Probability and Random Variables/Processes for Wireless Communications.
Professor Aditya K Jagannatham.
Department of Electrical Engineering.
Indian Institute of Technology Kanpur.
Lecture -18.
WSS Example - Narrowband Wireless Signal with Random Phase.

Hello, welcome to another module in this massive open online course on probability and random variables for wireless communication. In the previous model we had started looking at random processes and we have defined the mean and autocorrelation of the random process and also we look at a special kind of random process, wide sense stationary random process in which the mean is a constant that is it does not depend on time and the autocorrelation depends only on the timeshift that is the autocorrelation corresponding to T1 and T2 depends only on the timeshift τ between T1 and T2, let us write these definitions down once again.

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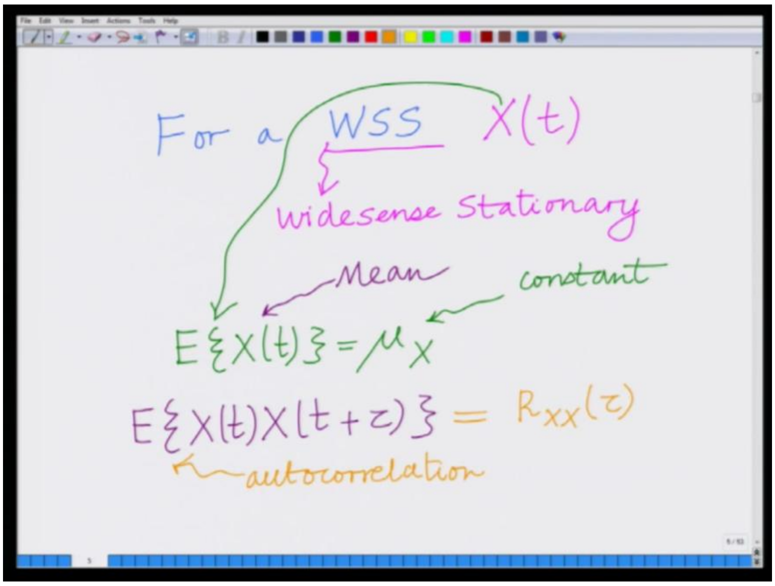

That is for a wide sense stationary, that is remember WSS is a wide sense stationary random process $X(t)$. For a wide sense stationary random process $X(t)$ we have,

$$E\{X(t)\} = \mu_X$$

which is a constant that is this is a constant and further we have,

$$E\{X(t)\, X(t+\tau)\} = R_{XX}(\tau)$$

So this is basically the mean and we have the autocorrelation depends only on timeshift, that is τ.

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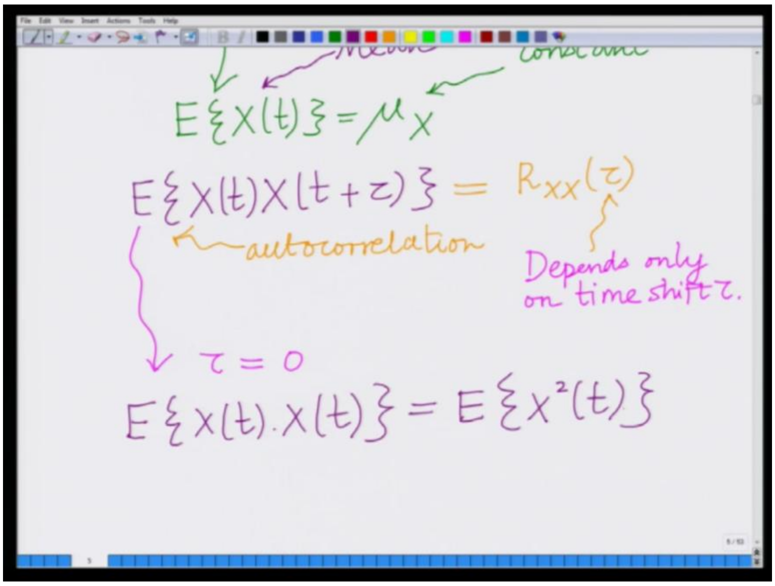

If I said this timeshift $\tau = 0$, what we get is,

$$E\{X(t)\, X(t+\tau)\} = E\{X^2(t)\}$$

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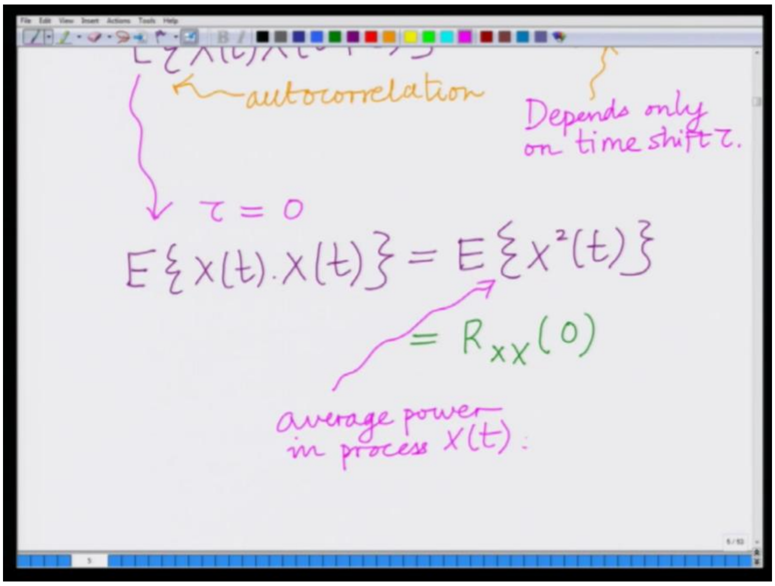

This is nothing but basically the average power.

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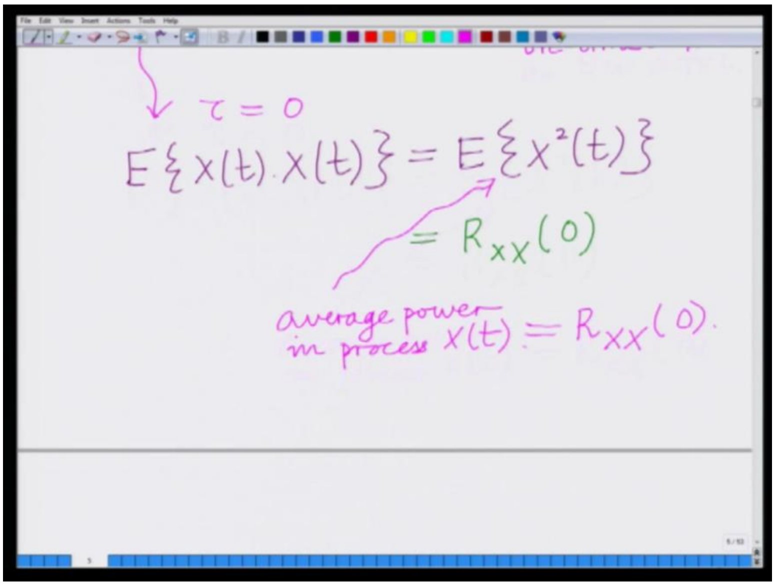

This is the average power in the random process X(t). So, if I said this τ = 0, what I get is

$$E\{X(t)\ X(t)\} = E\{X^2(t\ )\}$$

$$= R_{XX}(0)$$

Therefore $R_{XX}(0)$ denotes the power of the average power of the random process X(t). So, this average power of the random process X(t) equals $R_{XX}(0)$.

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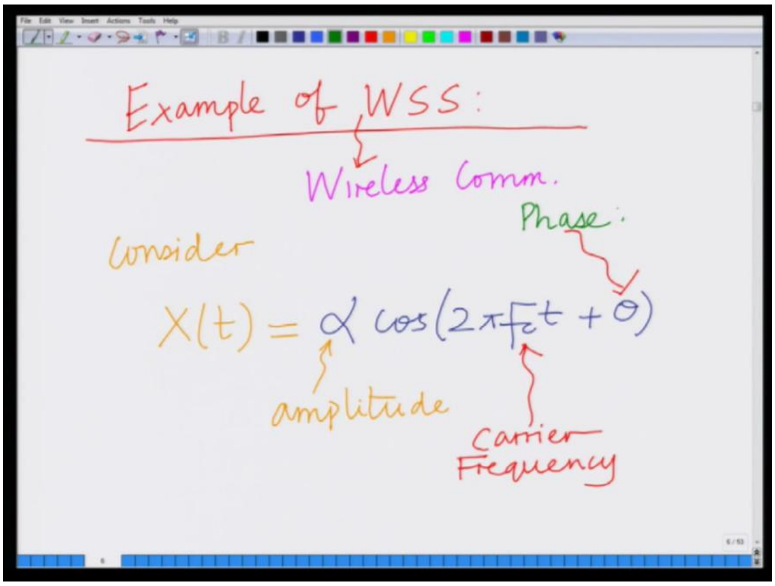

Let us now look at an interesting example in the context of this random process and wide sense stationarity in the context of wireless communication, let us look at an example of a

WSS, wide sense stationary random process in the context of wireless communication. Let us consider a signal $X(t)$ which is equal to –

$$X(t) = \alpha \cos(2\pi f_c t + \theta)$$

where α is our amplitude, f_c is the carrier frequency, everyone should be familiar with this in the context of communications or wireless communication. And θ is basically the phase.

This is a very common model, for the wireless communication single or for that matter any communication any narrowband communications signal.

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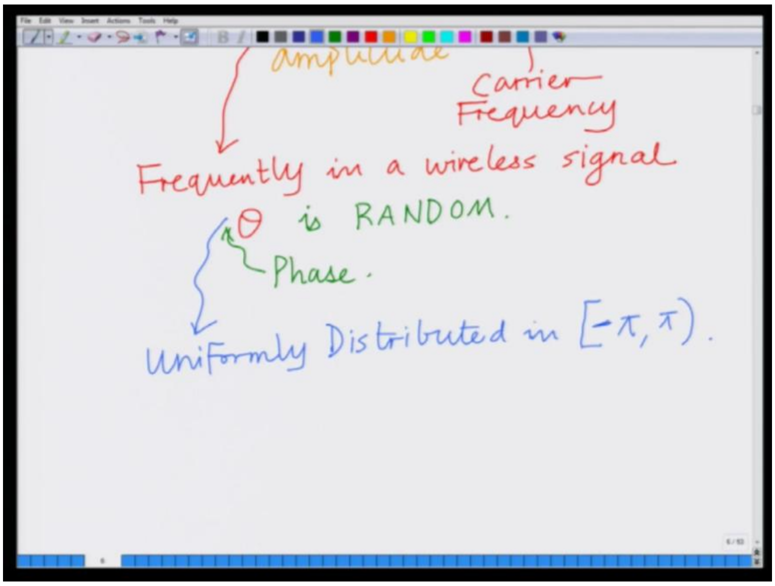

So, now let us look further, let us incorporate randomness in this. Now frequently what happens in such signal, frequently in wireless signal, frequently we have in a wireless communication, we have θ is random θ. The phase θ is random, further how do we model this randomness?

θ is uniformly distributed over the interval $[-\pi, \pi]$. So, what we are saying is we have the wireless communication signal $X(t) = \alpha \cos(2\pi f_c t + \theta)$ where θ is the phase and is random, how is it distributed and what is the probability density function ? It is uniform in the interval $[-\pi, \pi]$. So, it is a uniform random variable and that random variable is given as follows,

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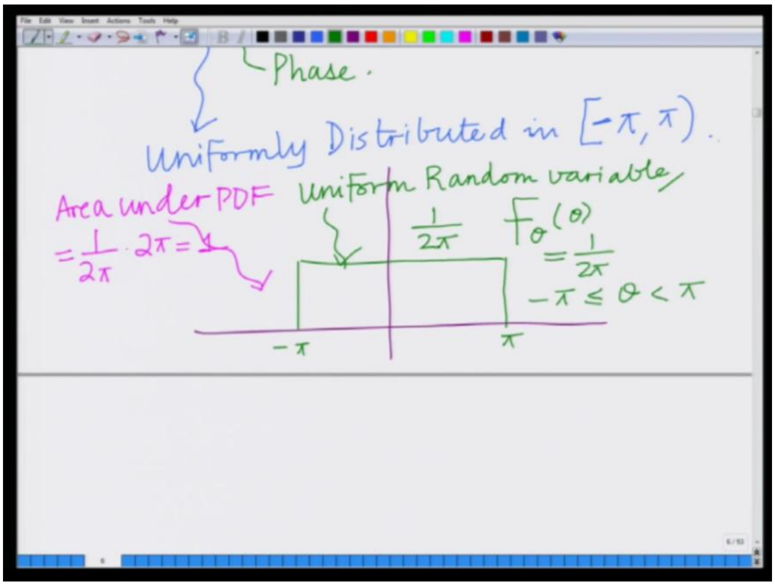

that is if I look at this interval $[-\pi, \pi]$, we have the magnitude of this is $1/2\pi$, so the probability density function,

$$F_{\theta}(\theta) = \frac{1}{2\pi} \qquad -\pi \le \theta < \pi$$

So, for this phase θ in the interval $[-\pi, \pi]$, the probability density function is $1/2\pi$ and outside of this interval of course this probability density function is 0 and now you can see this is a valid quality density function because 1st it is greater than 0.

Further if I look at the area under the probability density function, the area under PDF, that is equal to $1/2\pi$ times 2π which is basically 1. So, probability density function is basically integrates to 1, that is the total probability is basically one, therefore this is a valid probability density function. Further this is uniform, it takes the uniform value $1/2\pi$ over the entire domain that is $[-\pi, \pi]$, therefore this is a uniform random variable. Because this is uniform over the domain, this is a uniform random variable.

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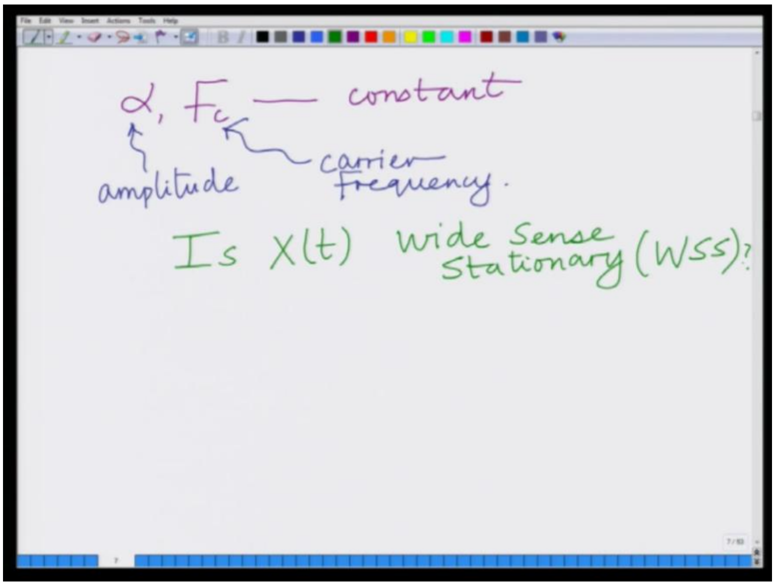

So, the phase θ is distributed uniformly in [-π, π]. Further we are also given that amplitude α, carrier frequency f_c, these are constant. Now the question that we want to ask is, is this signal X(t) wide sense stationary that is WSS. This is a random process, is this wide sense stationary? Let us examine this we are given basically X(t), 1st observe that if we look.

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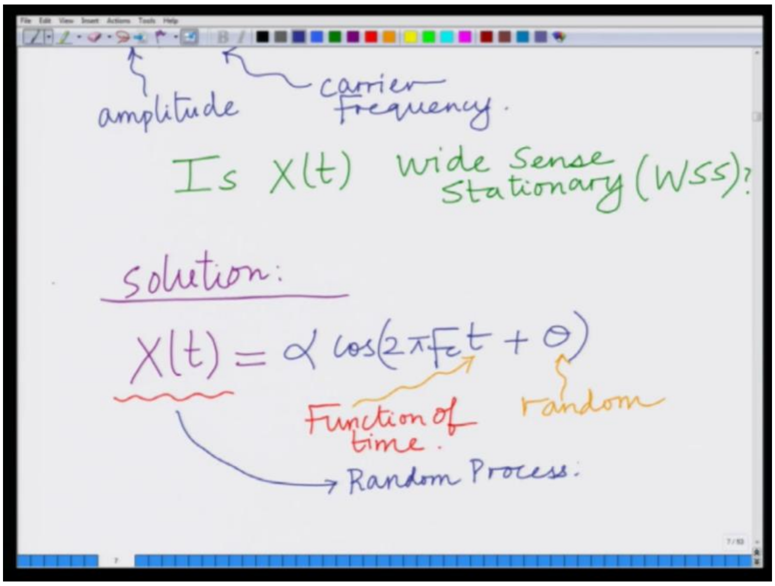

So let us look at the solution we have,

$$X(t) = \alpha \cos(2\pi f_c t + \theta)$$

and we also have θ is random in nature, θ is a uniformly distributed random variable. Further look at this time instant, this is a function of time t. So, we have X(t) which has the phase θ which is the uniformly distributed random variable. So, therefore it is a random variable which is a function of time, hence X(t) is a random process. So, 1st, we have established that this X(t) which is a random signal, is in fact a random process.

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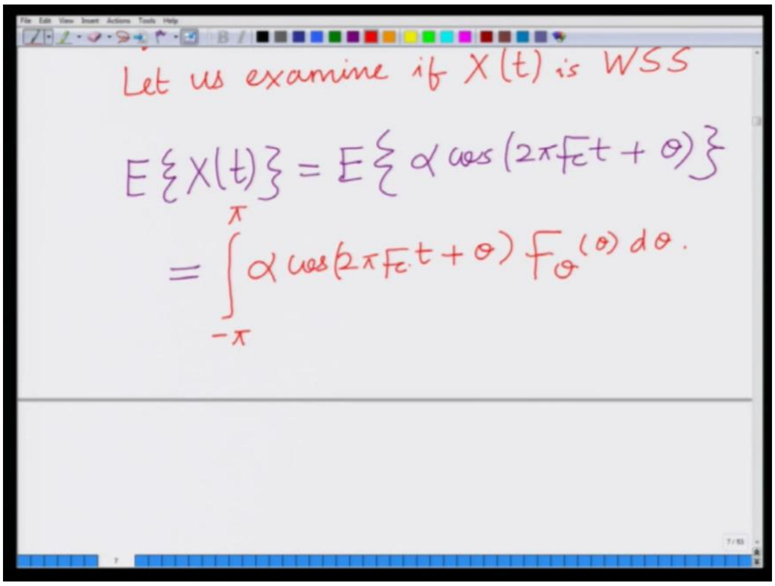

Now let us examine if basically it is a wide sense stationary process. 1st let us look at the mean, let us look at the expected value of X(t) is –

$$E\{X(t)\} = E\{\alpha \cos(2\pi f_c t + \theta)\}$$

$$= \int_{-\pi}^{\pi} \alpha \cos(2\pi fct + \theta) F_\theta(\theta) d\theta$$

So, how do we check if this random process X(t) is a wide sense stationary random process? 1st, remember, the wide sense stationary random process is stationary in the mean, therefore we have to check the mean. And the probability density function is a uniform probability density function which is 1/2π. So, I am going to replace $F_\theta(\theta)$ by 1/2π.

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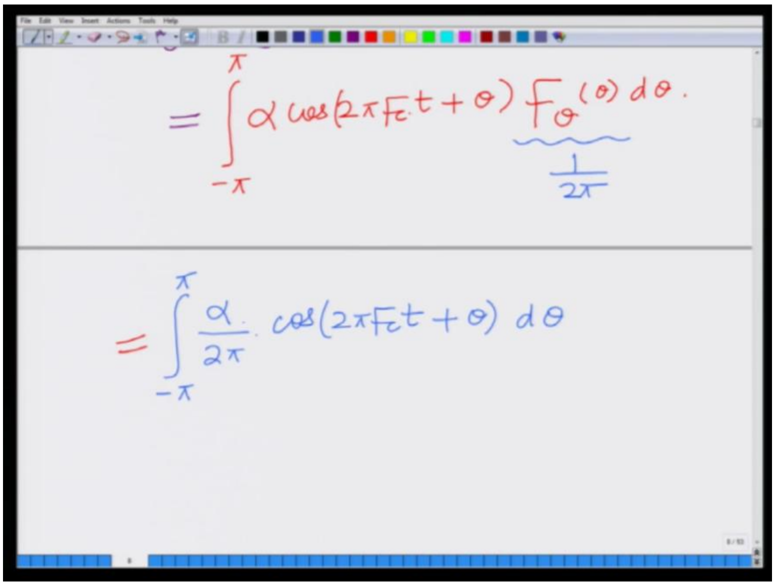

Therefore this mean is basically equal to –

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \alpha \cos(\, 2\pi fct \;+\; \theta\,) d\theta$$

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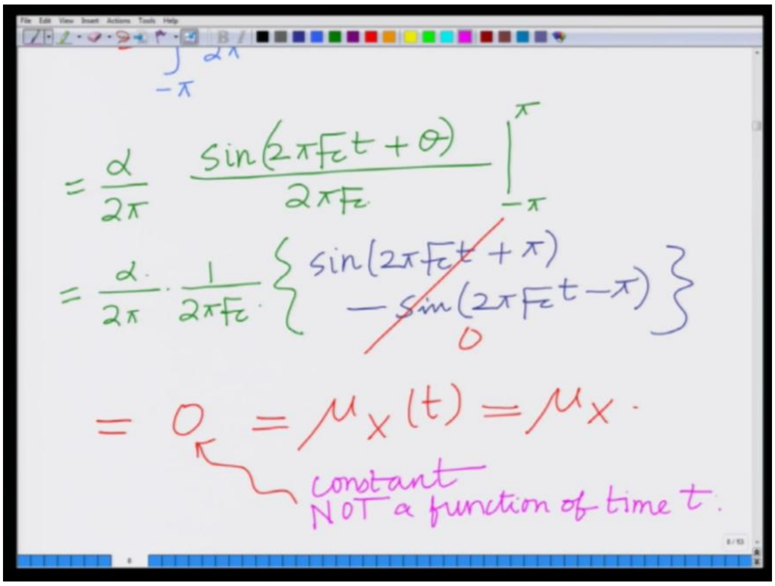

Now, if we look at this, we have –

$$= \frac{\alpha}{2\pi} \{ \frac{\sin(\, 2\pi fct + \theta\,)}{2\pi fc} |_{-\pi}^{\pi} \}$$

$$= \frac{\alpha}{2\pi} . \frac{1}{2\pi fc} \{\; \sin(\, 2\pi f_c t + \pi) - \sin(\, 2\pi f_c t \text{ - } \pi\,)\}$$

$$= 0 = \mu_X(t) = \mu_X$$

And this is not a function of the time t. So, what did we do, we looked at the average of this random process which is expected value of its $X(t)$, we performed, we have computed the mean of this random process and we have showed that $E\{X(t)\} = 0$ and therefore that since expected value of $X(t)$ is 0, it is a constant which does not depend on the time t, therefore this random process is in fact stationary in the mean.

It is not yet wide sense stationary random process, remember because for that we have to also examine the autocorrelation but 1st we have established that the mean μ_X is not a function of time, therefore this is stationary in the mean. So, this signal $X(t)$, the mean is not a function of time, therefore this is stationary.

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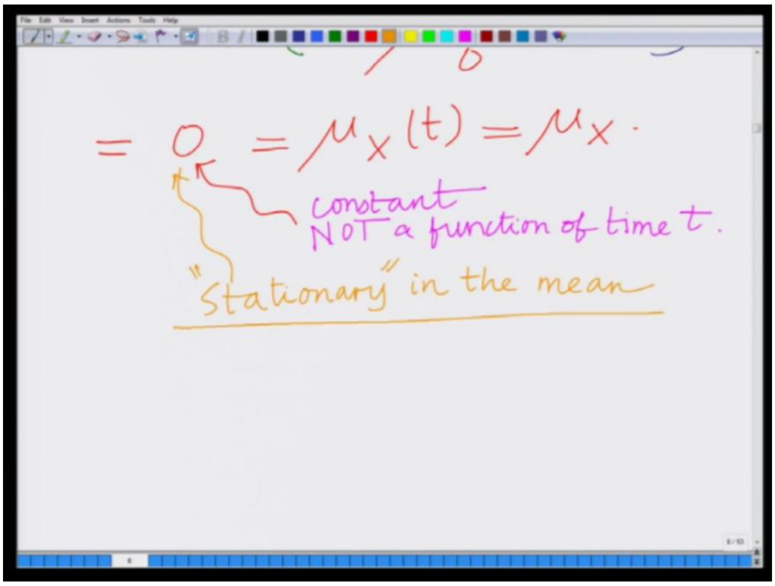

The mean is not changing as a function of time, therefore this is stationary in the mean. Let us now look at the autocorrelation of this random process.

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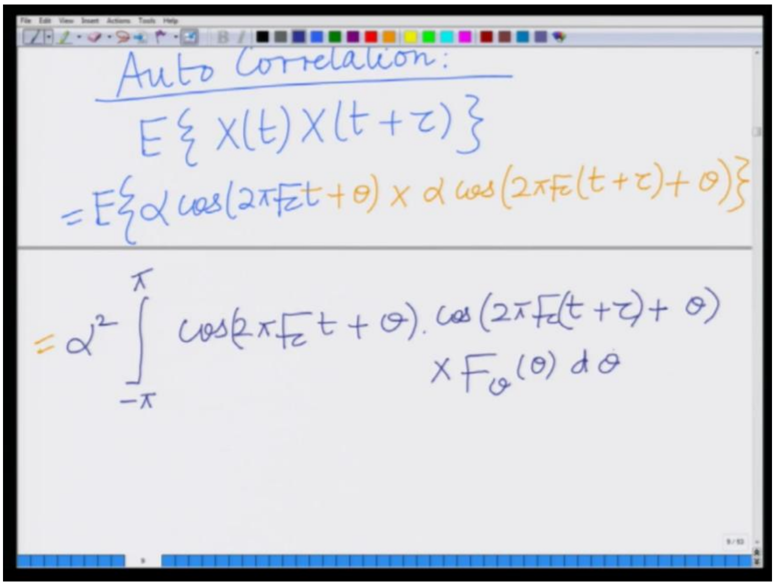

Let us now look at the autocorrelation, remember, the autocorrelation at 2 time instants $X(t)$ and $X(t + \tau)$ is –

$$E\{\, X(t)\, X(t+\tau)\} = E\{\, \alpha \cos(\, 2\pi fct \,+\, \theta\,) x\, \alpha \cos(\, 2\pi fc(t+\tau) +\, \theta\,) F_\theta(\theta)\}$$

$$= \alpha^2 \int_{-\pi}^{\pi} \cos(\, 2\pi fct \,+\, \theta\,)\; x\; \cos(\, 2\pi fc(t+\tau) +\, \theta\,) F_\theta(\theta) d\theta$$

$$= \frac{\alpha^2}{2} \int_{-\pi}^{\pi} \{\, \cos(\, 2\pi fc\tau) +\, \cos(\, 4\pi fct +\, 2\pi fc\tau +\, \theta\,)\} \frac{1}{2\pi} d\theta$$

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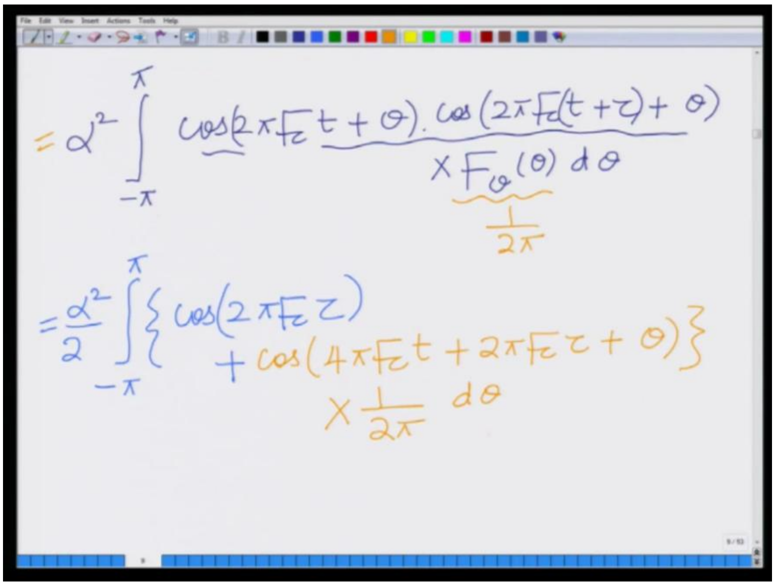

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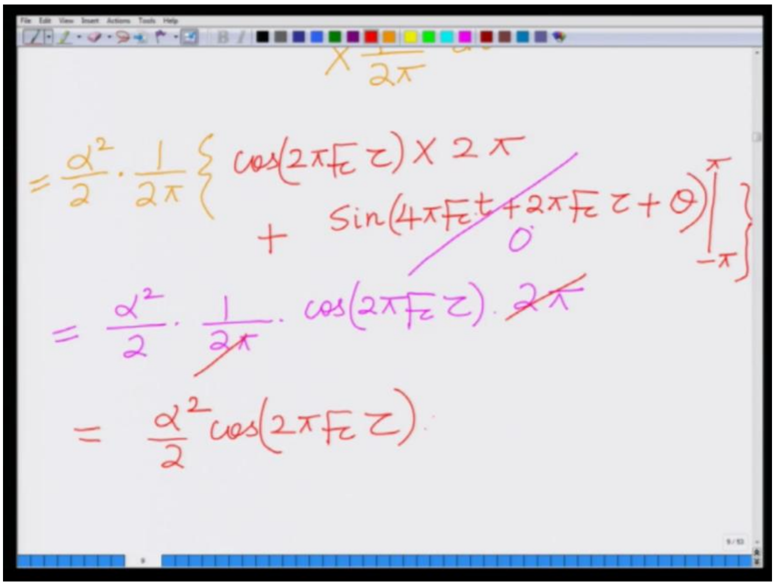

And this now if you look at this, this is –

$$= \frac{\alpha^2}{2} . \frac{1}{2\pi} \{\cos(2\pi f c \tau) x\, 2\pi \; + \; \sin(4\pi f c t + \; 2\pi f c \tau + \; \theta)|_{-\pi}^{\pi}\}$$

Again we have the difference between 2 sine at the sine at 2 phases which are separated by 2π, so therefore this is equal to 0. And what we are left with is basically,

$$= \frac{\alpha^2}{2} . \frac{1}{2\pi} \{\cos(2\pi f c \tau) x\, 2\pi$$

$$= \frac{\alpha^2}{2}.\cos(2\pi f c \tau)$$

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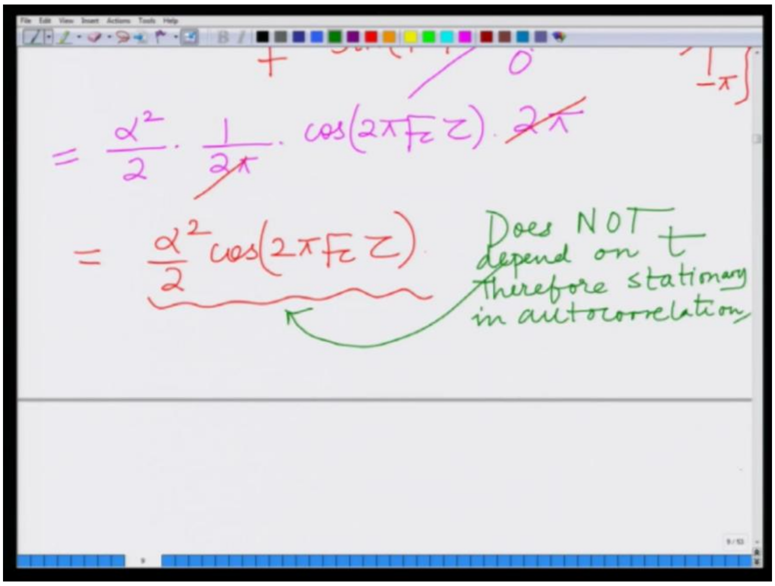

and therefore now you can see the autocorrelation at 2 points t and $t + \tau$ does not depend on this time difference τ and therefore this is stationary in the autocorrelation and stationary since it is stationary in the mean as well as stationary in the autocorrelation.

Therefore now you can see this does not depend on t, therefore it is stationary in the autocorrelation and stationary in the mean,

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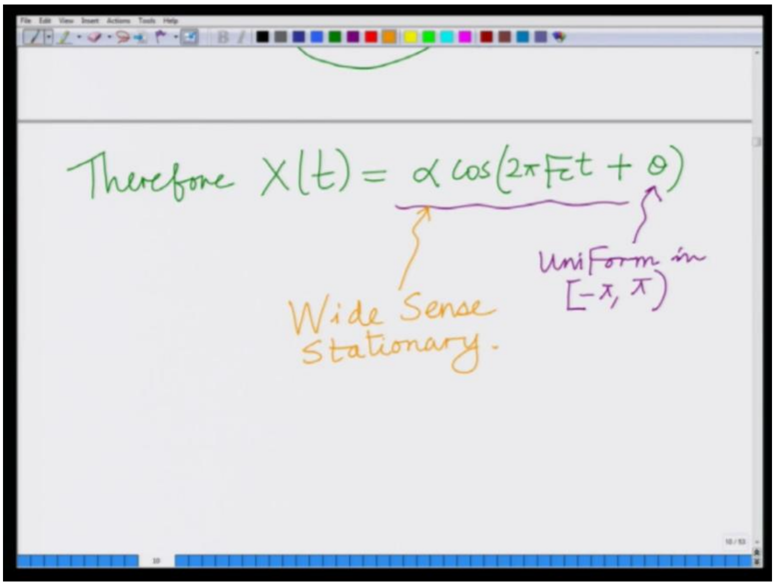

Therefore,

$$X(t) = \alpha \cos(2\pi fct + \theta)$$

where θ is uniformly distributed, uniform in the interval $[-\pi, \pi]$, this we are saying this signal or this is a random process which is wide this is a wide sense stationary random process.

So, we're saying $X(t)$ is stationary random process because the mean μ_X is constant which is 0, does not depend on time t and further the autocorrelation corresponding to 2 time instants t and $t + \tau$ depends only on the timeshift tau and that is equal to $\frac{\alpha^2}{2}.\cos(2\pi fc\tau)$.

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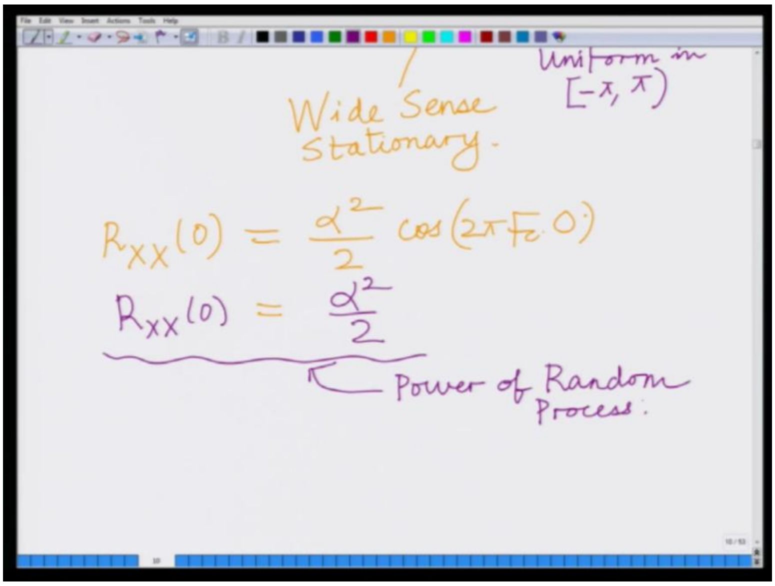

Okay, so this is a wide sense stationary random process and further now we can also see that,

$$R_{XX}(0) = \frac{\alpha^2}{2}.\cos(\ 2\pi fc0)$$

$$R_{XX}(0) = \frac{\alpha^2}{2}$$

Therefore the power equals $\frac{\alpha^2}{2}$. Remember when we have a sinusoid of amplitude α, the power is $\frac{\alpha^2}{2}$, however, we cannot directly compute the power that way because this is a random signal, this is a random process and therefore we had to define, we proved that this is a wide sense stationary random process, we computed the autocorrelation of this and then we derived the power of this random process as $R_{XX}(0)$. Where $R_{XX}(0)$ at $\tau = 0$, is the power of this random process $X(t)$.

So, in this module we have looked at an interesting application of this interesting example of wide sense stationarity, we have seen a wireless signal which is given at $\alpha \cos(\ 2\pi fct + \theta\ )$, where α, the amplitude is constant, fc is the carrier frequency of the signal and θ is a random phase which is uniformly distributed between $[\ -\pi, \pi\ ]$ and we said this signal or this random process is actually a wide sense stationary random process and we have say demonstrated by showing that this is basically stationary in the mean and the autocorrelation. And we also derived power of this random process as $R_{XX}(0)$ which is $\frac{\alpha^2}{2}$,.

So, let us end this module here, we will look at other aspects of random process in the subsequent modules. Thank you very much.