

Probability and Random Variables/Processes for Wireless Communications.

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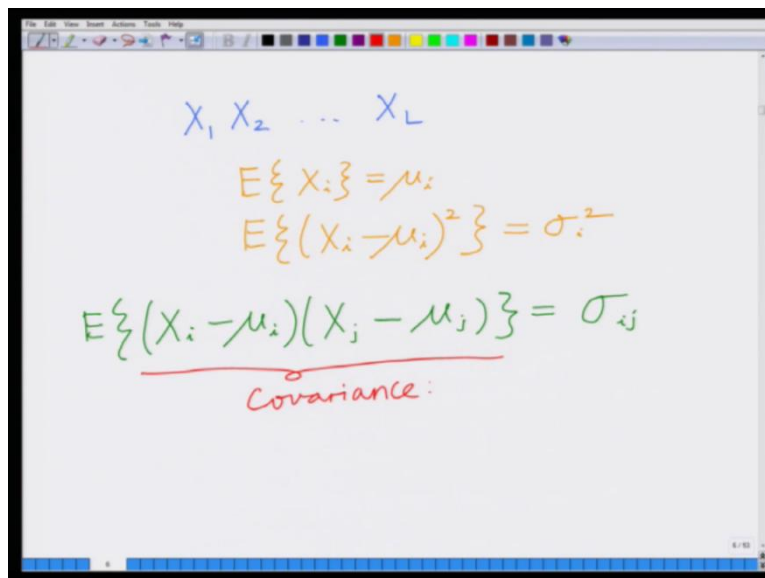
Indian Institute of Technology Kanpur.

Lecture -15.

Special Case: IID Gaussian Random Variables.

Hello, welcome to another module in this massive open online course on probability and random variables for wireless communication. So, in the previous module, we started looking at Gaussian random variables and the various key properties of Gaussian random variables. One of the important properties of the Gaussian random variable we said is the following thing that is,

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The image shows a whiteboard with handwritten mathematical formulas. At the top, it lists X_1, X_2, \dots, X_L . Below this, it states $E\{X_i\} = \mu_i$ and $E\{(X_i - \mu_i)^2\} = \sigma_i^2$. The bottom part of the whiteboard shows the covariance formula $E\{(X_i - \mu_i)(X_j - \mu_j)\} = \sigma_{ij}$, which is underlined and labeled "Covariance:" in red.

if I have L Gaussian random variables- $\{X_1, X_2, \dots, X_L\}$ are Gaussian random variables with

$$E(X_i) = \mu_i, \text{ and,}$$

$$\text{variance} = \sigma_{X_i}^2 = E\{(x_i - \mu_i)^2\}$$

Further if I look at the covariance,

$$\text{Covariance} = \sigma_{ij} = E\{(X_i - \mu_i)(X_j - \mu_j)\}$$

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Handwritten notes on a whiteboard:

$$E\{(X_i - \mu_i)^2\} = \sigma_i^2$$
$$E\{(X_i - \mu_i)(X_j - \mu_j)\} = \sigma_{ij}$$

Covariance.

$$X = a_1 X_1 + a_2 X_2 + \dots + a_L X_L$$

Linear combination of X_1, X_2, \dots, X_L

Now, we said if I generate a new random variable X which is generated as a linear combination of these Gaussian random variables as-

$$X = a_1 X_1 + a_2 X_2 + \dots + a_L X_L$$

then we said this X is a Gaussian random variable.

This Gaussian random variable which is generated as a linear combination of a group of random variables is in turn a gaussian random variable and we also calculated the mean and the variance of this new Gaussian random variable.

$$X = N(\mu_X, \sigma_X^2)$$

Where,

$$\mu = E\{X\} = \sum_{i=1}^L a_i \mu_i$$

And,

$$\sigma_X^2 = \sum_i a_i^2 E\{X_i - \mu_i\}^2 + \sum \sum a_i a_j \sigma_{ij}$$

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$$X = a_1 X_1 + a_2 X_2 + \dots + a_L X_L$$

Linear Combination of X_1, X_2, \dots, X_L

Gaussian

$$N\left(\underbrace{\sum_i a_i \mu_i}_{\mu}, \underbrace{\sum_i a_i^2 \sigma_i^2 + \sum_{i \neq j} a_i a_j \sigma_{ij}}_{\sigma^2}\right)$$

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Special Case:

$E\{X_i\} = \mu_i = 0$ — Zero mean

$E\{(X_i - \mu_i)^2\} = \sigma_i^2 = \sigma^2$ — Variance.

Let us now consider a special case of this linear combination, when

$$\text{Mean} = E(X_i) = \mu_i = 0 \quad \text{and,}$$

$$\text{variance} = \sigma_{X_i}^2 = E\{(x_i - \mu_i)^2\} = \sigma^2$$

Thus, all the Gaussian random variables have identical mean and in fact the mean is identically equal to 0. So, all the Gaussian random variables $\{X_1, X_2, \dots, X_L\}$ are Identical, i.e.

$$X_i = N(0, \sigma^2), \quad i = 1, 2, \dots, L$$

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Special Case:

$$E\{X_i\} = \mu_i = 0 \text{ — Zero mean}$$
$$E\{(X_i - \mu_i)^2\} = \sigma_i^2 = \sigma^2 \text{ — Variance.}$$

A pink arrow points from the first equation to the second.

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$E\{X_i\} = \mu_i = 0 \text{ — Zero mean}$

$$E\{(X_i - \mu_i)^2\} = \sigma_i^2 = \sigma^2 \text{ — Variance.}$$

A blue arrow points from the first equation to the second.

Identical Gaussian RVs.

So, these are bunch or a group of identical Gaussian random variables.

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Identical Gaussian
RVs.

$$E\{(X_i - \mu_i)(X_j - \mu_j)\}$$
$$= E\{X_i X_j\} = 0$$

Covariance = 0.

X_i, X_j are uncorrelated.

Further we are also going to assume that the covariance,

$$\text{Covariance} = \sigma_{ij} = E\{(X_i - \mu_i)(X_j - \mu_j)\} = 0$$

Such random variables are known as uncorrelated random variables, which means the covariance of 2 random variables is 0. So, we are assuming that all Gaussian random variables are identical and further, they are uncorrelated, and specifically for the case of Gaussian random variable, uncorrelated also implies independence.

This is not true for any random variable but specifically for the Gaussian random variable, the property of uncorrelated random variables imply that they are independent.

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The image shows a digital whiteboard with handwritten mathematical expressions. At the top, the covariance formula is written in green: $E\{(X_i - \mu_i)(X_j - \mu_j)\}$. Below it, this is equated to $E\{X_i X_j\} = 0$ in green. A red bracket connects the expression $E\{X_i X_j\}$ to the text "Covariance = 0." written in red. Below this, it is noted in purple that X_i, X_j are uncorrelated. Two downward arrows then point to the conclusion in purple: "For Gaussian Independence."

$$E\{(X_i - \mu_i)(X_j - \mu_j)\}$$
$$= E\{X_i X_j\} = 0$$

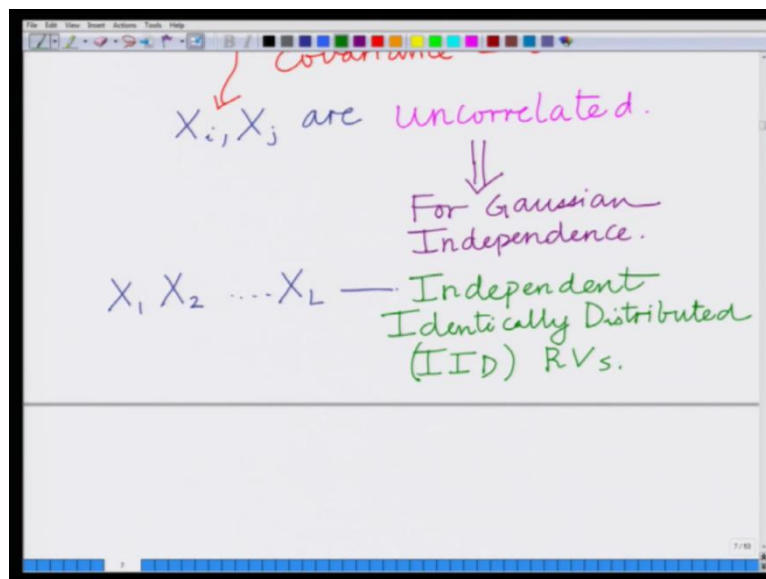
Covariance = 0.

X_i, X_j are uncorrelated.

↓
For Gaussian Independence.

So, therefore we are considering a group of L independent and identically distributed Gaussian random variables $\{X_1, X_2, \dots, X_L\}$.

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So, we are considering group of L random variables, i.e. $\{X_1, X_2, \dots, X_L\}$. In this context, these random variables are independent. Remember we have seen this nomenclature before, independent and identically distributed random variables.

So, we were considering L independent identically distributed Gaussian random variables. Let us now again consider X which is generated as a linear combination of these Gaussian random variables.

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$$\begin{aligned} X &= a_1 X_1 + a_2 X_2 + \dots + a_L X_L \\ &= \underbrace{[a_1 \ a_2 \ \dots \ a_L]}_{\bar{a}^T} \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_L \end{bmatrix}}_{\bar{X}} \\ &= \bar{a}^T \bar{X} \end{aligned}$$

where $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_L \end{bmatrix}$ and $\bar{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_L \end{bmatrix}$

$$X = a_1 X_1 + a_2 X_2 + \dots + a_L X_L$$

which we can now write using vector operations as

$$X = [a_1 \ a_2 \ \dots \ a_L] \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_L \end{bmatrix}$$

Denoting,

$$\bar{a}^T = [a_1 \ a_2 \ \dots \ a_L] \quad \text{and}$$

$$\bar{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_L \end{bmatrix}$$

Therefore,

$$X = \bar{a}^T \bar{X}$$

This is the new random variable X which we saw previously is Gaussian in nature.

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Handwritten derivation on a digital whiteboard:

$$\begin{aligned}
 E\{X\} &= \mu = \sum_i a_i \mu_i \\
 &= \sum_i a_i \times 0 = 0 \\
 \mu &= 0 \\
 E\{(X - \mu)^2\} &= E\{X^2\} \\
 &= \sum_i a_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} a_i a_j \sigma_{ij} \\
 &\quad \text{(Note: } \sigma_{ij} \text{ is crossed out and } 0 \text{ is written next to it)}
 \end{aligned}$$

Now what is the mean of this Gaussian random variable? The mean of this Gaussian random variable is

$$\mu = E\{X\} = \sum_{i=1}^L a_i \mu_i = 0,$$

$$\text{as, } \mu_i = 0$$

Similarly, we have

$$\begin{aligned} E\{(X - \mu)^2\} &= E\{(X)^2\} \\ &= \sum_i a_i^2 \sigma^2 + \sum \sum a_i a_j \sigma_{ij} \end{aligned}$$

Now as, we assumed, $\sigma_{ij} = 0$, as the random variables are uncorrelated

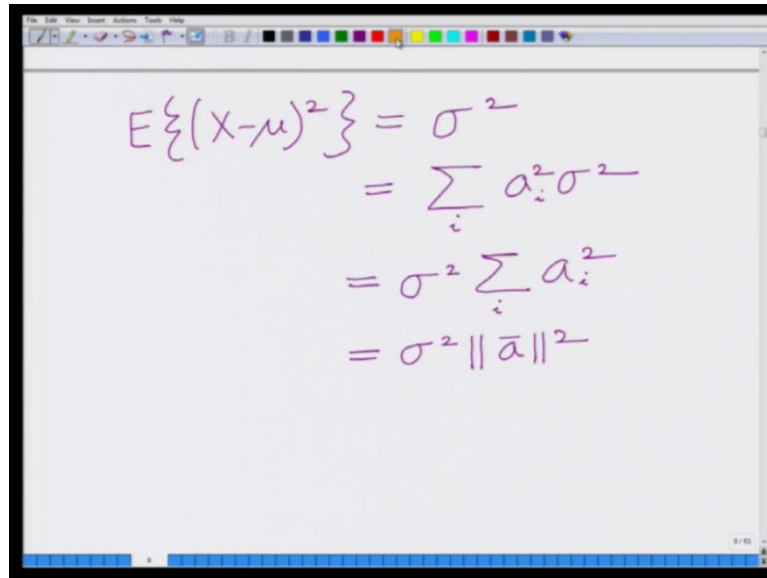
Therefore,

$$\begin{aligned} E\{(X - \mu)^2\} &= \sum_i a_i^2 \sigma^2 \\ &= \sigma^2 \sum_i a_i^2 \quad (\text{as variance is same for all } X_i) \\ &= \sigma^2 \|\bar{a}\|^2 \end{aligned}$$

Therefore,

$$E\{(X - \mu)^2\} = \sigma^2 \|\bar{a}\|^2$$

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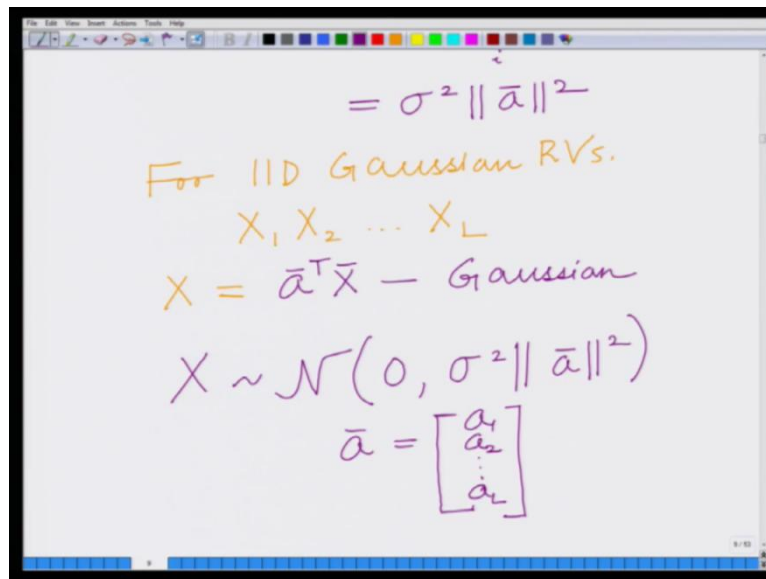


A digital whiteboard interface showing a handwritten mathematical derivation. The equations are written in purple ink and are as follows:

$$\begin{aligned} E\{(X-\mu)^2\} &= \sigma^2 \\ &= \sum_i a_i^2 \sigma^2 \\ &= \sigma^2 \sum_i a_i^2 \\ &= \sigma^2 \|\bar{a}\|^2 \end{aligned}$$

So, basically variance is equal to $\sigma^2 \|\bar{a}\|^2$.

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The image shows a digital whiteboard with handwritten mathematical notes. At the top, the expression $= \sigma^2 \|\bar{a}\|^2$ is written in purple. Below it, the text "For IID Gaussian RVs." is written in orange. This is followed by the sequence X_1, X_2, \dots, X_L in orange. Then, the expression $X = \bar{a}^T \bar{X}$ is written in orange, with the word "Gaussian" written in purple to its right. Below this, the probability distribution $X \sim \mathcal{N}(0, \sigma^2 \|\bar{a}\|^2)$ is written in purple. Finally, the vector \bar{a} is defined as $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_L \end{bmatrix}$ in purple.

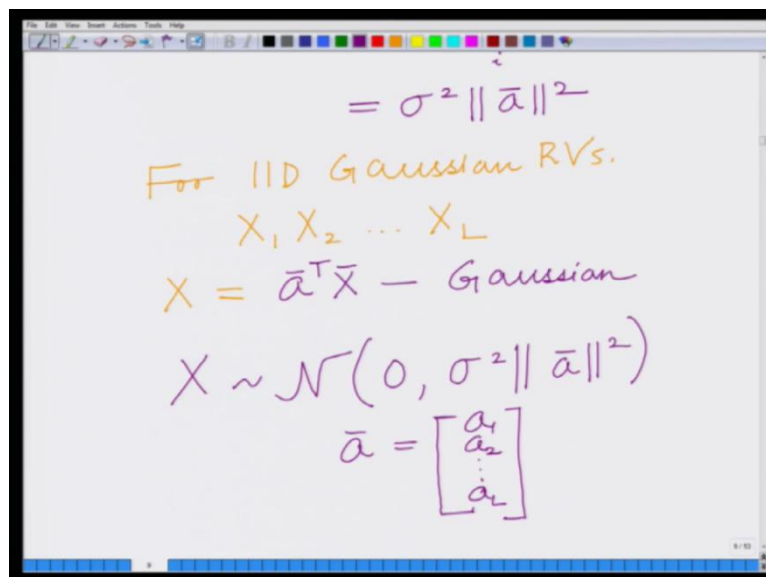
$$= \sigma^2 \|\bar{a}\|^2$$

For IID Gaussian RVs.
 X_1, X_2, \dots, X_L
 $X = \bar{a}^T \bar{X}$ - Gaussian
 $X \sim \mathcal{N}(0, \sigma^2 \|\bar{a}\|^2)$
 $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_L \end{bmatrix}$

Therefore for IID Gaussian random variables therefore for IID Gaussian random variables $\{X_1, X_2, \dots, X_L\}$, we have X which is defined as

$$X = \bar{a}^T \bar{X}, \text{ is a Gaussian random variable}$$

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This image is an identical copy of the one above, showing the same handwritten mathematical derivation on a digital whiteboard. It includes the expression $= \sigma^2 \|\bar{a}\|^2$, the text "For IID Gaussian RVs.", the sequence X_1, X_2, \dots, X_L , the expression $X = \bar{a}^T \bar{X}$ followed by "Gaussian", the distribution $X \sim \mathcal{N}(0, \sigma^2 \|\bar{a}\|^2)$, and the definition of the vector \bar{a} .

$$= \sigma^2 \|\bar{a}\|^2$$

For IID Gaussian RVs.
 X_1, X_2, \dots, X_L
 $X = \bar{a}^T \bar{X}$ - Gaussian
 $X \sim \mathcal{N}(0, \sigma^2 \|\bar{a}\|^2)$
 $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_L \end{bmatrix}$

Further X can now be represented as

$$X \sim N(0, \sigma^2 \|\bar{a}\|^2),$$

$$\text{Where, } \bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_L \end{bmatrix}$$

So, what we have done is we have considered the special case that, all the Gaussian random variables $\{X_1, X_2, \dots, X_L\}$ are 0 mean, they have identical variance σ^2 and further they are uncorrelated, that is their covariance σ_{ij} is equal to 0, and for the Gaussian random variables, we said uncorrelated also means that these Gaussian random variables are independent, therefore were considering a group of IID (i.e. independent identically distributed) Gaussian random variables and then we said if we generate a new Gaussian random variable X which is a weighted combination of these IID Gaussian random variables, that is

$$X = a_1 X_1 + a_2 X_2 + \dots + a_L X_L$$

then the mean of X is 0, the variance is $\sigma^2 \|\bar{a}\|^2$.

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Handwritten derivation on a whiteboard:

$$= \sigma^2 \|\bar{a}\|^2$$

For IID Gaussian RVs.

$$X_1, X_2, \dots, X_L$$
$$X = \bar{a}^T \bar{X} \text{ — Gaussian}$$
$$X \sim \mathcal{N}(0, \sigma^2 \|\bar{a}\|^2)$$
$$\bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_L \end{bmatrix}$$

So, this is an interesting property which will in fact come very handy, which is in fact very useful when we consider, when we **analyze** various communications, the **behavior** and the performance of various communication system as well as wireless a medication systems. So, we have seen a special case of linear combination of a group of random variables. So, we will end this module here and proceed with other topics in subsequent modules. Thank you very much.