Probability and Random Variables/Processes for Wireless Communications.
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Lecture -16.
Application: array processing and Array Gain with uniform linear.

Hello, welcome to another module in this massive open online course on probability and random variables for wireless communication. In the previous modules, we have looked at the Gaussian random variables and properties of Gaussian random variables. Specially the linear combination of Gaussian random variables which results in another Gaussian random variable. Let us look today at an application, at a novel application of this property of Gaussian random variables from wireless communication in the context of antenna arrays or array processing or beamforming.

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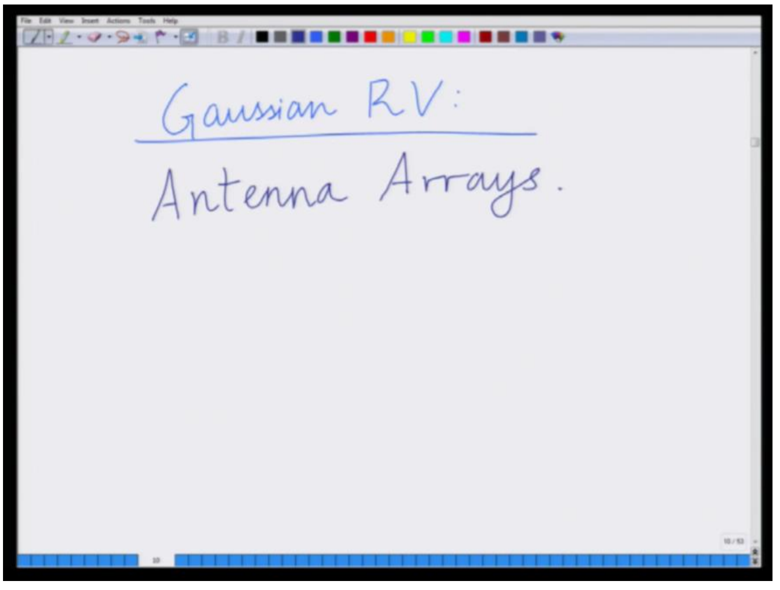

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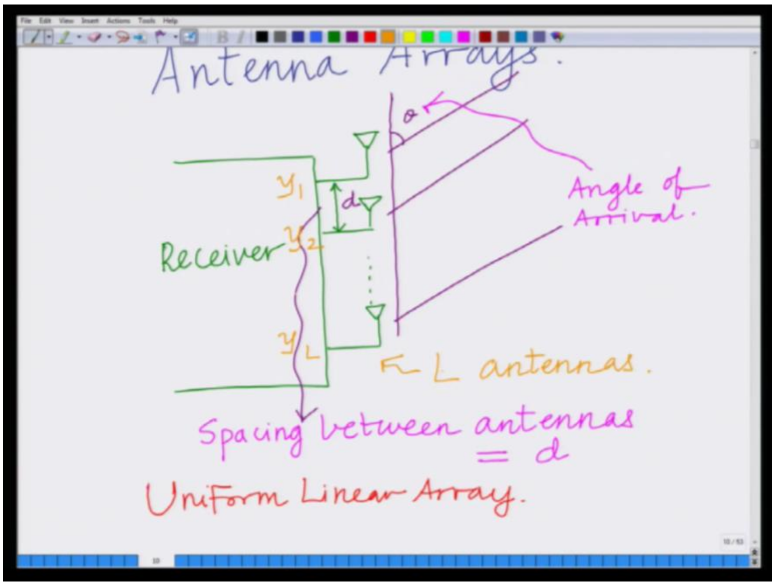

So, let me consider a receiver with multiple antennas, so what I am drawing here is a wireless receiver with multiple antennas, such that the antennas are arranged in a line or these antennas are arranged linearly i.e. the spacing between antenna is constant.

$$\textit{Spacing between antennas} = d.$$

So, this is the linear array of antenna.

Now that each antenna is spaced at a distance d from the next antenna, so the spacing between these antennas is uniform and this array of antennas, is known as uniform linear array. Such a spacing of antenna, i.e. antenna configuration, is known as a uniform linear array and in this new uniform linear array, let us consider a signal which is arriving at an angle θ.

This angle θ, this is known as the angle of arrival. So, I am considering a uniform linear antenna array with antenna spacing d and the signal which is arriving at an angle of θ with the vertical, this is known as the angle of arrival of the signal. Okay. And let Y_i denote the signal at the i^{th} antenna, so which means Y_1 is the signal at the 1^{st} antenna, Y_2 is the signal at the 2^{nd} antenna and so on if we have L antenna, so we are considering a scenario with L antennas which implies that Y_L is the signal at the L^{th} antenna. So what I am saying is that Y_i is the signal received at the i^{th} antenna and the spacing between the antennas is d.

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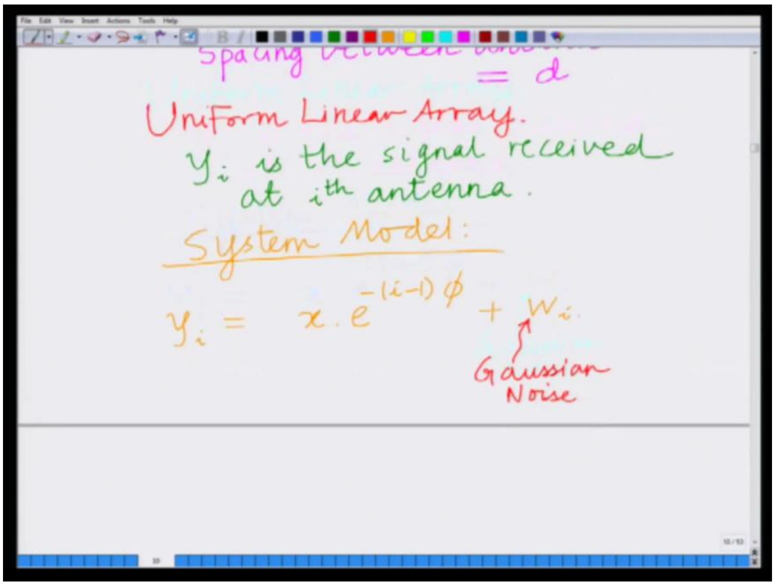

θ is the angle of arrival.

Now let us write the system model for this uniform linear array. The system model for this uniform linear array is given as follows.

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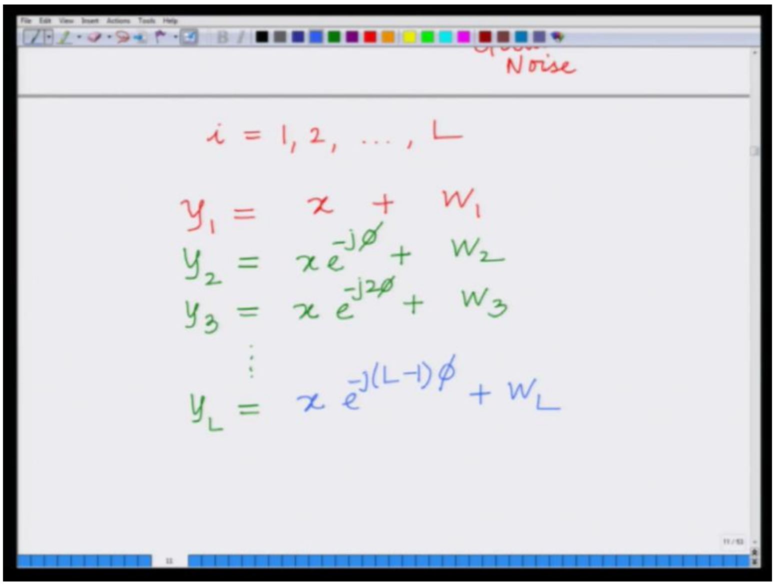

This system follows the model for uniform linear array given as the signal received at the i[th] antenna is given as

$$y_i = xe^{-j(n-1)\emptyset} + w_i$$

where W_n is the Gaussian noise.

Now let us substitute the different values of i

$$i = 1,2, \dots , L$$

$$y_1 = x + w_1$$

$$y_2 = xe^{-j\emptyset} + w_2$$

$$y_3 = xe^{-j2\emptyset} + w_3$$

.

.

.

$$y_L = xe^{-j(L-1)\emptyset} + w_L$$

Okay, so this is the system model.

We are saying the signal Y_i received that i^{th} antenna is

$$y_i = xe^{-j(i-1)\emptyset} + w_i$$

therefore now you can see therefore for each of the received signal, if you remove the noise, you see that the received signal at the 1st antenna is x, 2nd antenna, it is $xe^{-j\emptyset}$ at the 3rd antenna, it is $xe^{-j2\emptyset}$, so the signal at each successive receive antenna is delayed by a phase factor $e^{-j\emptyset}$ corresponding to the signal at the previous receive antenna.

So this signal at each successive receive antenna is delayed by, or has an additional phase factor of $e^{-j\emptyset}$.

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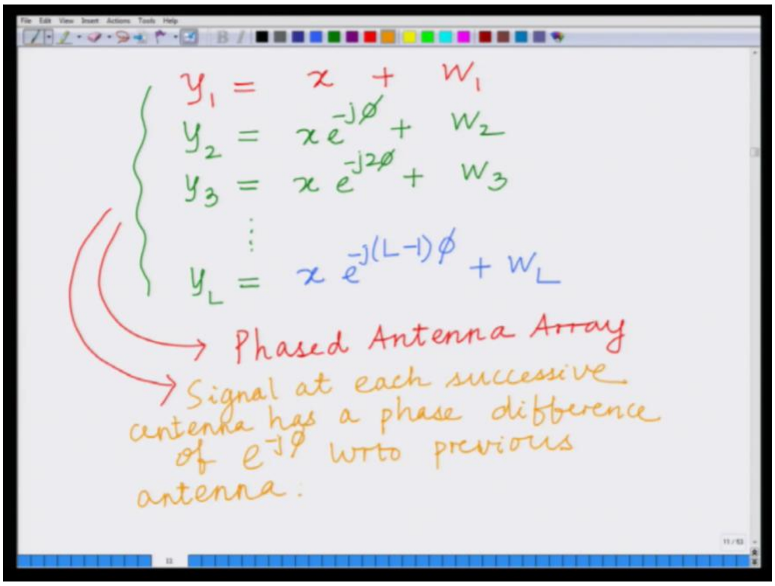

Therefore, this is also known as a phased antenna arrays. This is a phased array. So, as in this uniform linear array, the signal at each successive antenna has a phase with respect to the signal at the previous antenna, is known as the phased antenna array or simply known as phased array.

So, in this phased array, signal at each successive antenna has a phase difference of $e^{-j\phi}$ with respect to the previous antenna.

Now, let us look at the signal processing for this phased antenna array. How do we process the received signal $[Y_1, Y_2 \ldots Y_L]$ in this phase array or in this uniform linear array?

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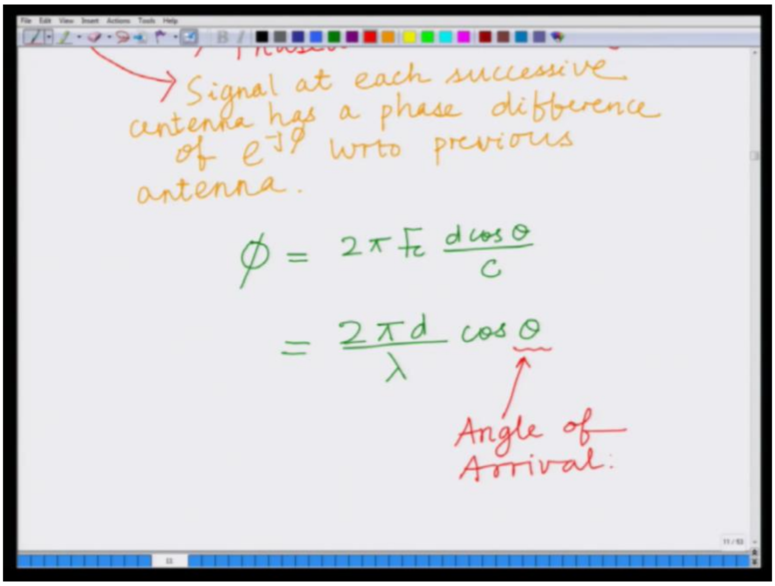

Now, before I write the system model, let me describe to you what this angle ∅ is? This angle ∅ is related to the angle of arrival θ as

$$\emptyset = 2\pi F_c \frac{d\ cos\theta}{C}$$

$$= \frac{2\pi d}{\lambda}\ cos\theta$$

where, $\theta = Angle\ of\ arrival\ for\ this\ uniform\ linear\ array\ or\ this\ phased\ array.$

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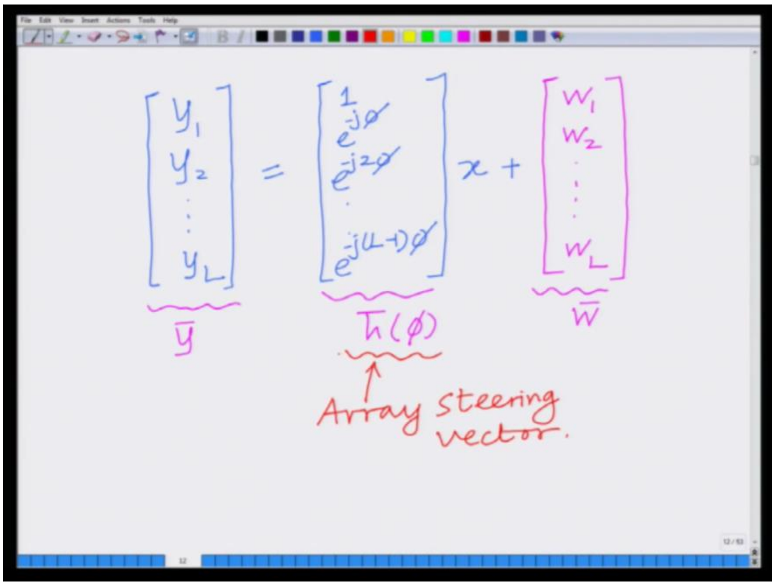

Okay, now let us look at signal processing for this phased array. The system model, the vector system model for this phased array, is given as

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_L \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-j\phi} \\ \cdot \\ \cdot \\ \cdot \\ e^{-j(L-1)\phi} \end{bmatrix} x + \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_L \end{bmatrix}$$

$$\boxed{\Longrightarrow \ \bar{y} = \bar{h}(\phi)x + \bar{w}}$$

Here, $\bar{h}(\phi)$ *is called the array steering vector*.

It is as if you are steering the array in the direction of θ. So, I write this system model, i.e. received signal vector,

$$\bar{y} = \bar{h}(\phi)x + \bar{w}$$

Where,

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_L \end{bmatrix}, y_i = signal\ received\ at\ the\ i^{th}\ antenna.$$

$$\bar{w} = noise\ vector,$$

$$\bar{h}(\phi) = array\ steering\ vector,$$

$$where\ \phi\ is\ function\ of\ the\ angle\ of\ arrival\ \theta.$$

Thus, I have expressed this system model in terms of the array steering vector.

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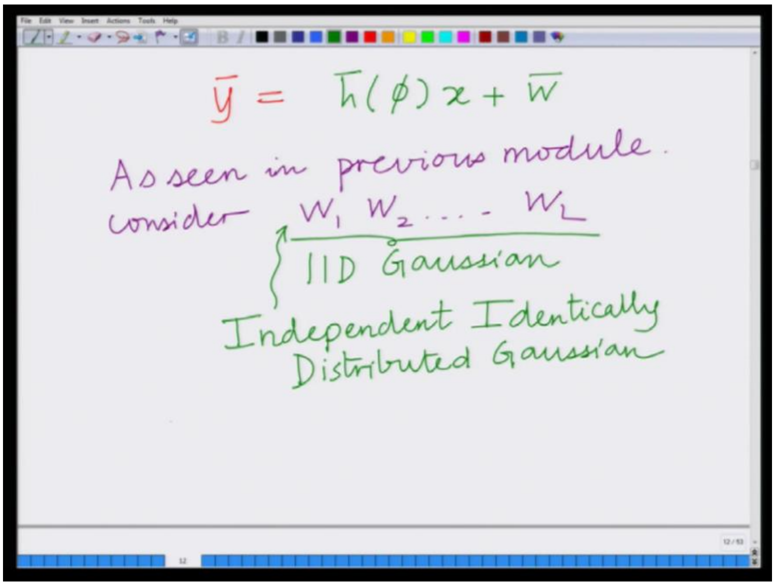

Now let us look at the signal processing, so as we have model as-

$$\bar{y} = \bar{h}(\phi)x + \bar{w}$$

Further, now let us look at the properties of noise as seen in the special case in previous module. Consider noise samples W_1, W_2, $\ldots W_L$ to be IID Gaussian, i.e. independent identically distributed Gaussian. Remember IID means, independent identically distributed Gaussian with 0 mean. That is each Gaussian random variable, each noise sample W_i has mean equal to 0.

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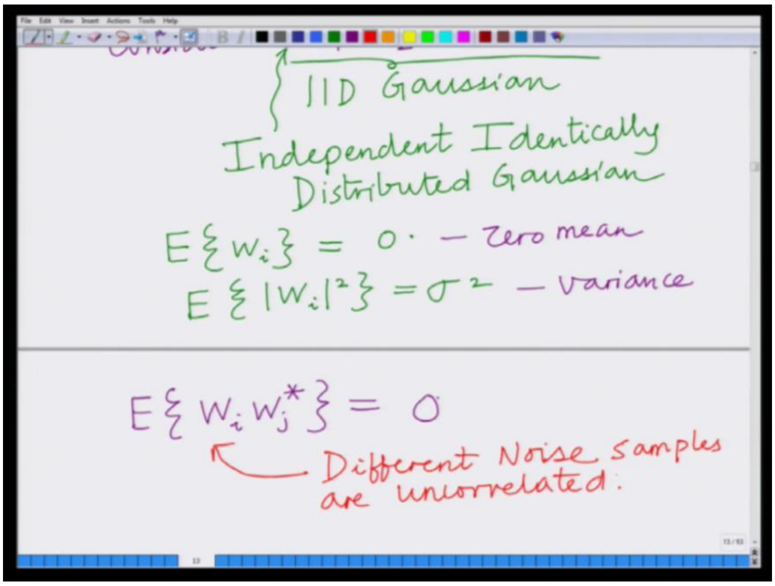

Further, let us assume that each noise sample has power or variance equal to

$$E\{|w_i|^2\} = \sigma^2$$

So this is basically your variance or noise power.

Further we also assume that the noise samples are not independent, i.e. they are uncorrelated, so

$$E\{w_i w_j^*\} = 0$$

However remember, only in the case of Gaussian, uncorrelated also means that they are independent.

So, we are considering independent identically distributed Gaussian noise samples with

$$mean = 0,$$

$$variance = \sigma^2,$$

$$and, covariance\ between\ the\ noise\ samples\ with\ two\ different\ antennas,$$

$$i.e.\quad E\{w_i w_j^*\} = 0$$

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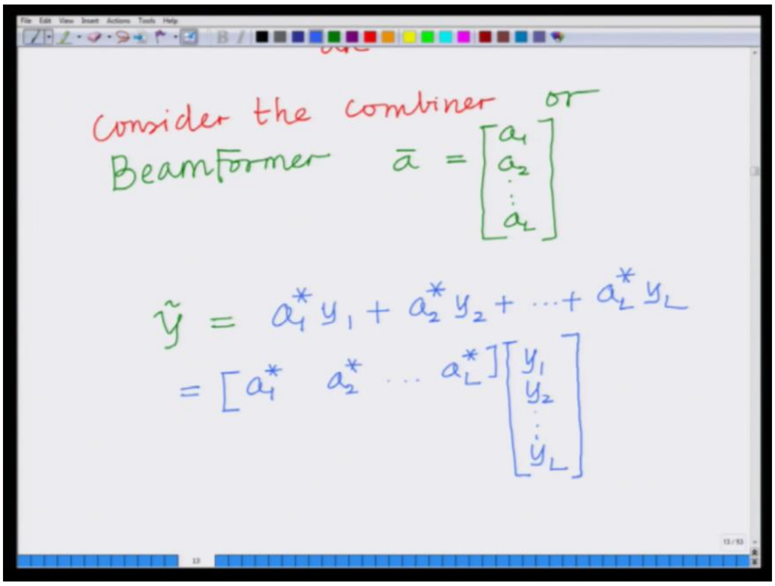

Now let us consider the combining of the received samples Y_1, Y_2, up to Y_L. So what I am going to do is, consider the combiner vector $\bar{a}$,

$$combiner\ vector\ \bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_L \end{bmatrix}$$

This is a vector of size L, and now I am going to form $\tilde{y}$ equals,

$$\tilde{y} = a_1^* y_1 + a_2^* y_2 + \cdots + a_L^* y_L$$

Where, $a_i^* = a_i\ conjugate$

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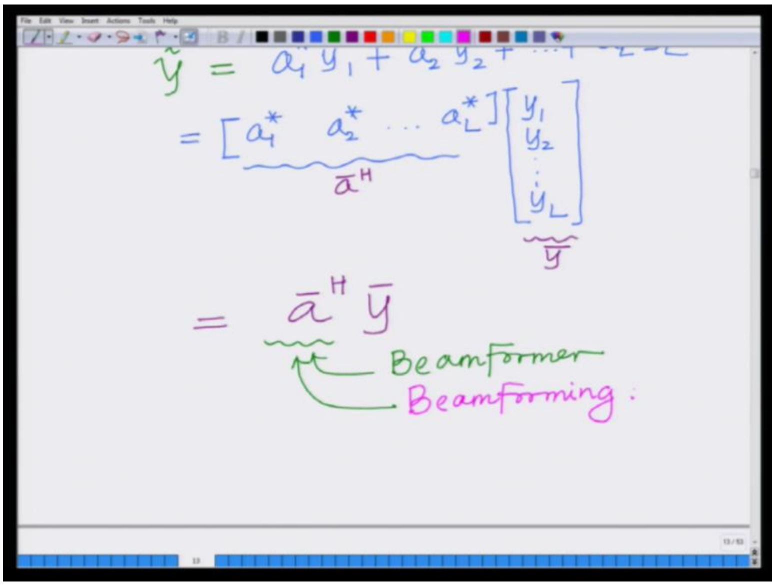

Now you can see this is basically nothing but,

$$\tilde{y} = a_1^* y_1 + a_2^* y_2 + \cdots + a_L^* y_L$$

$$\tilde{y} = [a_1^* \quad a_2^* \quad \ldots \quad a_L^*] \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_L \end{bmatrix}$$

$$here, [a_1^* \quad a_2^* \quad \ldots \quad a_L^*] = \bar{a}^H, i.e. \bar{a}\ Hermitian$$

$$Therefore,$$

$$\boxed{\tilde{y} = \bar{a}^H \bar{y}}$$

$$Here, \bar{a}^H \text{ is called a beamformer vector, and the process is called as beamforming.}$$

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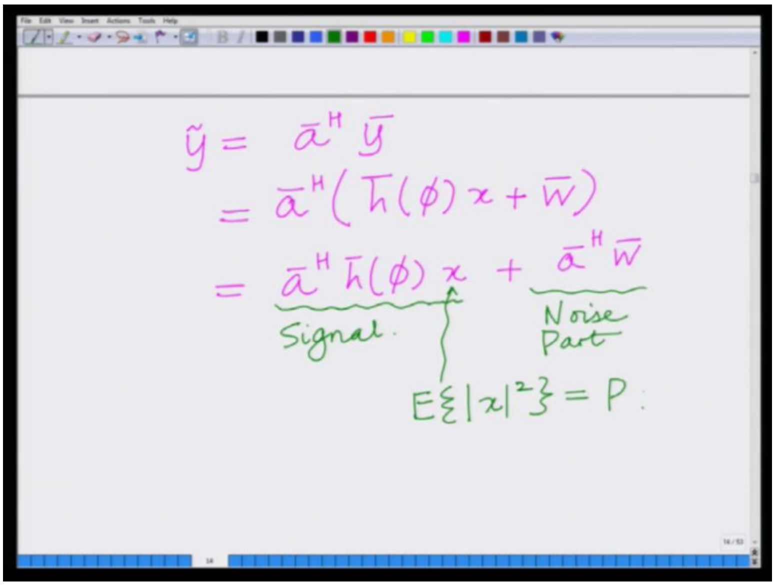

So now, I have

$$\tilde{y} = \bar{a}^H \, \bar{y}$$

As we remember, $\bar{y}$ is the array steering vector given as,

$$\bar{y} = \bar{h}(\phi)x + \bar{w}$$

$$Thus, \qquad \tilde{y} = \bar{a}^H \left(\bar{h}(\phi)x + \bar{w}\right)$$

$$= \bar{a}^H \, \bar{h}(\phi)x + \bar{a}^H \, \bar{w} \qquad (1)$$

$$we\ can\ separate\ the\ equation\ (1)\ into\ two\ parts, given\ as -$$

$$signal = \bar{a}^H \, \bar{h}(\phi)x$$

$$noise\ part = \bar{a}^H \, \bar{w}$$

In addition let us say that the transparent signal power is P, that is

$$E\{|x|^2\} = P$$

Also, remember the power in the transmitted signal is nothing but the variance of the signal, expected value of magnitude X square equals P, let us assume that the transmitted signal power is P.

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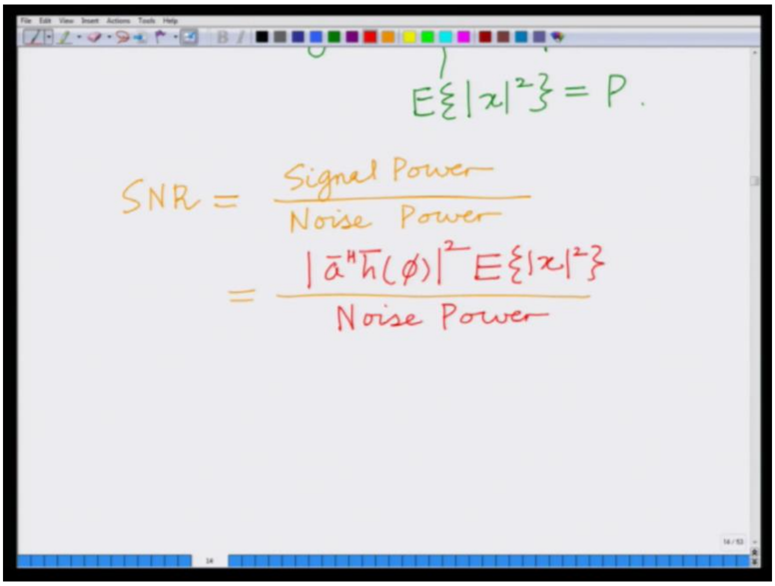

Let us now look at what is the signal to noise power ratio of this system. So, we are looking at a beam former, we are looking at performing $\bar{a}^H \bar{y}$. From this we have isolated the signal part and the noise part. Let us now look at the signal-to-noise power ratio, which is

$$SNR = \frac{Signal\ Power}{Noise\ Power}$$

i.e. the ratio of power in the signal to the power in the noise.

Here,

$$Signal\ Power = \left|\bar{a}^H\,\bar{h}(\phi)\right|^2 E\{|x|^2\}$$

So,

$$SNR = \frac{\left|\bar{a}^H\,\bar{h}(\phi)\right|^2 E\{|x|^2\}}{Noise\ Power}$$

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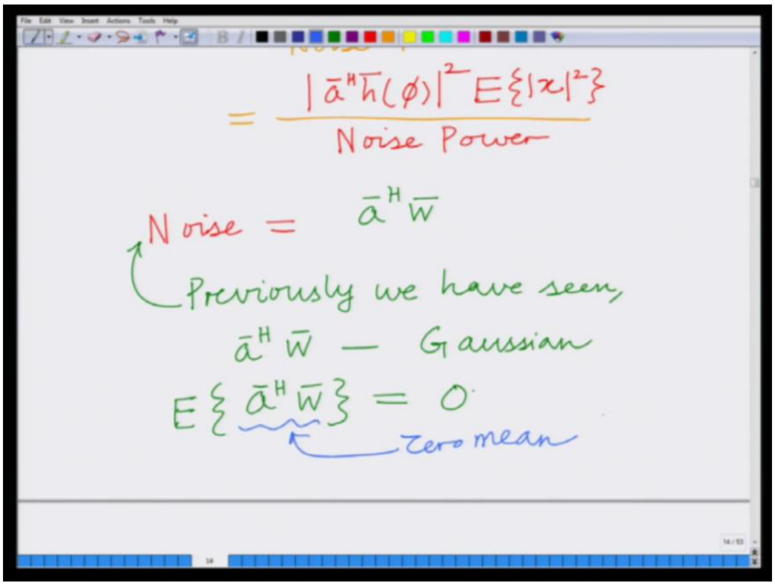

Now let us look at what is the noise power of this system. Now previously we have seen and this is when our properties of Gaussian will come handy, remember the noise is given as

$$Noise = \bar{a}^H \, \bar{w}$$

which is a linear combination of Gaussian random variable. Previously we have seen that if $W_1, W_2, \ldots, W_L$ are Gaussian and if they are IID Gaussian with 0 mean, we have

$$E\{\bar{a}^H \, \bar{w}\} = 0$$

i.e. this basically is 0 mean noise.

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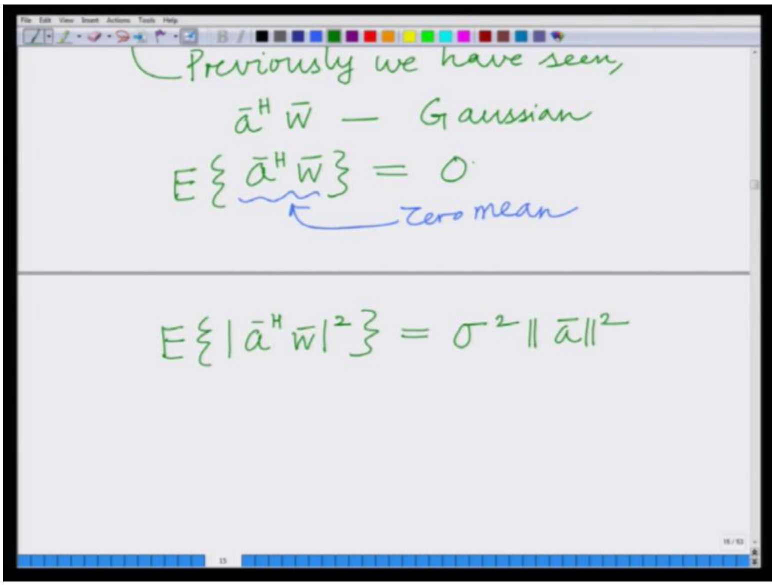

Further the noise variance of the resulting noise power, i.e.

$$E\{|\bar{a}^H \bar{w}|^2\} = \sigma^2 \|\bar{a}\|^2$$

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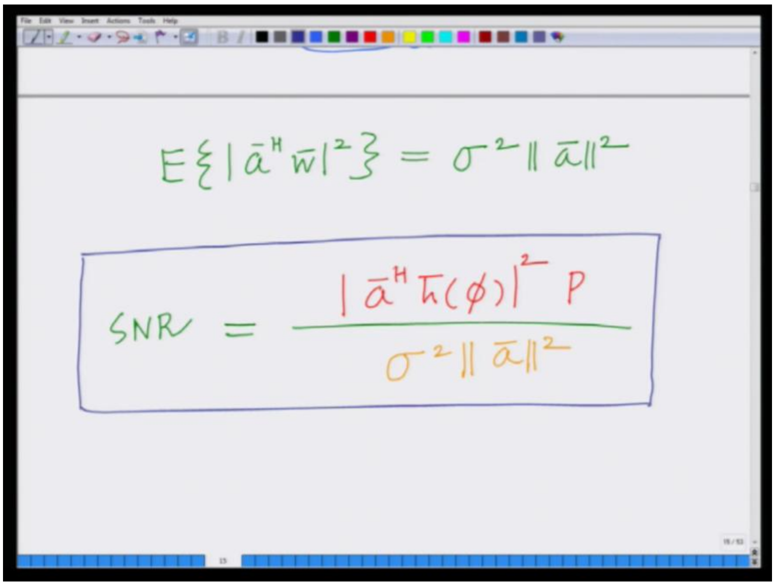

Therefore SNR becomes,

$$SNR = \frac{\left|\bar{a}^H \bar{h}(\phi)\right|^2 E\{|x|^2\}}{\sigma^2 \|\bar{a}\|^2}$$

Therefore this is the expression for general beam former A and this is the expression we derived for the SNR at the output.

Now we want to find the beamforming vector $\bar{a}$ or the beamforming weights $A_1, A_2, \ldots, A_L$ which will maximise the SNR at the output of this uniform linear array or this phased array i.e. we want to find the beam former A bar which will yield the maximum SNR so that we enhance the signal to noise power ratio at the output.

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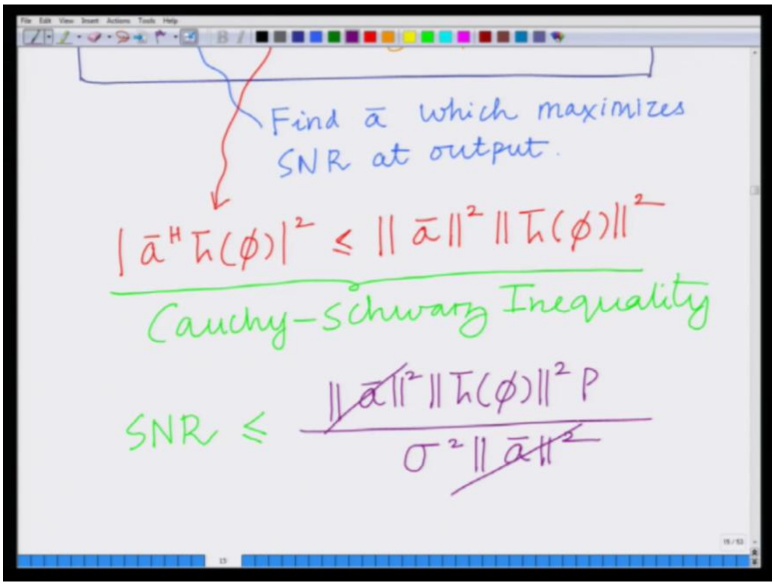

So, we want to find we want to find $\bar{a}$ which maximizes SNR at the output. Now you can see from this expression from the Cauchy Schwarz inequality,

$$\left|\bar{a}^{H}\,\bar{h}(\phi)\right|^{2} \leq \|\bar{a}\|^{2}\left\|\bar{h}(\emptyset)\right\|^{2}$$

$$SNR \leq \frac{\|\bar{a}\|^{2}\left\|\bar{h}(\emptyset)\right\|^{2}P}{\sigma^{2}\|\bar{a}\|^{2}} = \frac{\left\|\bar{h}(\emptyset)\right\|^{2}P}{\sigma^{2}}$$

$$thus, \qquad SNR \leq \frac{\left\|\bar{h}(\emptyset)\right\|^{2}P}{\sigma^{2}}$$

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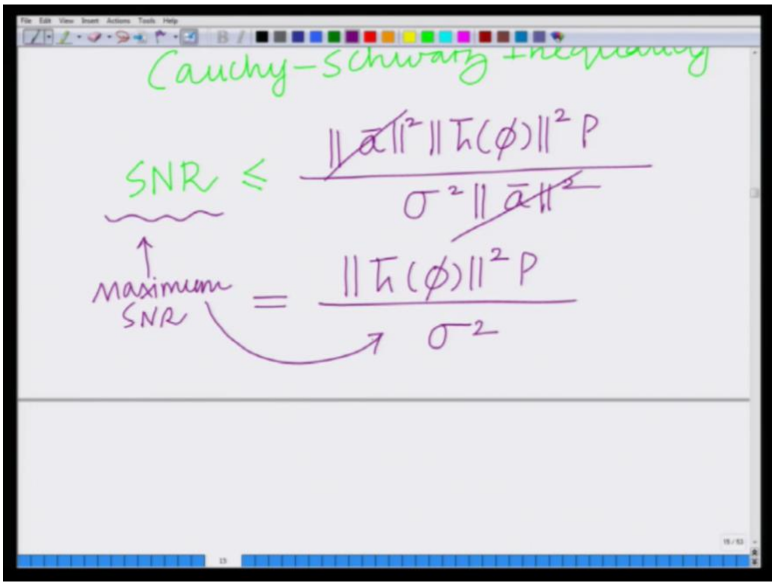

Therefore the maximum SNR is given as-

$$SNR_{max} = \frac{\left\|\bar{h}(\emptyset)\right\|^{2}P}{\sigma^{2}}$$

This is what we derived from the Cauchy Schwarz inequality.

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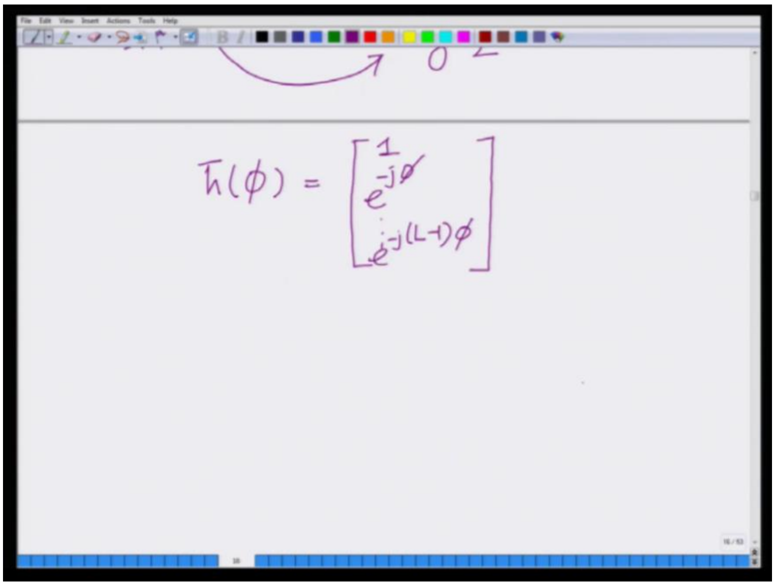

Now let us see what this maximum SNR is. As we showed, $\bar{h}(\emptyset)$ is

$$\bar{h}(\emptyset) = \begin{bmatrix} 1 \\ e^{-j\emptyset} \\ . \\ . \\ . \\ e^{-j(L-1)\emptyset} \end{bmatrix}$$

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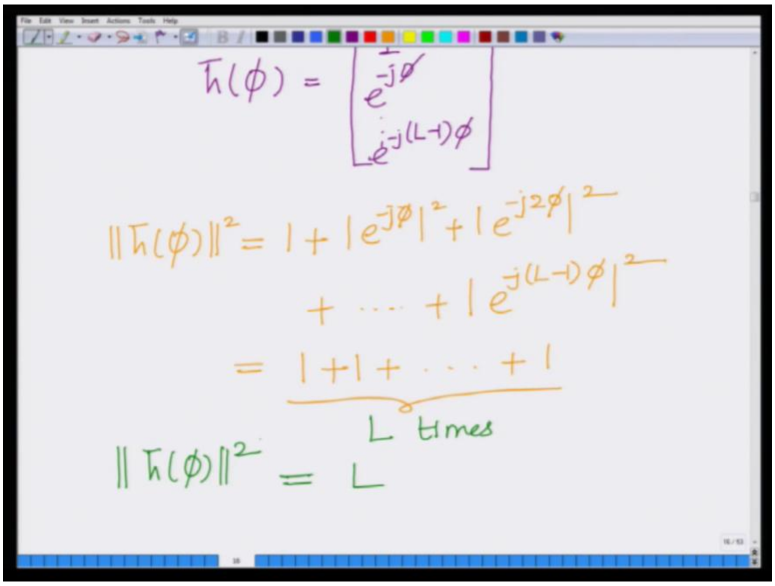

Therefore the maximum SNR,

$$\left\|\bar{h}(\emptyset)\right\|^2 = 1 + \left|e^{-j\emptyset}\right|^2 + \left|e^{-j2\emptyset}\right|^2 + \cdots + \left|e^{-j(L-1)\emptyset}\right|^2$$

$$= 1 + 1 + 1 + \cdots + 1$$

$$= L$$

Thus, maximum SNR is given as,

$$SNR_{max} = L.\frac{P}{\sigma^2}$$

here, L is the array Gain of the system.

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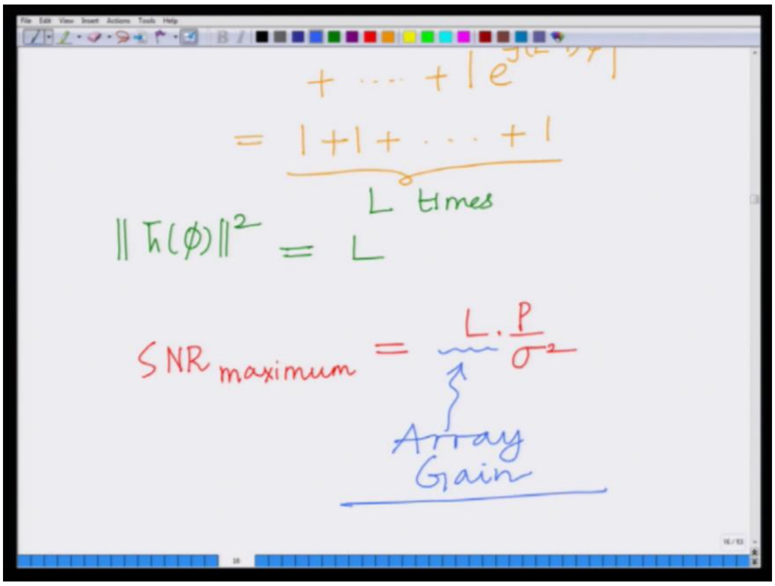

Remember, without the array, the initial signal to noise power ratio is simply

$$SNR_{without\ array} = \frac{P}{\sigma^2}$$

So using this phased array, i.e. uniform linear array, we are able to multiply this signal to noise power ratio by a factor of L. So, this factor of L, where L is the number of antennas is gain in the signal to noise power ratio at the output of this uniform linear array. Thus this is known as the array gain.

So, we are able to multiply the initial SNR by the factor of L, known as the array gain of the system and therefore the uniform linear array is a very important or is a very novel technology because it results in the gain of a factor of L in the signal to noise power ratio at the receiver.

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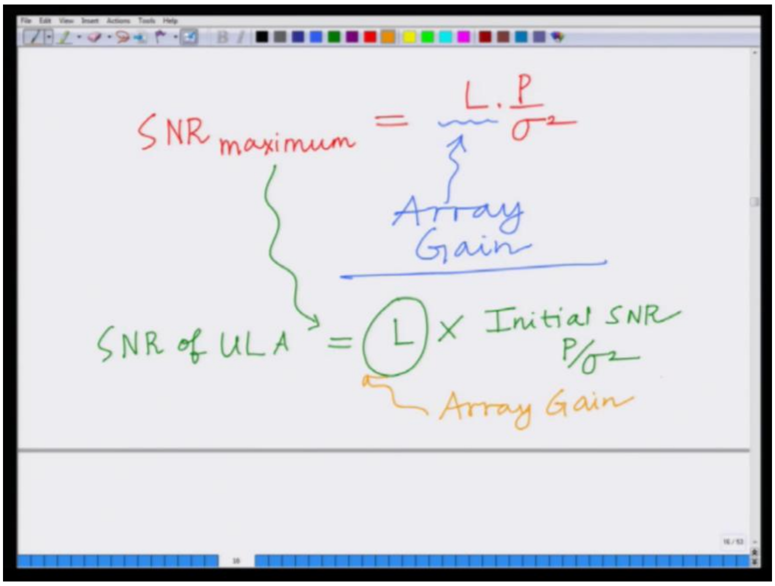

So, what do we have, we have the SNR of the phased array that is SNR of the uniform linear array equals L Times initial SNR. What do I mean by the initial SNR? That is initial SNR is

$$SNR_{initial} = SNR_{without\ array} = \frac{P}{\sigma^2}$$

Thus, i.e. without the phased array, therefore in this we have the gain of L and this is the array gain. This is the array gain of the system.

Now, how do we choose the weights A_1, A_2,.., A_L, remember from Cauchy Schwarz inequality…

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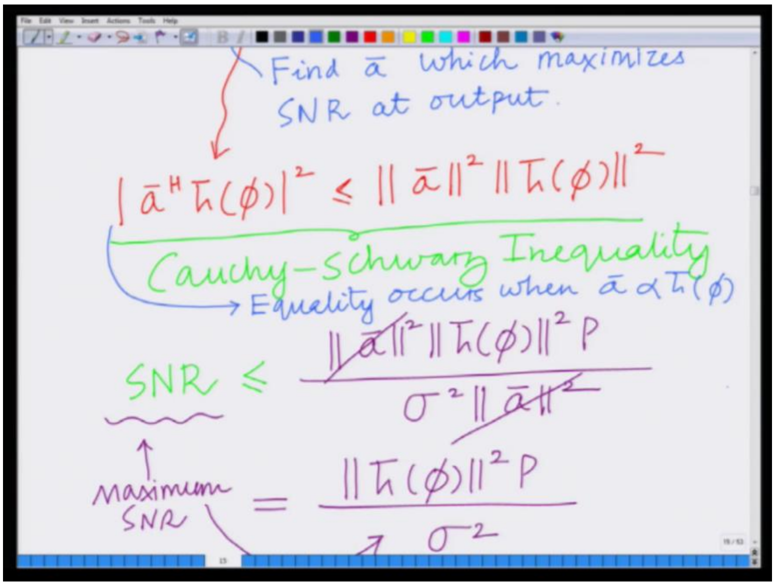

if we look at the Cauchy Schwarz inequality, we have

$$\left|\bar{a}^H\,\bar{h}(\phi)\right|^2 \leq \|\bar{a}\|^2\left\|\bar{h}(\emptyset)\right\|^2$$

Where, equality occurs only when $\bar{a}$, that is the combining vector is proportional to the vector $\bar{h}$.

The equality occurs when

$$\bar{a} \;\propto\; \bar{h}(\emptyset)$$

one way to achieve this Cauchy Schwarz inequality therefore is to set $\bar{a}$ simply equal to $\bar{h}(\emptyset)$.

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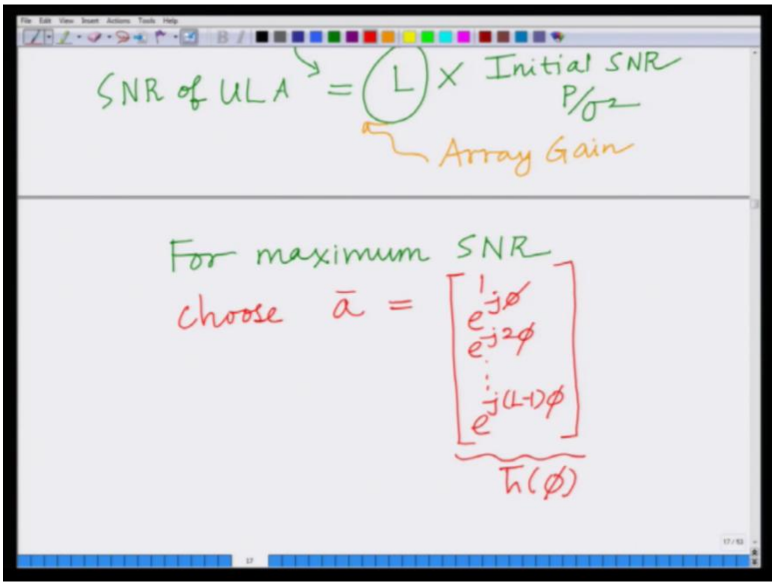

Therefore if we choose

$$\bar{a} = \bar{h}(\emptyset)$$

for maximum SNR, which is basically

$$\bar{a} = \begin{bmatrix} 1 \\ e^{-j\emptyset} \\ . \\ . \\ . \\ e^{-j(L-1)\emptyset} \end{bmatrix} = \bar{h}(\emptyset)$$

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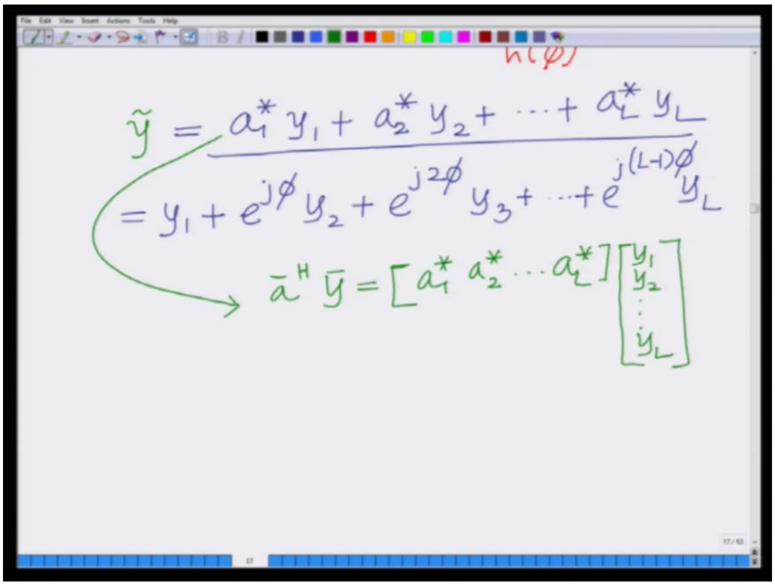

Therefore, now the optimal beam former will be given or optimal combining will be given as

$$\tilde{y} = a_1^* y_1 + a_2^* y_2 + \cdots + a_L^* y_L$$

$$= y_1 + e^{j\emptyset} y_2 + e^{j2\emptyset} y_3 + \cdots + e^{j(L-1)\emptyset} y_L$$

And therefore this is the optimal combiner, this basically, you remember is also equal to-

$$\tilde{y} = a_1^* y_1 + a_2^* y_2 + \cdots + a_L^* y_L$$

$$= \bar{a}^H \bar{y} = \begin{bmatrix} a_1^* & a_2^* & \ldots & a_L^* \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_L \end{bmatrix}$$

$$\boxed{\tilde{y} = y_1 + e^{j\emptyset} y_2 + e^{j2\emptyset} y_3 + \cdots + e^{j(L-1)\emptyset} y_L}$$

The last relation is also known as optimal combiner which results in the maximum SNR, remember what is the maximum SNR,

$$SNR_{max} = L.\frac{P}{\sigma^2}$$

Thus it is known as the maximal ratio combiner, for this array processing system or this is also the matched filter, since it maximizes the ratio of the signal power to the noise power at the receiver. Remember, since

$$\bar{a} = \bar{h}(\emptyset)$$

i.e. receiver filter is matched to the array steering vector, this is also known as the specially matched filter. This is also known as matched filter or as the maximal ratio combiner.

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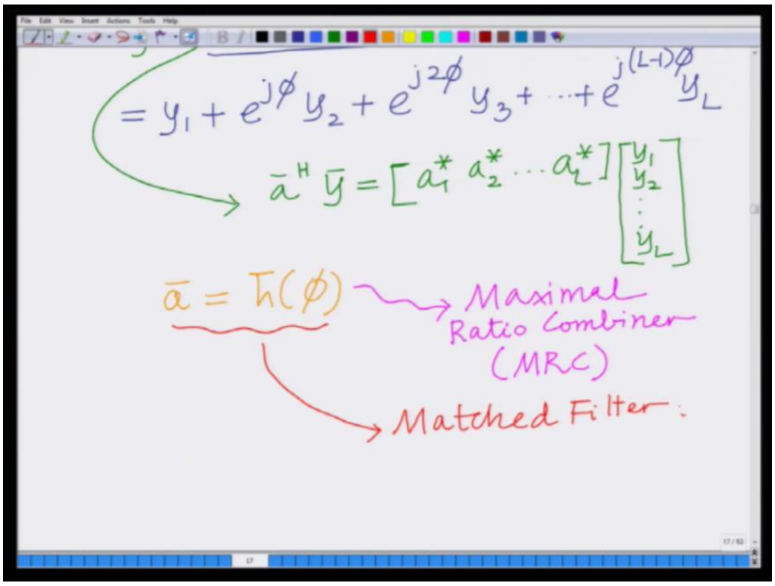

Alright, so this is a unique application of the properties of Gaussian in the context of signal processing, in the context of wireless communications. What we have said, we have considered a multiple receive antennas so, there are L receive antennas with equal antenna spacing, that is in turns of a phased array, we have looked at a signal which is arriving at an angle of Θ with respect to the vertical, that is angle of arrival Θ, we have considered independent identically distributed noise elements at the different receive antennas and we have demonstrated that using the maximal ratio combiner or using the optimal beam former or the matched filter, one can enhance the signal to noise power ratio in this uniform linear array or phased array system by a factor of L, where L is the number of antennas, therefore this L is also known as the array gain of the system.

And this is basically possible because of the coherent combining of the signals received across the various receive antennas. That is, the member were performing

$$y_1 + e^{j\emptyset} y_2 + e^{j2\emptyset} y_3 + \cdots + e^{j(L-1)\emptyset} y_L,$$

that is where inverting the phase of the received signal on each antennas combining them, that is the signal combines coherently, where is the noise which is random, the Gaussian noise combines incoherently, therefore we are able to enhance the signal to noise power ratio by a factor of L and which is the array gain and this is an important property, this demonstrates an important application of both signal processing in Wireless Systems and also the principal of combination of these Gaussian noise samples across the various receive antennas.

So, with this let us end this module here, we will look at other aspects in the subsequent lectures. Thank you very much.