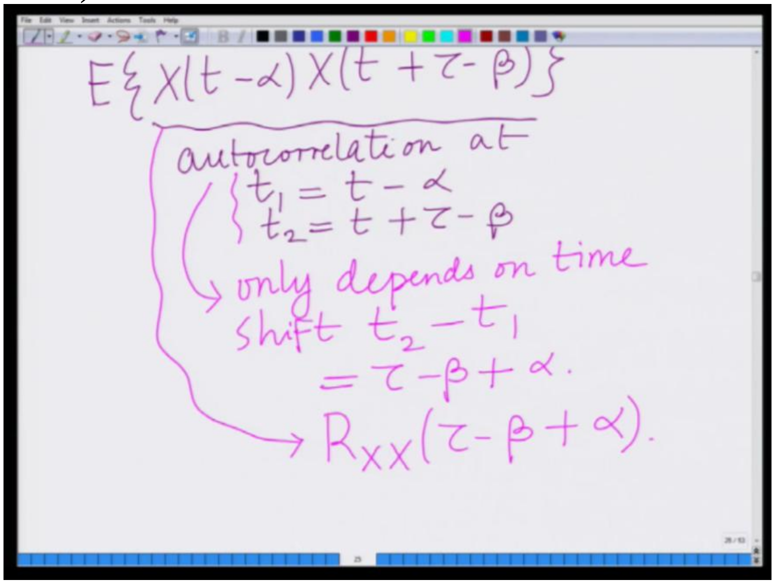
And remember, $X(t)$ the random process, is a wide sense stationary random process. Therefore the autocorrelation depends only on the timeshift or the time difference between these 2 time instances.

(Refer Slide Time: 17:50)



Therefore, since X(t) is wide sense stationary. This is autocorrelation at T1 = $t-\alpha$, T2 = $t+\tau-\beta$.

(Refer Slide Time: 18:44)



So it only depends on, the timeshift between these 2 instants T2 -T1 between these 2 instants which is equal to –

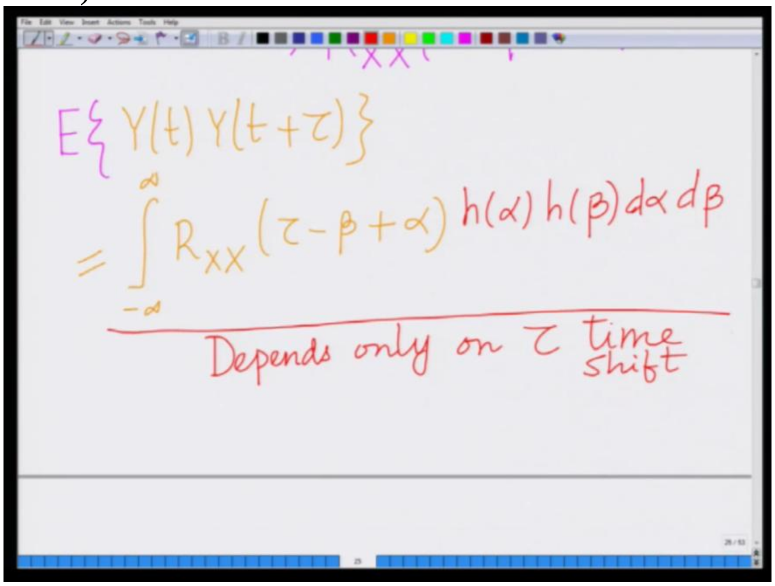
$$T_2 - T_1 = \tau - \beta + \alpha$$

Therefore,

$$E\{X(t-\alpha)X(t+\tau-\beta)\} = R_{XX}(\tau - \beta + \alpha)$$

the expected value of $X(t-\alpha)X(t+\tau-\beta)$ depends only on the timeshift between these 2 time instants. Therefore, it is $R_{XX}(\tau - \beta + \alpha)$ because X is a wide sense stationary random process.

(Refer Slide Time: 19:46)

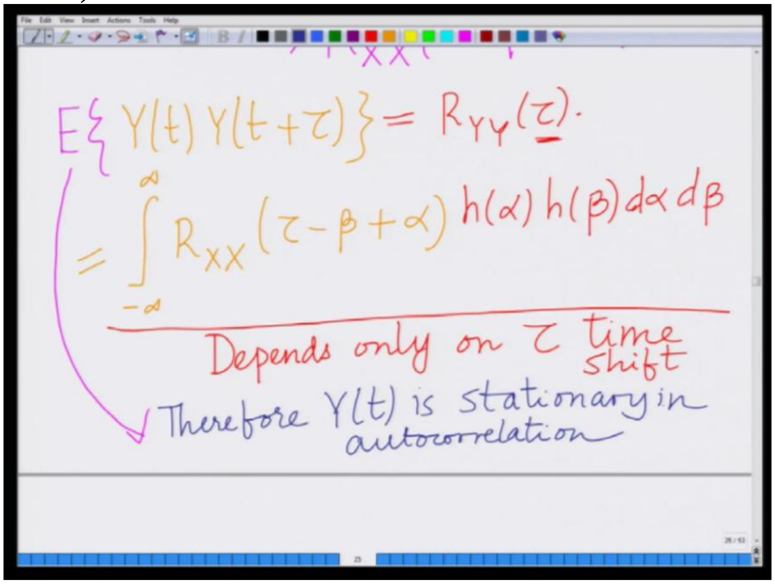


And now, substituting this in here, I have,

$$E\{Y(t)\ Y(t+\tau)\} = \iint_{-\infty}^{\infty} R_{XX}(\tau - \beta + \alpha)\ h(\alpha)\ h(\beta)\ d\alpha\ d\beta$$

Now you can see, this whole thing depends only on the shift, τ. That is $E\{Y(t)\ Y(t+\tau)\}$ depends only on τ, that is the timeshift.

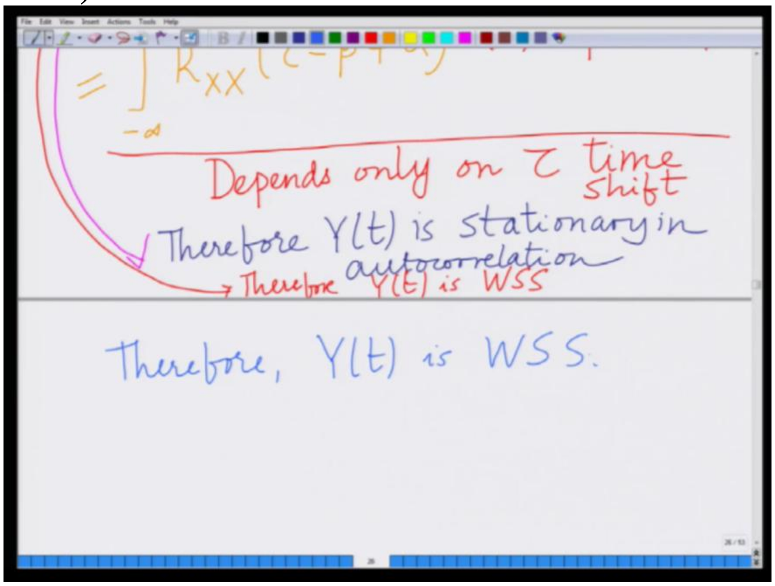
(Refer Slide Time: 21:01)



Therefore, I can denote this by $R_{YY}(\tau)$, which is the autocorrelation Y which depends only on the timeshift, τ.

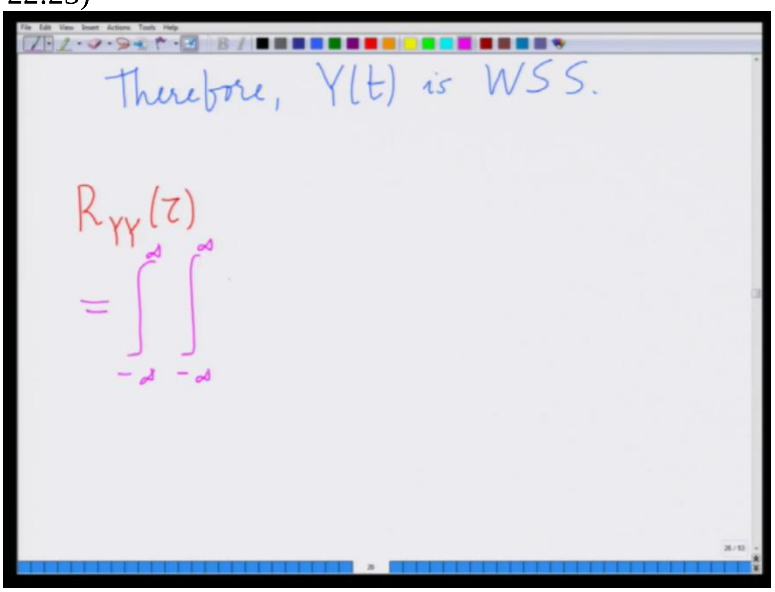
Therefore, it follows that $Y(t)$ is stationary in the autocorrelation therefore it follows that you get this, $E\{Y(t)\ Y(t+\tau)\}$ depends only on the timeshift τ. Therefore it is stationary on the autocorrelation. Previously, we have shown that $Y(t)$ is stationary in the mean. Therefore, which means therefore, these 2 together demonstrate that $Y(t)$ is also a wide sense stationary random process.

(Refer Slide Time: 21:53)



Therefore, Y(t) is WSS. So therefore let me write this again here. Therefore, Y(t) is a wide sense stationary random process.

(Refer Slide Time: 22:23)

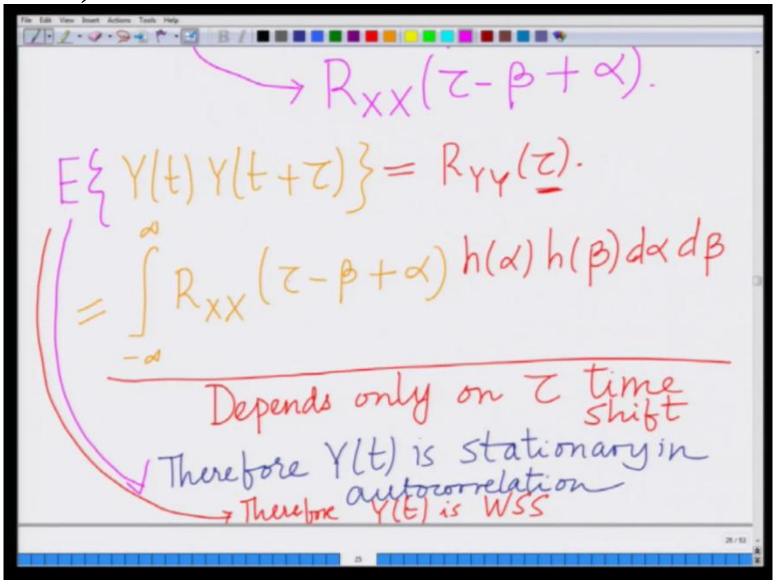


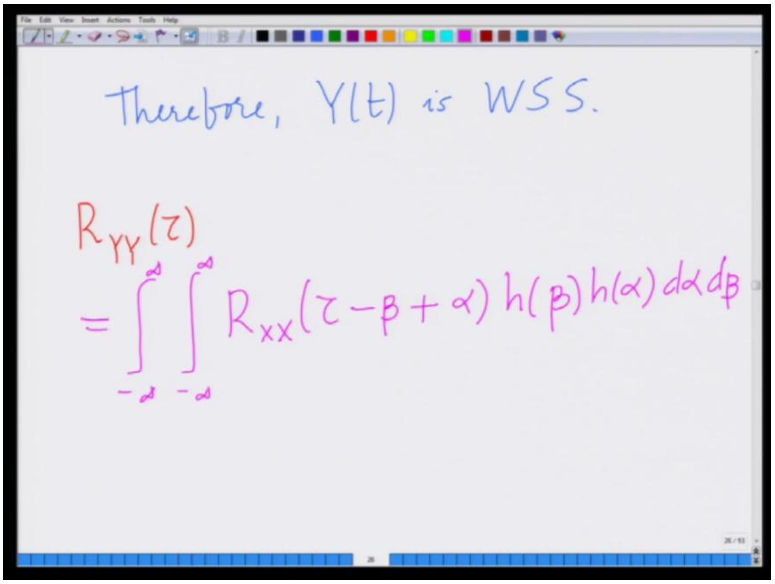
Now further look at this. Let me read this again. I have the autocorrelation $R_{YY}(\tau)$ is equal to –

$$E\{Y(t)\,Y(t+\tau)\} = R_{YY}(\tau)$$

$$= \iint_{-\infty}^{\infty} R_{XX}(\tau - \beta + \alpha)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$

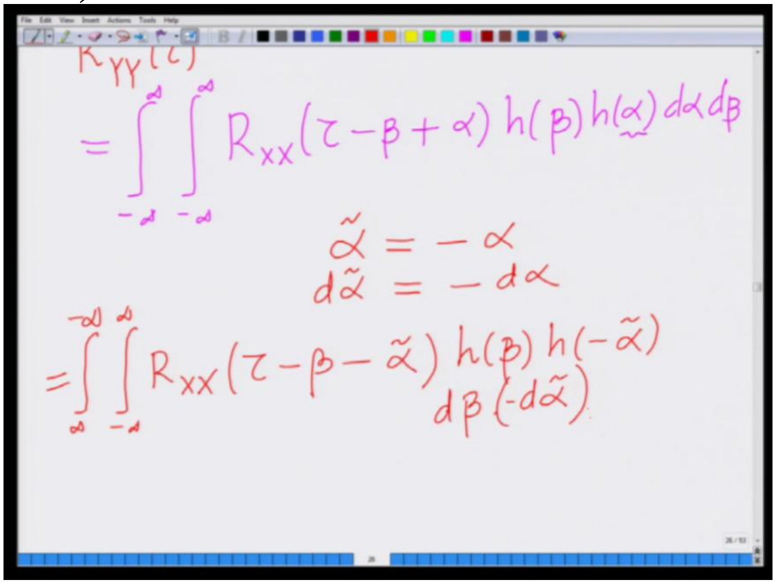
(Refer Slide Time: 22:55)





Now let us do a simple thing.

(Refer Slide Time: 23:29)



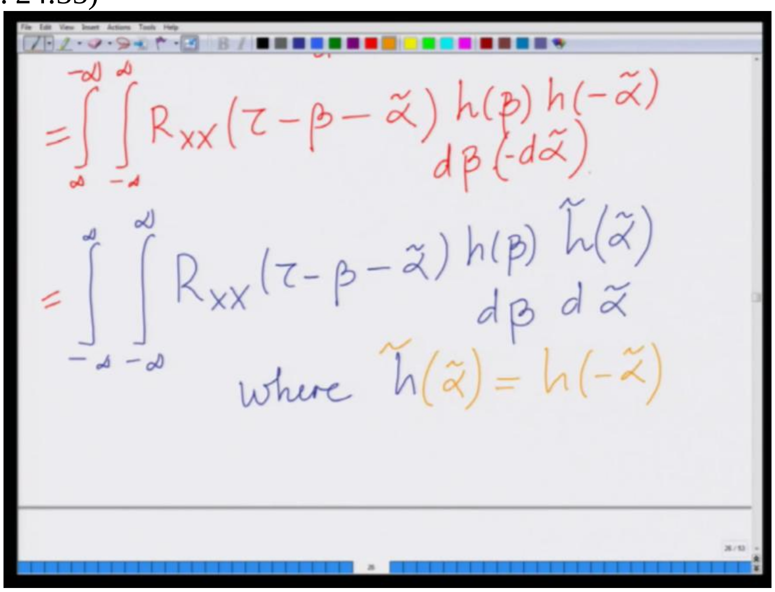
Let us replace alpha by $\tilde{a}$. So let us do the substitution. $\tilde{a} = -\alpha$. So,

$$d\tilde{a} = -d\alpha$$

So I can write this thing as,

$$R_{YY}(\tau) = -\iint_{-\infty}^{\infty} R_{XX}(\tau - \beta - \tilde{a})h(-\tilde{a})\, h(\beta)(-d\tilde{a})\, d\beta$$

(Refer Slide Time: 24:35)

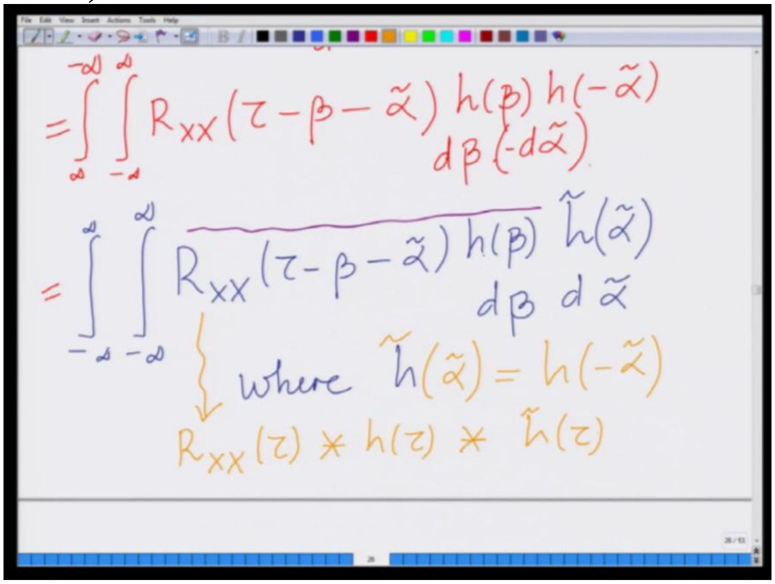


$$R_{YY}(\tau) = -\iint_{-\infty}^{\infty} R_{XX}(\tau - \beta - \tilde{a})h(-\tilde{a})\, h(\beta)(-d\tilde{a})\, d\beta$$

$$= \iint_{-\infty}^{\infty} R_{XX}(\tau - \beta - \tilde{\alpha})\tilde{h}(\tilde{\alpha})\, h(\beta)(-d\tilde{\alpha})\, d\beta$$

Where $\tilde{h}(\tilde{\alpha}) = h(-\tilde{\alpha})$

(Refer Slide Time: 25:42)



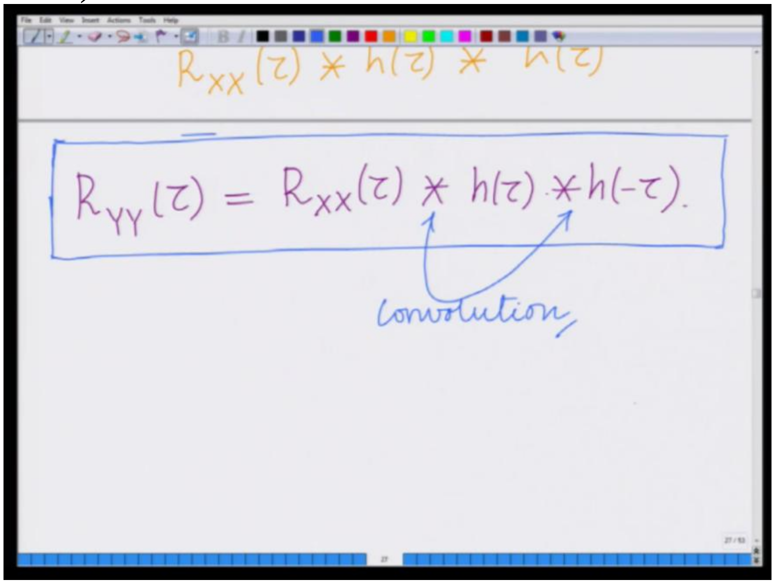
And now you can see from this. From this expression,

$$R_{XX}(\tau - \beta - \tilde{\alpha})\, h(\beta) = R_{XX}(\tau) * h(\tau) * \tilde{h}(\tau)$$

So this you can clearly see is the output autocorrelation that is –

$$R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

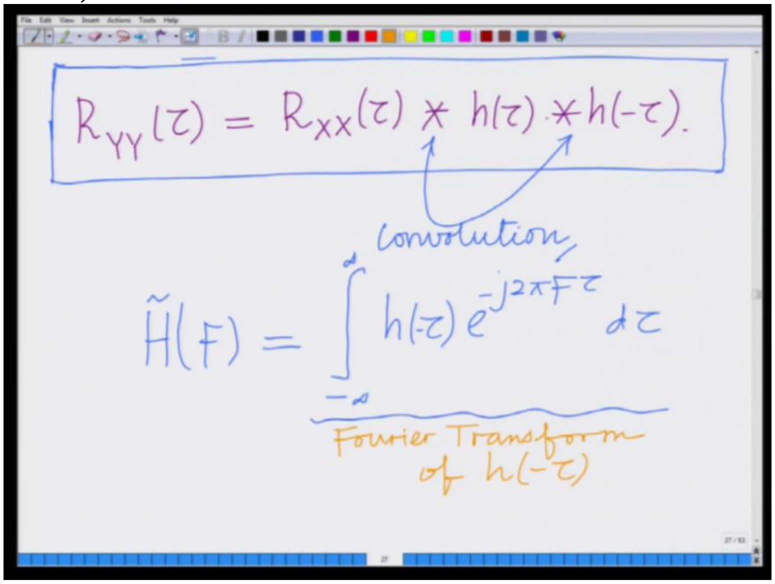
(Refer Slide Time: 26:42)



This is the relation I have in the time domain. Not exactly the time domain but the shift domain, the domain τ. That is the convolution in τ domain. These are your basic convolution operators. Now let us look at what happens in the frequency domain.

To understand what happens in the frequency domain, let us look at what is the frequency response of $h(\tau)$.

(Refer Slide Time: 27:51)

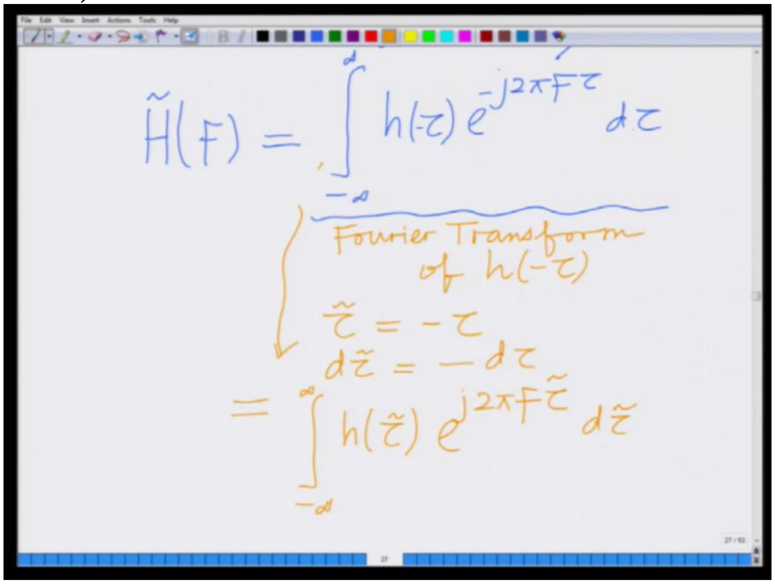


Now if I look at the frequency response of $h(-\tau)$. Let me call that $\tilde{H}(f)$. That is,

$$\tilde{H}(f). = \int_{-\infty}^{\infty} h(-\tau)\, e^{-j2\pi f\tau} d\tau$$

This is the Fourier transform of $h(-\tau)$.

(Refer Slide Time: 28:33)



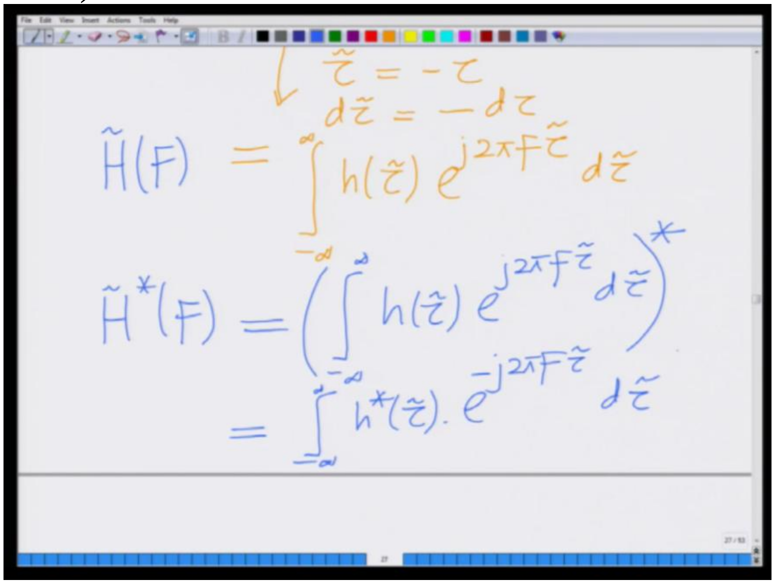
Now I substitute,

$$\tilde{\tau} = -\tau$$

$$d\tilde{\tau} = -d\tau$$

$$\tilde{H}(f). = \int_{-\infty}^{\infty} h(\tilde{\tau})\, e^{j2\pi f\tilde{\tau}} d\tilde{\tau}$$

(Refer Slide Time: 29:12)



Now if I consider $\tilde{H}^*(f)$, that is equal to –

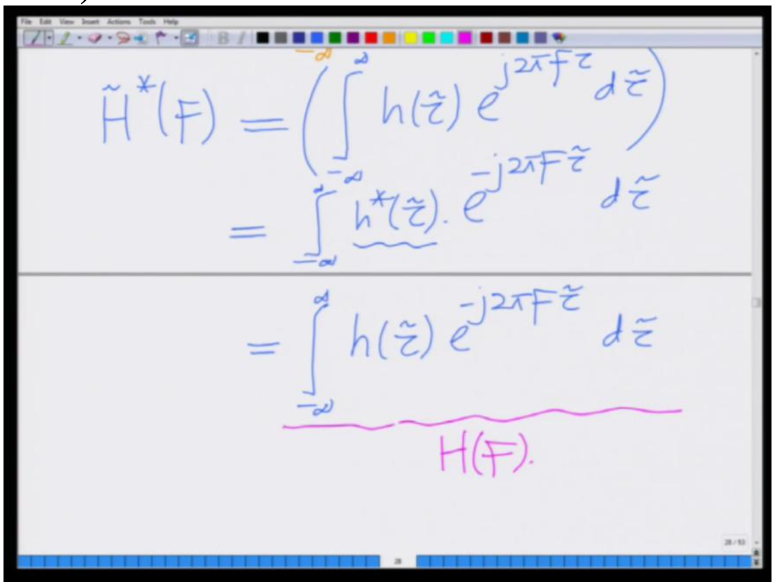
$$\tilde{H}^*(f) = \left(\int_{-\infty}^{\infty} h(\tilde{\tau})\, e^{j2\pi f\tilde{\tau}} d\tilde{\tau}\right)^*$$

$$= \int_{-\infty}^{\infty} h^*(\tilde{\tau})\, e^{-j2\pi f\tilde{\tau}} d\tilde{\tau}$$

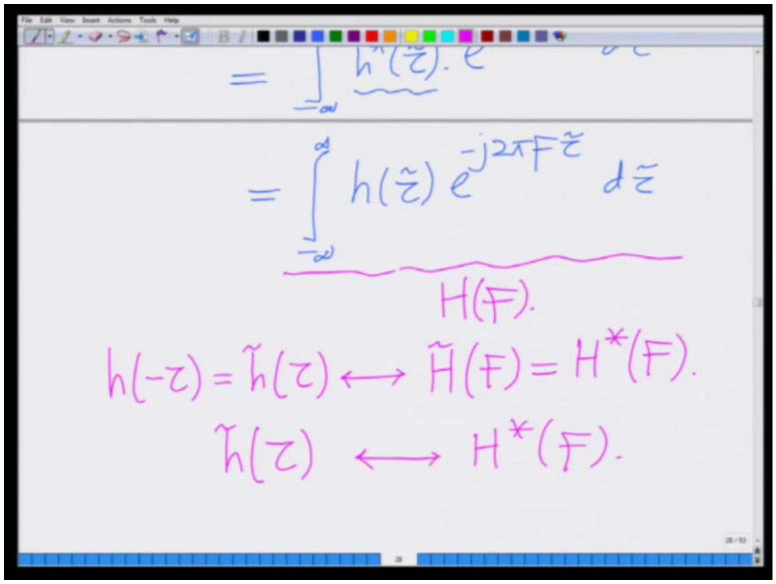
$$H(f) = \int_{-\infty}^{\infty} h(\tilde{\tau})\, e^{-j2\pi f\tilde{\tau}} d\tilde{\tau}$$

but we're considering a system with a real impulse response $h(t)$.

(Refer Slide Time: 30:21)



(Refer Slide Time: 30:56)



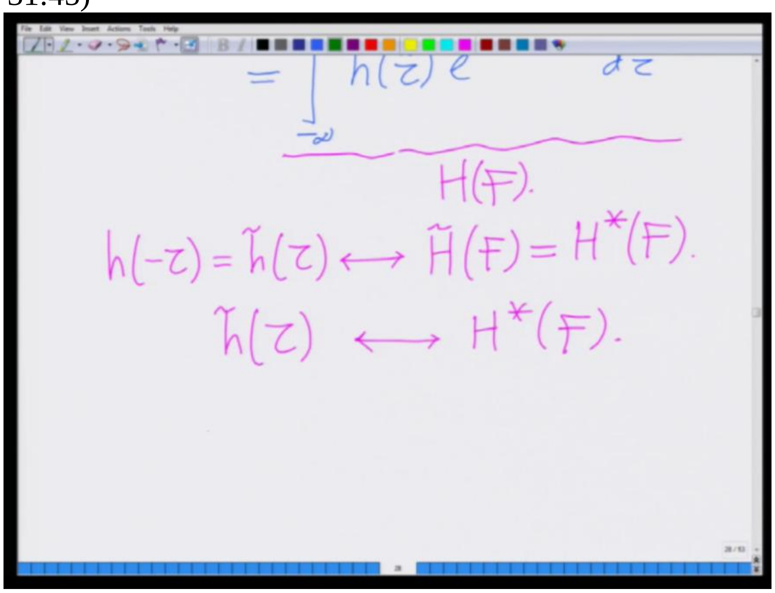
Therefore,

$$h(-\tau) = \tilde{h}(\tau) \leftrightarrow \tilde{H}(f) = H^*(f)$$

$$\tilde{h}(\tau) \leftrightarrow H^*(f)$$

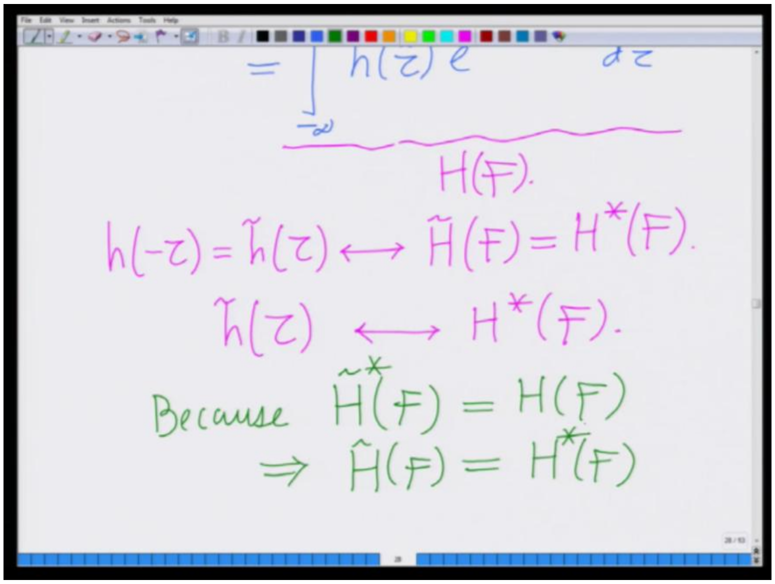
Why is that the case?

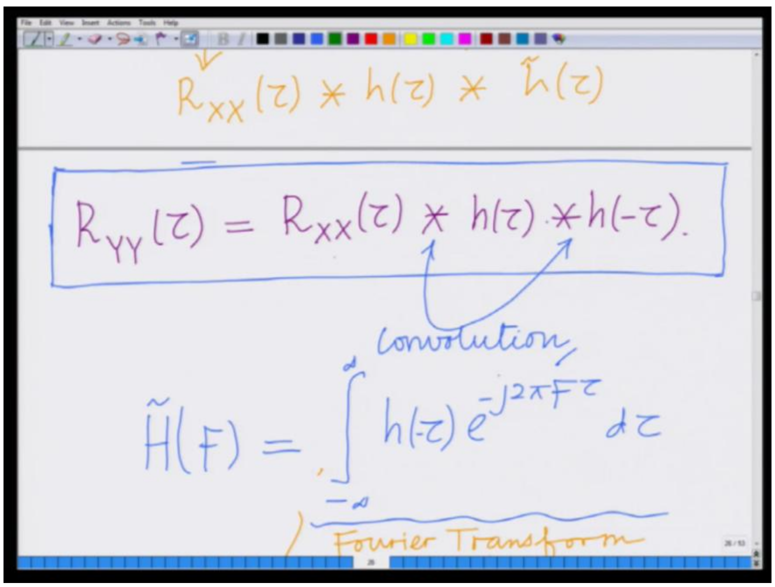
(Refer Slide Time: 31:45)



Because we have shown that $\tilde{H}^*(f) = H(f)$.

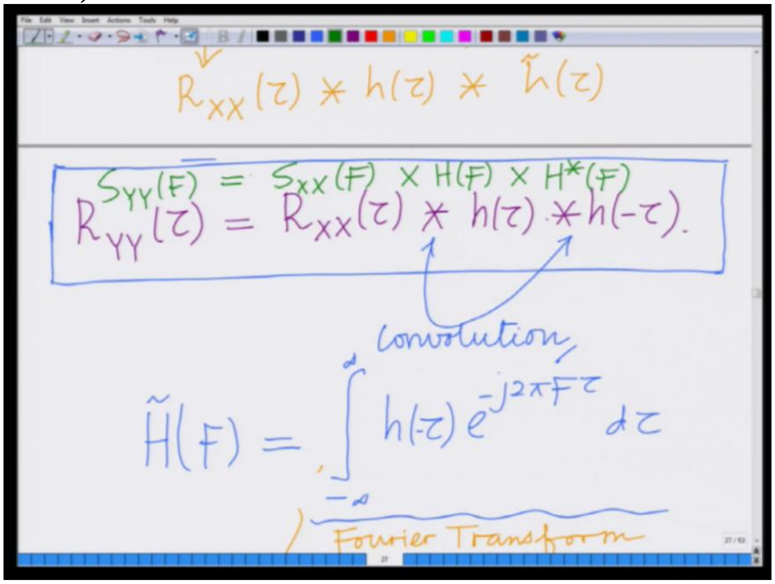
Refer Slide Time: 32:09)





Now, if I substitute this in this, I have $R_{XX}(\tau)$ whose Fourier transform is the power spectral density of X.

(Refer Slide Time: 32:12)



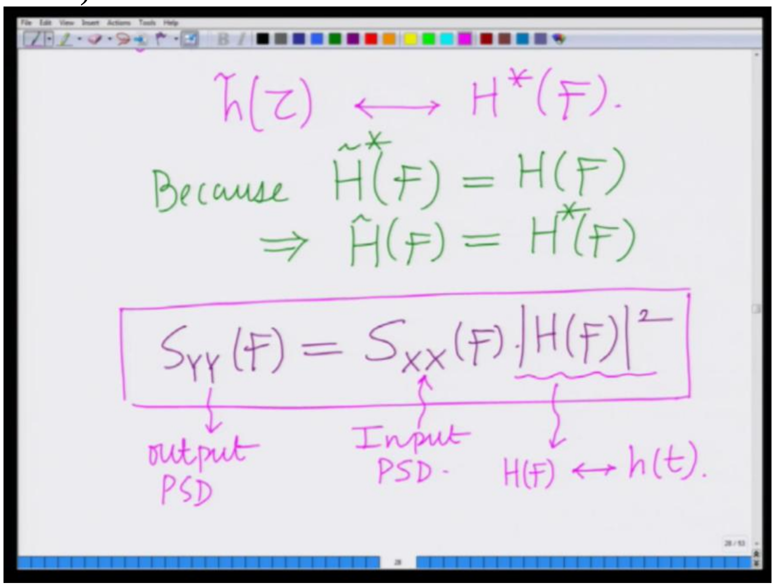
That is $S_{XX}(f)$. Remember,

$$S_{YY}(f) = S_{XX}(f) \times H(f) \times H^*(f)$$

$$R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

The frequency response of the LTI system that is the Fourier transform of the impulse response H(f) times H*(F).

(Refer Slide Time: 33:14)



Therefore we can write,

$$S_{YY}(f) = S_{XX}(f) \,.\, |H(f)|^2$$

So this is my output PSD. This is the PSD of the output. This is the input PSD. Remember, PSD stands for power spectral density.

So what we have shown is that the magnitude that is the output power spectral density $S_{YY}(f)$ is equal to the input power spectral density $H(f)$ times magnitude square of the Fourier transform of the channel frequency response. That is $|H(f)|^2$ and this is a very interesting result and a very important result which relates the input wide sense stationary random process to the output wide sense stationary random process. So we have demonstrated something very interesting. We have considered the transmission of a random process, wide sense stationary random process $X(t)$ through an LTI system which is characterized by an impulse response $h(t)$.

We have shown that the output random process Y of this LTI system is itself wide sense stationary with stationary in the mean, stationary in the autocorrelation and further, we have demonstrated something very interesting that is the output power spectral density $S_{YY}(f)$ is the input power spectral density $S_{XX}(f)$ times $|H(f)|^2$ where $H(f)$ is the frequency response of the LTI system. That is $H(f)$ is the Fourier transform of the impulse response, $h(t)$ of the LTI system. So we will stop this module here and we will look at other applications of this in the subsequent what use. Thank you very much.