Probability and Random Variables/Processes for Wireless Communication
Professor Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Module No. 4
Lecture 19
Power Spectral Density (PSD) for WSS Random Process.

Hello, welcome to another module in this massive open online course on probability and random variables for wireless communications. In the recent previous modules, we have seen random processes and a special kind of specific subclass of random processes that is wide sense stationary random processes. We have defined this property of wide sense stationarity of a random process. Now, let us look at another interesting property of a wide sense stationary random process which is known as the power spectral density and this is a very important quantity for a wide sense stationary random process.

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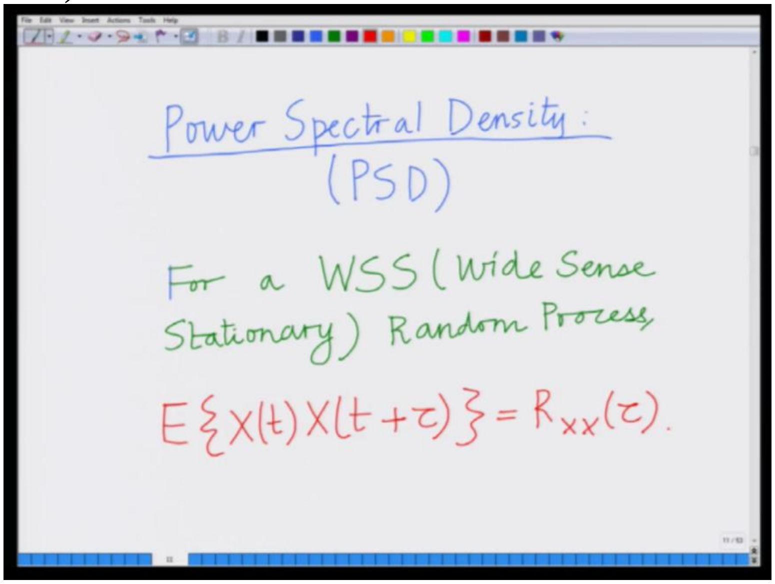

So what we want to start looking at today is another new and important concept which is the power spectral density and I'm also going to abbreviate this as PSD the power spectral density. And what is the power spectral density? We have already seen that. For a wide sense stationary random process, for a WSS which is basically for a wide sense stationary ~~for a wide sense stationary for our wide sense stationary~~ random process we have the autocorrelation function which is defined as –

$$E\{ X(t) \times X(t+\tau) \} = R_{XX}(\tau)$$

that is depends only on the timeshift τ.

That is, if we consider a random process $X(t)$ the co-relation at two time instances, t and $t+\tau$ that is $E\{ X(t) \times X(t+\tau) \}$ depends only on the timeshift τ and is denoted by $R_{XX}(\tau)$.

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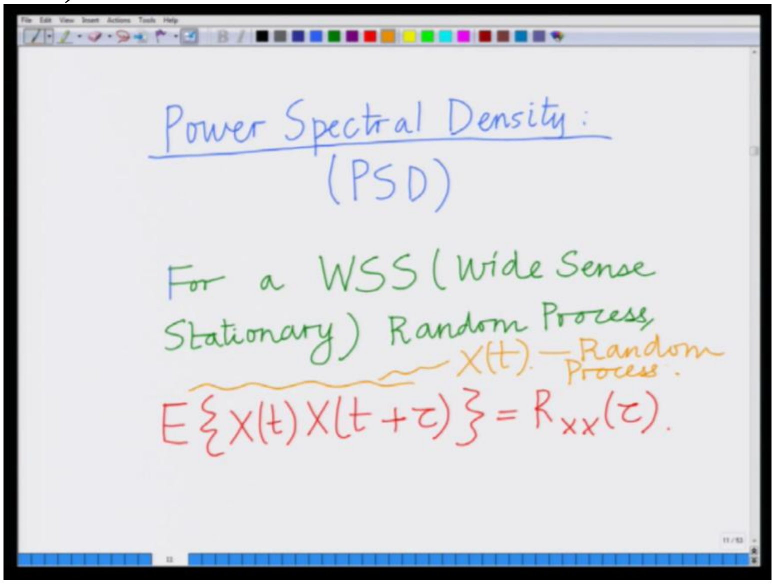

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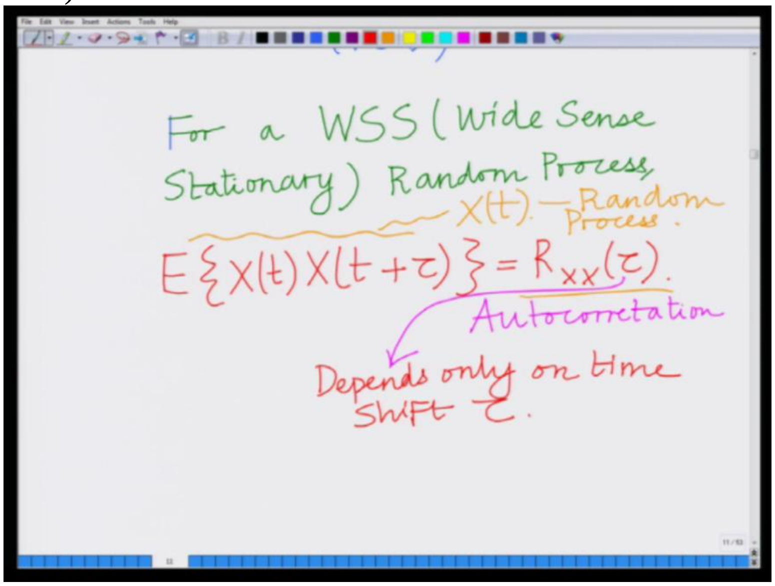

Now, remember this is also known as the autocorrelation function and this depends only on the timeshift τ. That is remember the autocorrelation for a wide sense stationary random process, depends only on the timeshift τ and not the specific time instant, t. And now for this wide sense stationary random process, we can find the power spectral density and the power spectral density is simply given as the fourier transform of the autocorrelation function.

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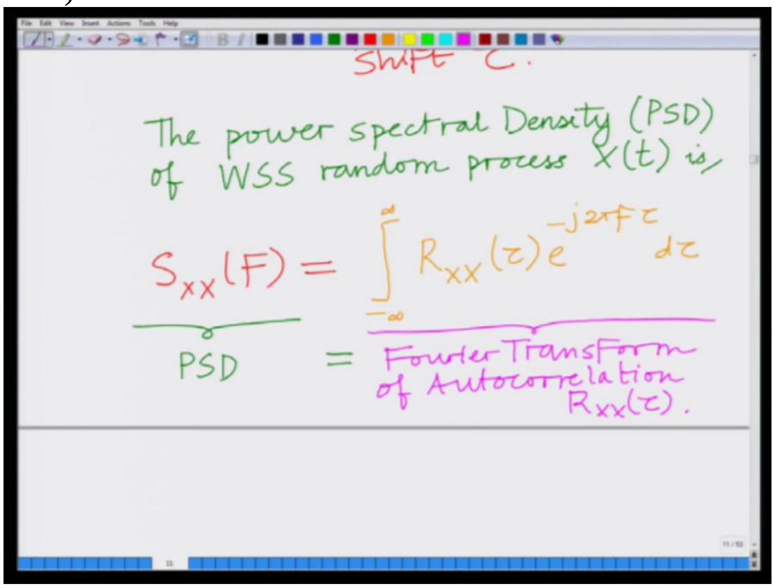

So, in the frequency domain, the power spectral density, or the PSD of a wide sense stationary random process $X(t)$ is denoted by $S_{XX}(f)$. This is the power spectral density and it is defined as the Fourier transform of the autocorrelation function. That is –

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j2\pi f\tau} d\tau$$

We have the PSD, the power spectral density is basically equal to what is this equal to?

This is equal to the Fourier transform of the autocorrelation function of $R_{XX}(\tau)$. So $S_{XX}(f)$ is, in the frequency domain, the power spectral density, for the random process, Okay? So naturally, now using the inverse Fourier transform, one can say that the autocorrelation function is the inverse Fourier transform of the power spectral density. Since the power spectral density is the Fourier transform of the autocorrelation, so naturally it follows that the autocorrelation is the inverse Fourier transform of the power spectral density.

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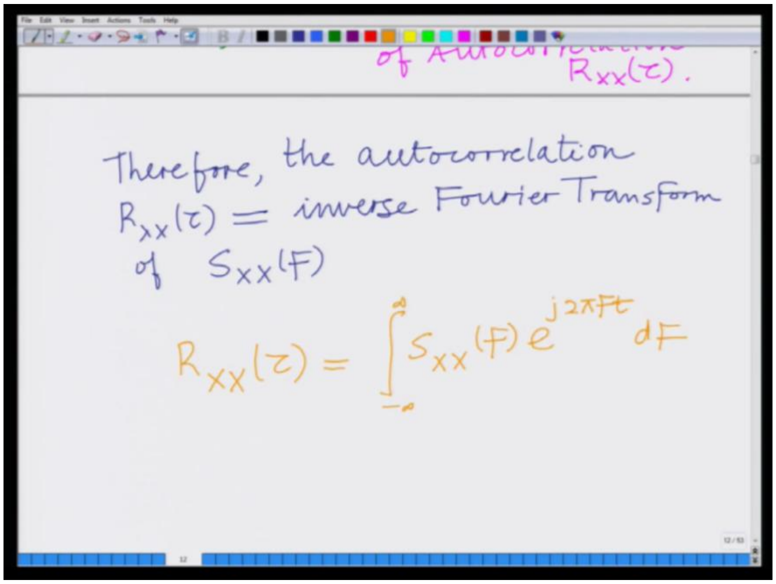

And therefore, PSD, the power spectral density is the Fourier transform or $S_{XX}(f)$. Therefore the autocorrelation that is $R_{XX}(\tau)$ equals inverse Fourier transform of the power spectral density $S_{XX}(f)$ and this can be denoted as –

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)\, e^{j2\pi ft} df$$

that is integration with respect to f, since this is the inverse Fourier transform.

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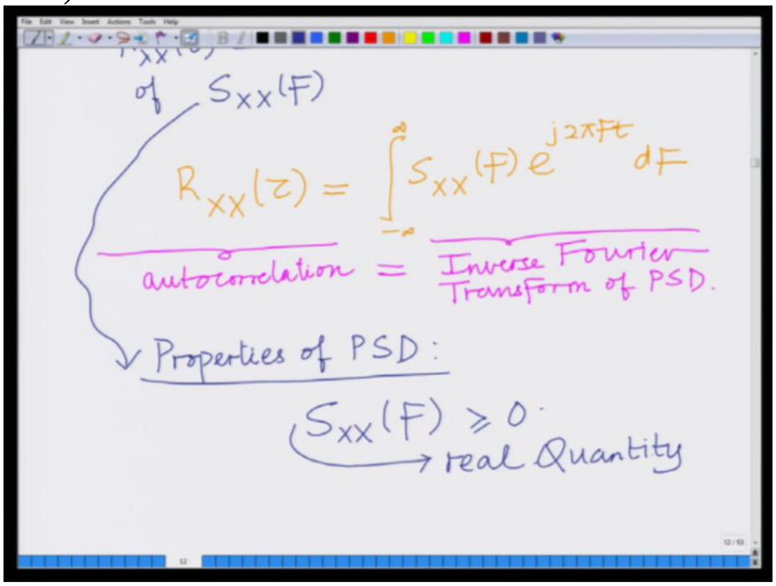

One property of the power spectral density is, it is a power so naturally it can be shown that the power spectral density has to be greater than or equal to 0. Further naturally, it is power so it has to be a real quantity. Right? It cannot be a complex quantity.

$$S_{XX}(\mathrm{f}) \geq 0$$

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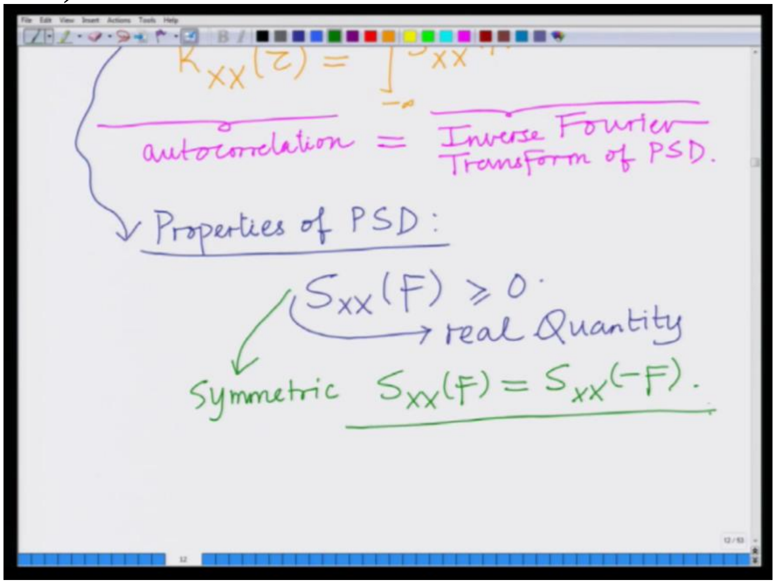

Further, it is symmetric about the frequency domain. Further, it is symmetric in the sense that –

$$S_{XX}(\mathrm{f}) = S_{XX}(-\mathrm{f})$$

So we are saying that the power spectral density is a real quantity which is always greater than or equal to 0 and it is also symmetric. That is $S_{XX}(\mathrm{f}) = S_{XX}(-\mathrm{f})$. Okay?

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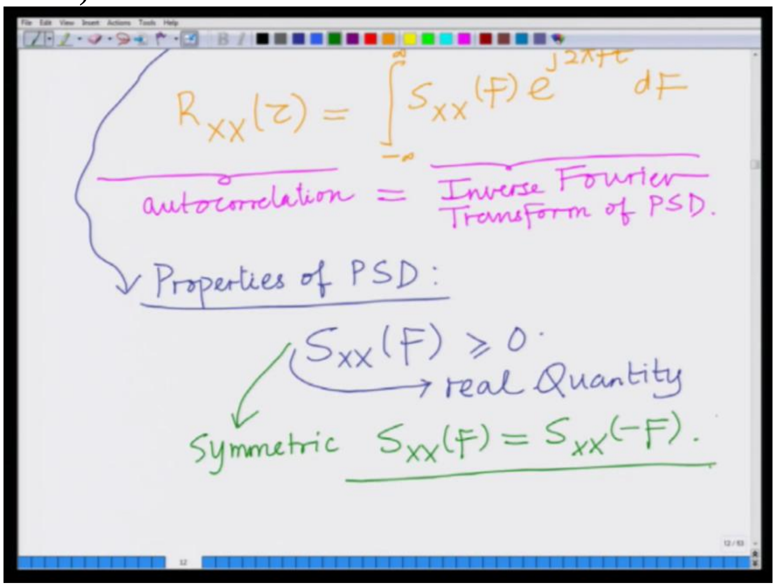

And another interesting property if you can look at it from this integral for the inverse Fourier transform, we have $R_{XX}(\tau)$ is the inverse Fourier transform of the power spectral density.

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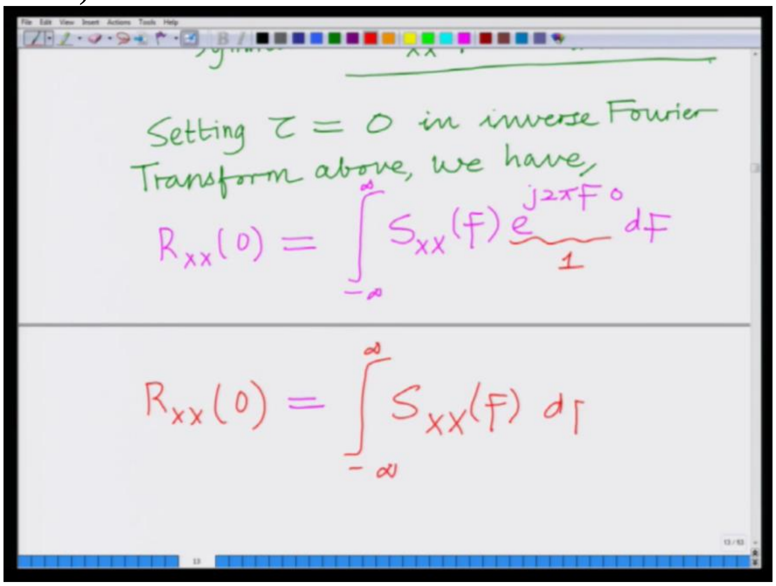

Now if I say, $\tau = 0$, in the inverse Fourier, we have, $R_{XX}(0)$ is the inverse Fourier transform evaluated at $\tau = 0$.

$$R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\mathrm{f})\, e^{j2\pi \mathrm{f}0} d\mathrm{f}$$

$$R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\mathrm{f})\, df$$

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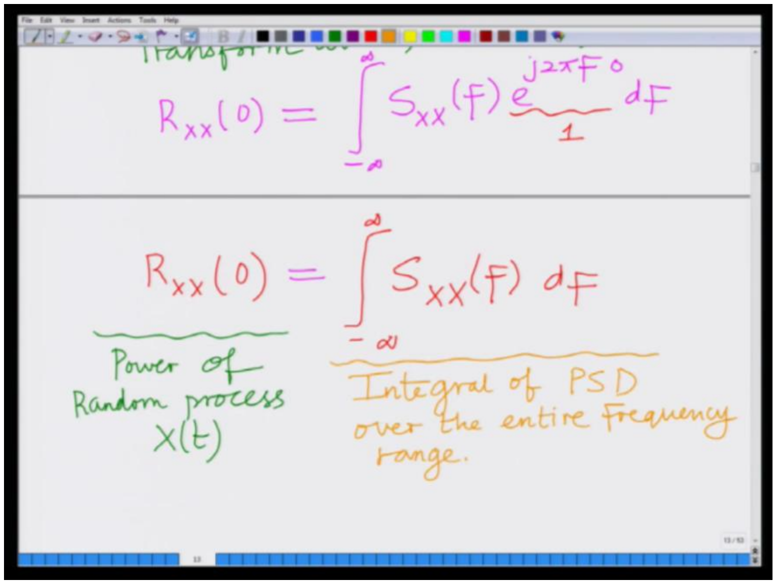

Now we can clearly see, $R_{XX}(0)$ is the autocorrelation at τ = 0, this is E{ $X^2(t)$}. This is the power of the random process X(t) and this is simply integral of the power spectral density over the frequency domain. That is what we are seeing is $R_{XX}(0)$ that is autocorrelation of the random process delay τ = 0, is equal to the integral of the power spectral density over the entire frequency domain or basically the area under the power spectral density from [-∞, ∞] okay?

So that is basically another formula for the power of the random process X(t) in terms of the power spectral density S_{XX}(f). And not only that there is another interesting property of the power spectral density which can be explained as follows. That is, if we want to find the power in a frequency band $[F_1 , F_2]$.

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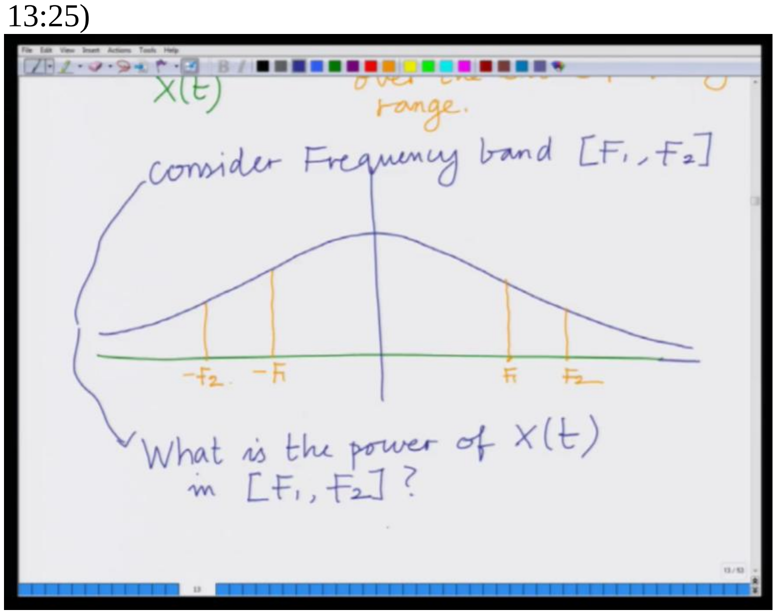

What is the power of random process X(t) in the frequency band $[F_1, F_2]$ and the answer to the question is very interesting. The power in the frequency band $[F_1, F_2]$ is basically the area under the power spectral density from $[-F_2, -F_1]$ and $[F_1, F_2]$.

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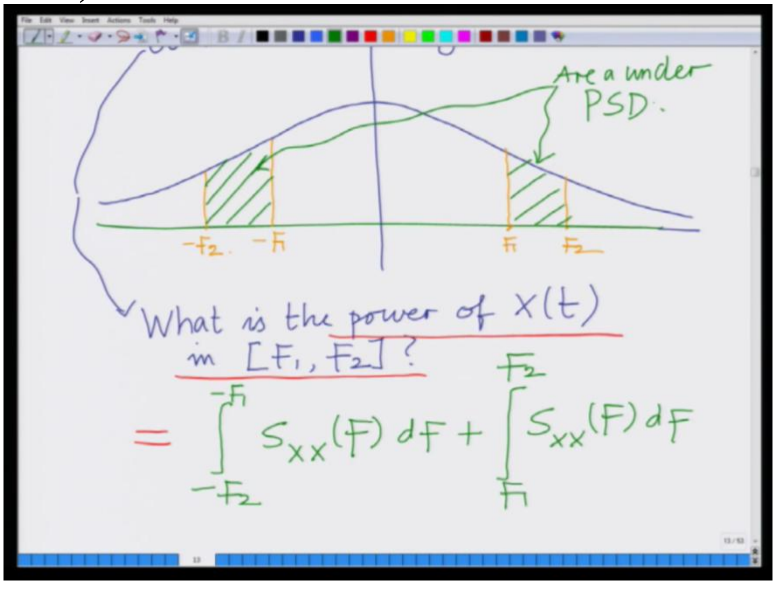

And therefore this power in the frequency band power of random process X(t) in spectral band $[F_1, F_2]$, this is equal to –

$$= \int_{-F_2}^{-F_1} S_{XX}(f)\, df + \int_{F_1}^{F_2} S_{XX}(f)\, df$$

In other words, the area under the power spectral density from $[-F_2, -F_1]$ and $[F_1, F_2]$.

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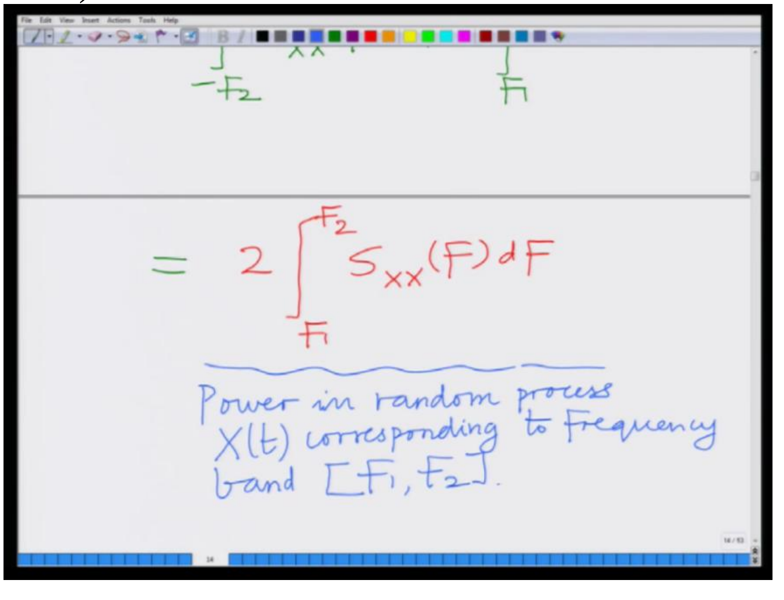

$$= 2 \int_{F_1}^{F_2} S_{XX}(f)\, df$$

This is the power in the random process X(t) corresponding to frequency or the spectral band $[F_1, F_2]$.

So, the total power is basically the integral of the power spectral density over the entire frequency band that is $[-\infty, \infty]$.

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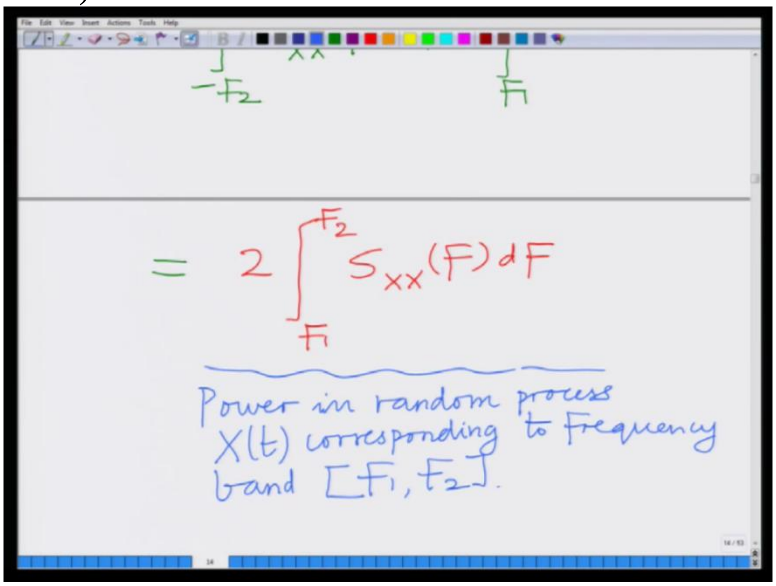

So what we have seen in this module is basically we have seen various aspects of the power spectral density. We have said that this is the power spectral density of a wide sense stationary random process which is basically defined in the terms of the autocorrelation function that is the power spectral density $S_{XX}(f)$ of random wide sense stationary random process X(t) is nothing but the Fourier transform of the autocorrelation function $R_{XX}(\tau)$.

And therefore similarly the autocorrelation is given as the inverse Fourier transform of the power spectral density. Further, the power spectral density is symmetric and it is always greater than or equal to 0 since it is a quantity associated with the power. More importantly what we have seen is basically that the power of the random process X(t) is basically the integral of the power spectral density over the entire frequency range $[-\infty, \infty]$ and specifically the power in any frequency band, $[F_1, F_2]$ can be calculated as twice the area under power spectral density between $[F_1, F_2]$. Power spectral density is a very key quantity, a very important quantity, to both analyse and examine and understand the random process $X(t)$. So we will stop this module here and look at an application in the subsequent class. Thank you very much.