Probability and Random Variables/Processes for Wireless Communication
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Module No. 1
Lecture 2
Axioms of probability

Hello, welcome to this massive open online course on probability and random variables for wireless communications. In the previous module, we had looked at the experiments and outcomes of the experiment and the concepts of sample space, events and other things.

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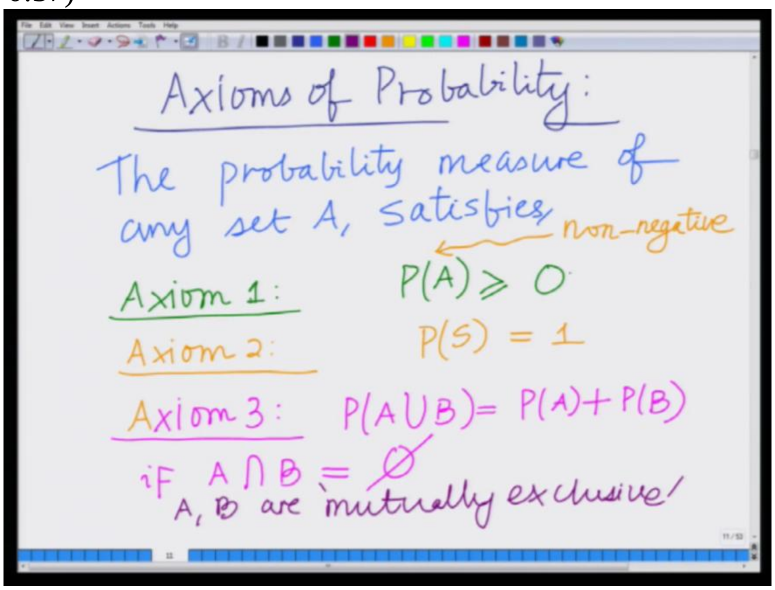

Let us now start with a formal definition of the axioms of probability. There are 3 axioms of probability which are very important and I'm going to list these. So, I have the 3 axioms of probability. What is probability? So let us start with the probability formal definition. The probability is a measure of any event A, the probability measure of any set A satisfies the following axioms and those axioms are as follows.

1st axiom : Probability measure of any event A is greater than or equal to 0.

i.e. $$P(A) \geq 0$$

This basic axiom states that this probability measure is nonnegative.

2nd axiom: The probability measure of the entire sample space is unity or one,

i.e. $P(S) = 1, \quad S = \text{sample space}$

3rd axiom: Probability of A union B equals the probability of A plus the probability of B if A intersection B is the null event phi or basically A, B are mutually exclusive.

i.e. $P(A \cup B) = P(A) + P(B), \quad A, B \subseteq S \text{ and } A \cap B = \emptyset$

So we have 3 axioms that the probability measure satisfies.

1. $P(A) \geq 0$
2. $P(S) = 1, \quad S = \text{sample space}$
3. $P(A \cup B) = P(A) + P(B), \quad A, B \subseteq S \text{ and } A \cap B = \emptyset$

So now, using these probability axioms, let us look at some basic properties.

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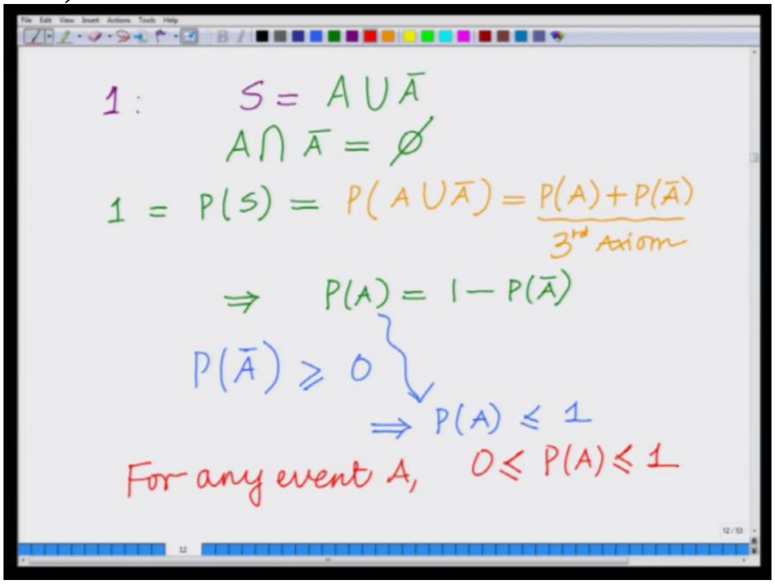

First, basic properties which follow from the probability axioms. Note that I have from my previous proof, that Sample space, S is given as-

$$S = A \cup \bar{A}$$

Further, any A and $\bar{A}$ are also mutually exclusive, i.e.

$$A \cap \bar{A} = \emptyset$$

That's what we saw in the previous module. Therefore I have 1 which is equal to the probability space, S is equal to A union A complement. But A and A complement are also mutually exclusive. Therefore from the third axiom,

$$P(A) + P(\bar{A}) = P(S) = 1$$

this is probability of A plus probability of A complement. This follows from the third axiom. And this is therefore also now you can see, this implies that

$$P(A) = 1 - P(\bar{A})$$

But once again, from the First axiom, we have,

$$P(\bar{A}) \geq 0$$

And from Second axiom we have,

$$P(\bar{A}) \leq 1$$

Therefore, $\Rightarrow 0 \leq P(A) \leq 1$

Thus, we have the result that the probability of any event A must lie between 0 and 1. That is the First important result that we have. Let us now look at the next result.

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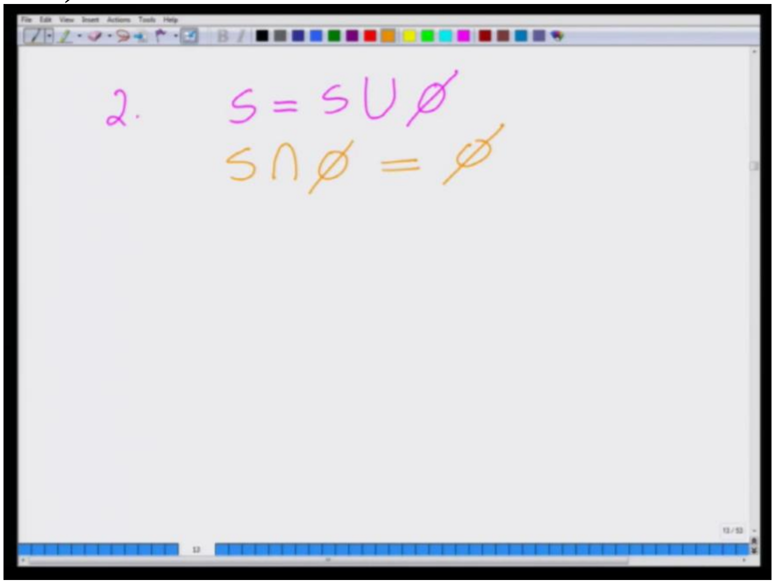

The union of the samples Sample space with the null event, is again the Sample space.

i.e. $S \cup \emptyset = S$

That is, if I take the intersection of the Sample space and the null event, that gives the null event because the null event does not contain any event. So its intersection with any event in particular intersection with the Sample space is again the null event because the intersection is basically Sample points which are both in the Sample space and the null event. And the null event does not contain any Sample points. Therefore, the intersection is the empty set or the null event.

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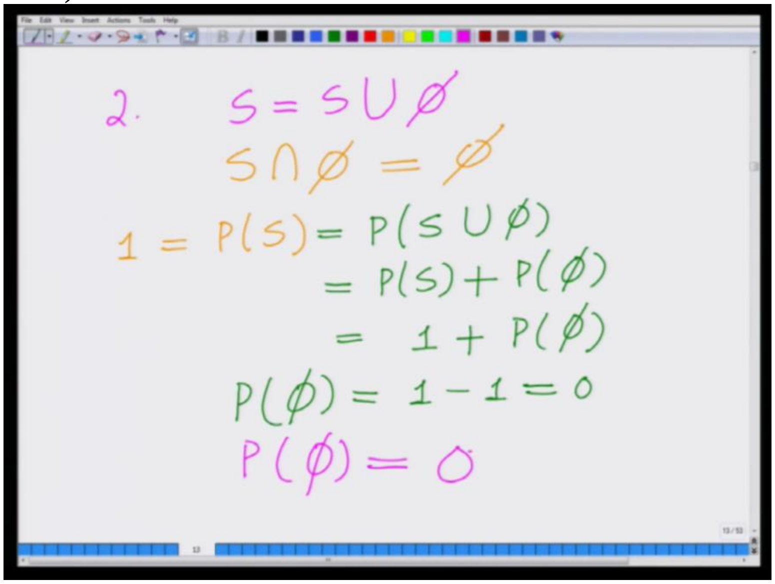

Now again, using the Second axiom, I have

$$P(S) = 1 = P(S \cup \emptyset)$$

and as, $S \cap \emptyset = \emptyset$

$$\Leftrightarrow P(S \cup \emptyset) = P(S) + P(\emptyset)$$

As, $P(S) = 1$,

$$= 1 + P(\emptyset)$$

$$\Rightarrow \quad 1 = 1 + P(\emptyset)$$

$$\Rightarrow \quad P(\emptyset) = 0$$

Therefore we have our Second property that we have derived that is the probability of null event is 0 or this is also basically an impossible event.

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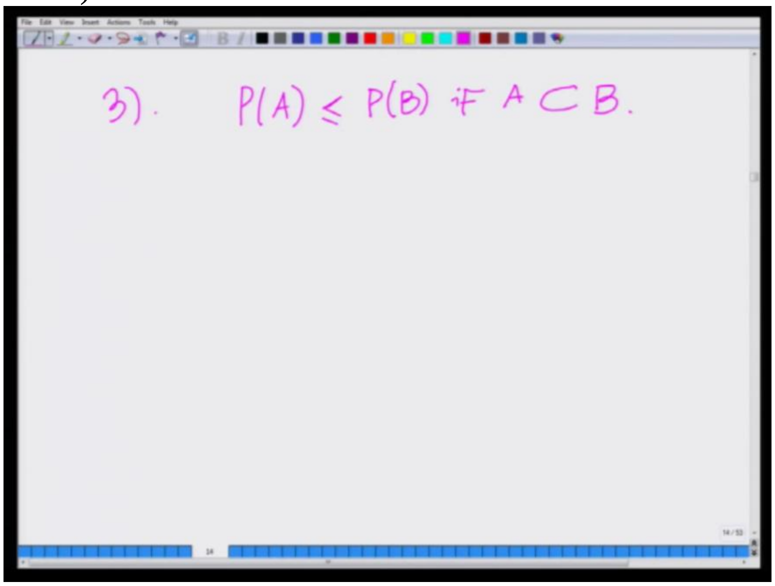

And 3rd important property is, if A and B are any events, and P(A) is less than P(B) if A is a subset of B. Thus if event A is the subset of event B, then the probability measure of the event A is less than or equal to the probability measure of B.

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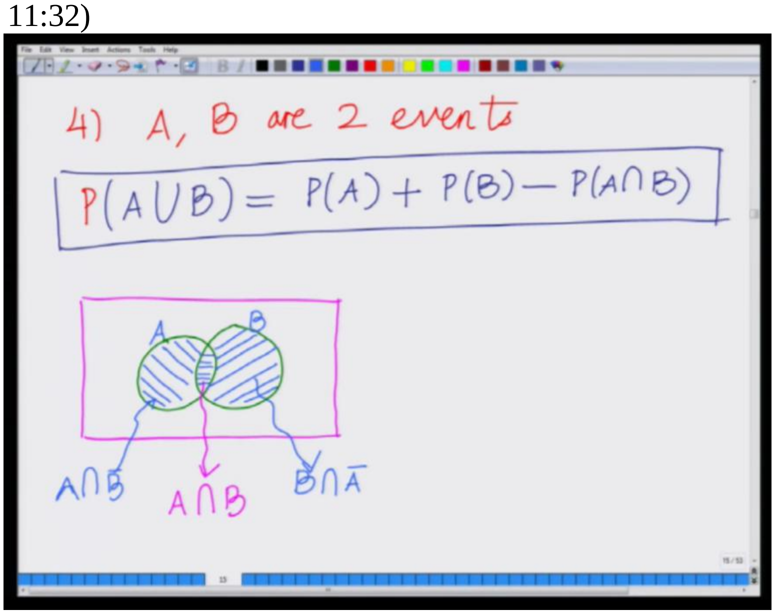

4th important property which I would like to prove, which is the following, If A and B are any events, then I have the probability of A union B equals the probability of A plus the probability of B minus the probability of A intersection B.

i.e. $\forall$ A, B $\subseteq$S $\Rightarrow$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

And this can be seen as follows. For instance, if I have 2 events A, B if I look at, this is my set A, this is my set B. This is $A \cap B$. One of the region which is present in B but not in A and this region is therefore $B \cap \bar{A}$. and another is the region which is present in A but not in B. Therefore this is the region $A \cap \bar{B}$.

So I have 2 sets A, B or 2 events A and B and I can decompose these 2 events A and B into 3 distinct regions. One is $A \cap B$, that is the overlap between these 2 events. $A \cap \bar{B}$ is all the Sample points which are in A but not in B, and $B \cap \bar{A}$ which is all the points which are in B but not in A.

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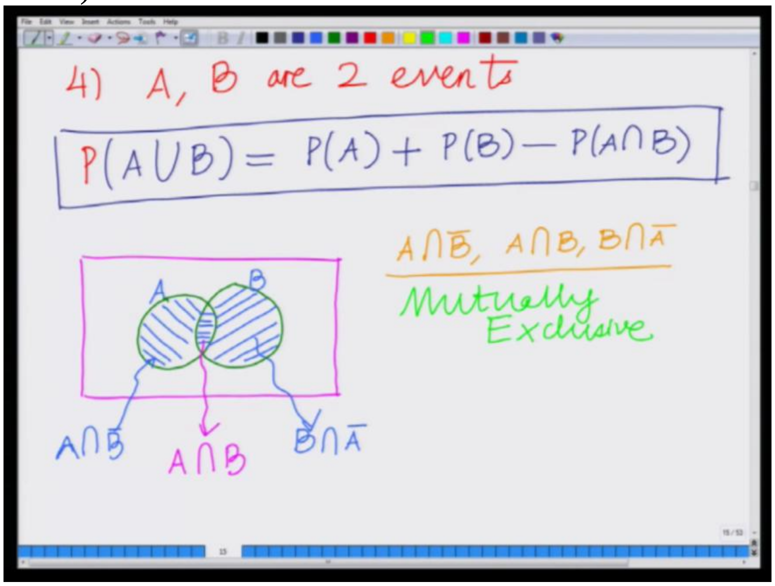

And you can see from this picture that these 3 events, $A \cap \bar{B}$, $A \cap B$ and $B \cap \bar{A}$ are mutually exclusive. These are mutually exclusive events because they do not have any overlap.

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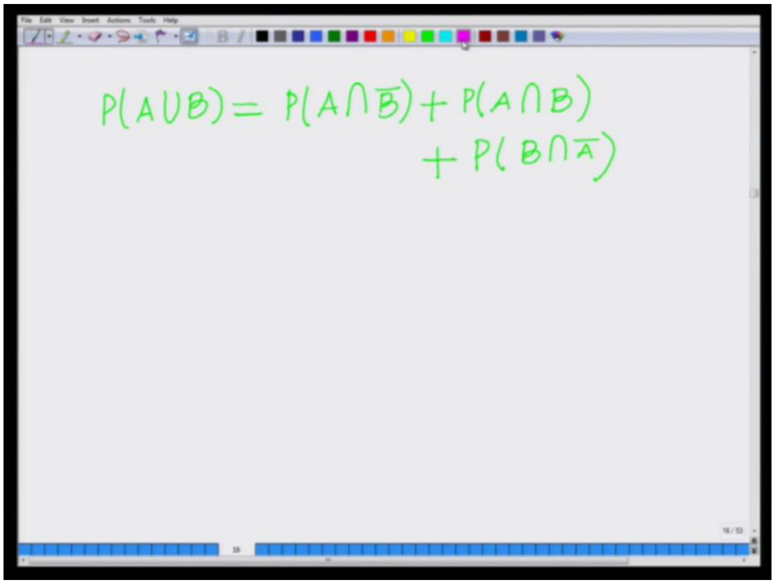

And therefore, naturally, it follows that I have the probability of $A \cup B$ equals

i.e. $P(A \cup B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B)$

Correct? So what did we say? We said that any 2 sets A and B can be represented as 3 distinct parts. That is $A \cap \bar{B}$, $A \cap B$ and $\bar{A} \cap B$. These are disjoint and further, their union is equal to $A \cup B$. Therefore the probability of $A \cup B$ is the sum of these 3 distinct mutually exclusive components.

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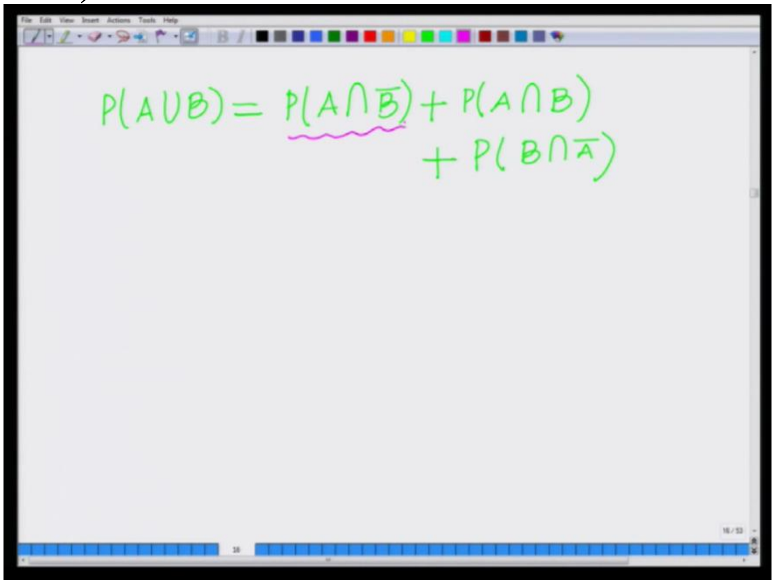

Now let us look at this event, probability of $A \cap \bar{B}$.

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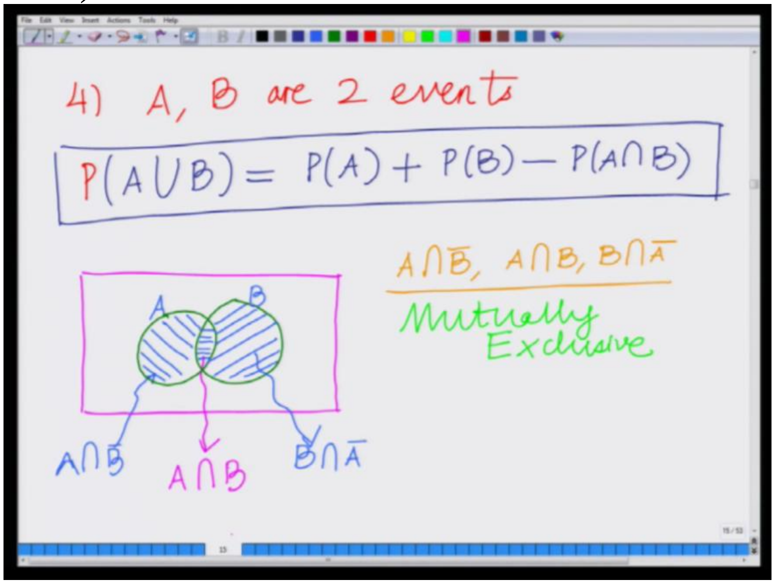

If you can see, A intersection, now if you look at A, the event A is the union of $A \cap \bar{B}$ and $A \cap B$.

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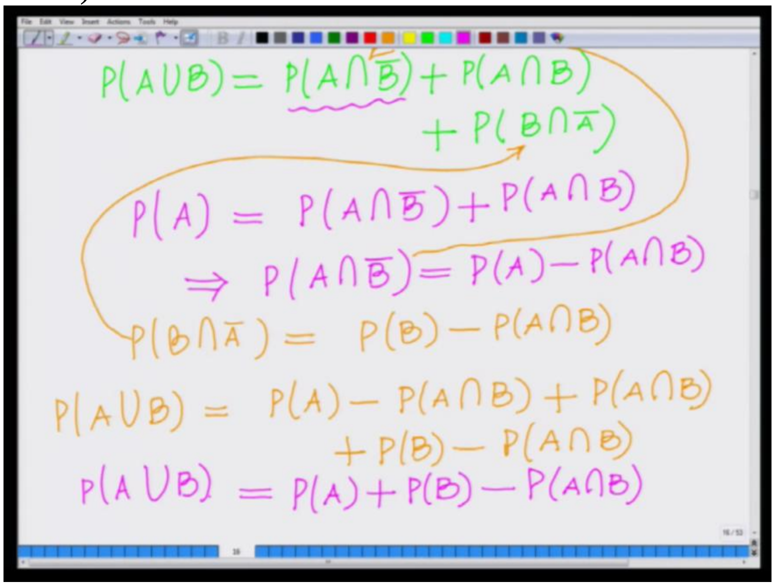

Therefore I have

$$P(A) = P(A\cap\bar{B}) + P(A\cap B)$$

$$\Rightarrow P(A\cap\bar{B}) = P(A) - P(A\cap B)$$

Now, as

$$P(A\cup B) = P(A\cap\bar{B}) + P(A\cap B) + P(\bar{A}\cap B)$$

So, substituting, value of $P(A\cap\bar{B})$, we have,

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

Therefore the probability of $A\cup B$ equals probability of A plus probability of B minus the probability of A intersection B. This is one of the fundamental results in probability.

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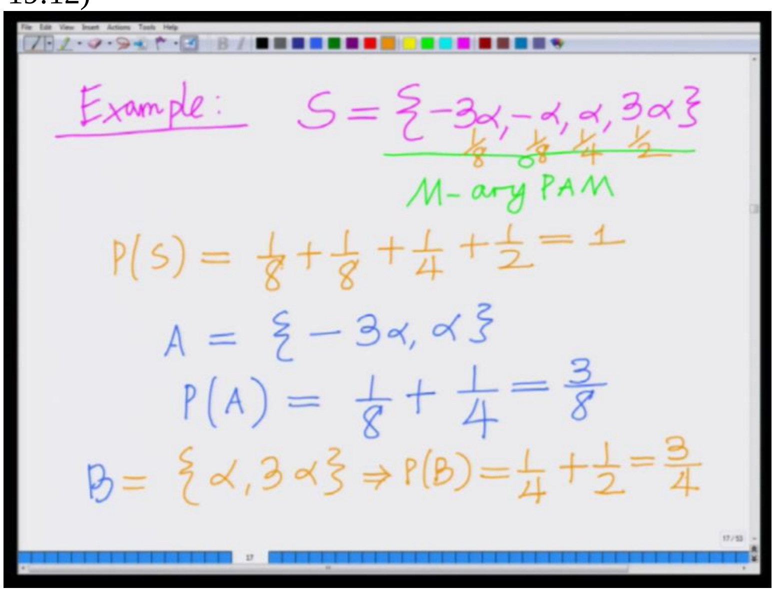

Let us take a simple example to understand this. It is look at a simple example. Let us go back to our M ary PAM. M ary pulse amplitude modulation. Let us take a simple example to understand this. Let us go back to our pulse amplitude, M ary pulse amplitude modulation which we introduced in the previous module.

This is our M ary PAM and let us say that we have the following probabilities.

Let us say,

$P(-3\alpha) = 1/8$

$P(-\alpha) = 1/8$

$P(\alpha) = 1/4$

$P(3\alpha) = 1/2$

Therefore now if you look at the sample set,

$$P(-3\alpha)+P(-\alpha)+P(\alpha)+P(3\alpha) = 1/8 + 1/8 + 1/4 + 1/2 = 1$$

Now, as

$$S = \{-3\alpha, -\alpha, \alpha, 3\alpha\}$$

Thus,

$$P(S) = 1$$

So the probability of the entire sample space is equal to 1. So that satisfies the probability axiom.

Further, let A be the event,

$$A = \{-3\alpha, \alpha\}$$

Then

$$P(A) = P(-3\alpha)+P(\alpha) = 1/8 + 1/4 = 3/8$$

Let B be the event, defined as

$$B = \{\alpha, 3\alpha\}$$

$$\Rightarrow P(B) = P(\alpha)+P(3\alpha)$$

$$\Rightarrow P(B) = 1/4 + 1/2 = 3/4$$

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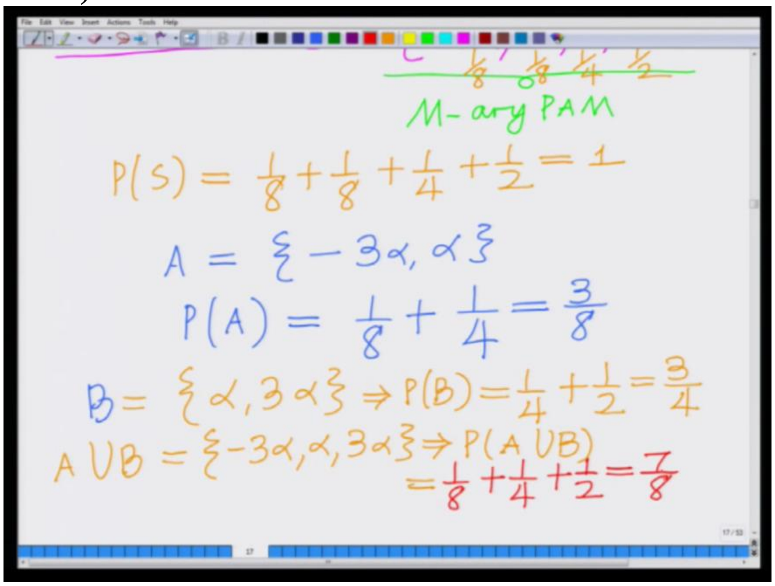

Now if you look at this, I have A union B which is,

$$A \cup B = \{-3\alpha, \alpha, 3\alpha\}$$

This implies the probability of A union B is given as

$$P(A \cup B) = P(-3\alpha) + P(\alpha) + P(3\alpha) = 1/8 + 1/4 + 1/2 = 7/8$$

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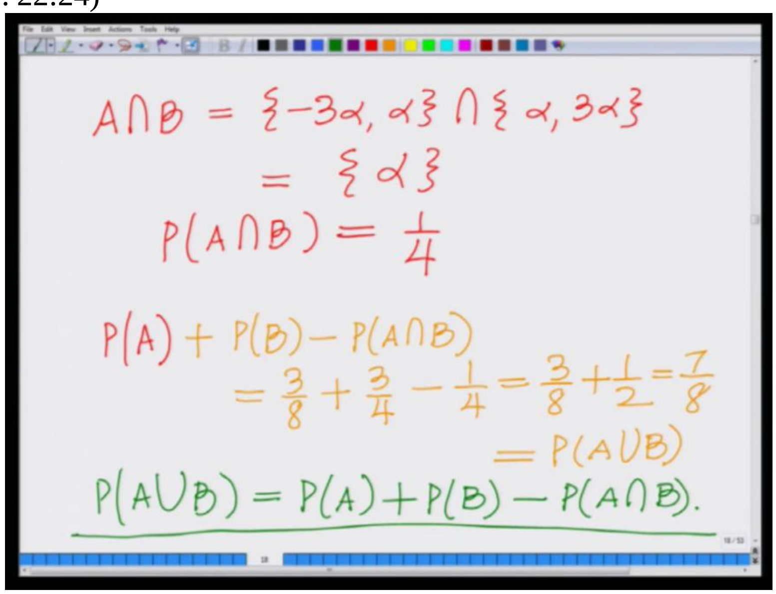

Now, $P(A\cap B)$, for $A\cap B$ you have to look at A intersection B,

And, since

$$A\cap B = \{\alpha\},$$

$$\Rightarrow P(A\cap B) = P(\alpha) = 1/4$$

And now,

$$\Rightarrow P(A)+P(B)-P(A\cap B) = 3/8 + 3/4 - 1/4 = 7/8$$

Which is equal to $P(A\cup B)$, as calculated above.

So what we have seen is, we have considered this simple examples example from M ary PAM, M ary pulse rate would modulation for a wireless communication system and we have derived that the probability, we have shown simply, we have verified the result that we had derived earlier that is the probability measure of A union B is the probability measure of A plus the probability measure of B minus the probability measure of A intersection B. So let us conclude this module with this example. We will look at some other aspects in subsequent modules. Thank you very much.