Probability and Random Variables / Processes for Wireless Communications
Professor Aditya K Jagannathan
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Module Number 03
Lecture Number 14
Gaussian Random Variable and Linear Transformation

Hello! Welcome to another module, in this massive open online course, on probability and random variables of wireless communications. So today, let us look at another important concept in probability and random variables, that is a 'Gaussian random variable'. So today, we're going to focus on the concept of a 'Gaussian random variable'.A Gaussian random variable is a very important random variable in the context of communications and wireless communication systems as it arises very frequently.

So it's important to know the Gaussian random variable and the various properties of the Gaussian random variable, alright? So the Gaussian random variable is also known as the normal random variable. So let 'x' be a Gaussian random variable. Then the probability density function is given as

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

so this is the probability density function,

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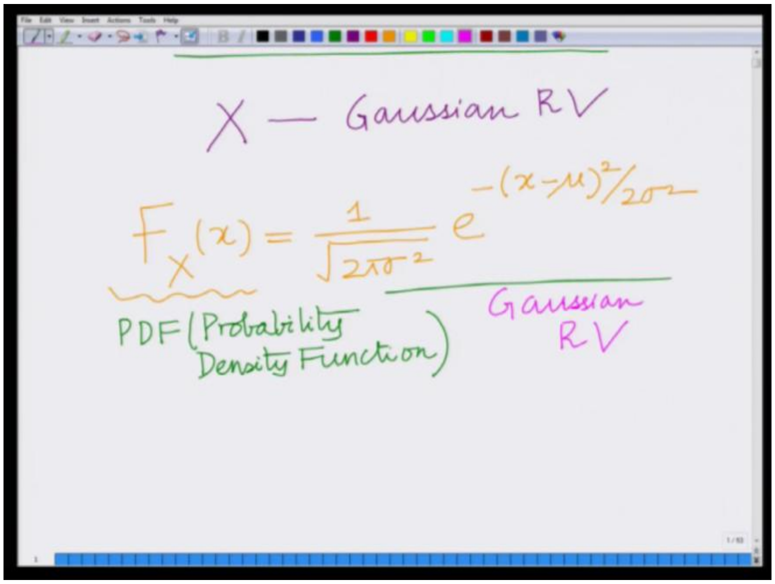

for the Gaussian random variable. where the quantities ‘μ’ and ‘σ’, are the average of the Gaussian random variable (‘μ’) and, the quantity ‘σ^2’ denotes the variance.

And if I plot this probability density function, if I plot the probability density function of this Gaussian random variable it is going to look something like this. Most of you should be familiar with this, this centre, the maximum of this occurs at this is at $x = \mu$ that is the peak.

This is the peak or the maximum. This is the peak of the Gaussian random variable, and it occurs at the mean that is x equal to the mean (μ) and the width of this distribution, the width of this curve is proportional to sigma square that is the variance (σ^2), and this shape is of course also known as a ‘bell-shape’ because it looks like a bell.

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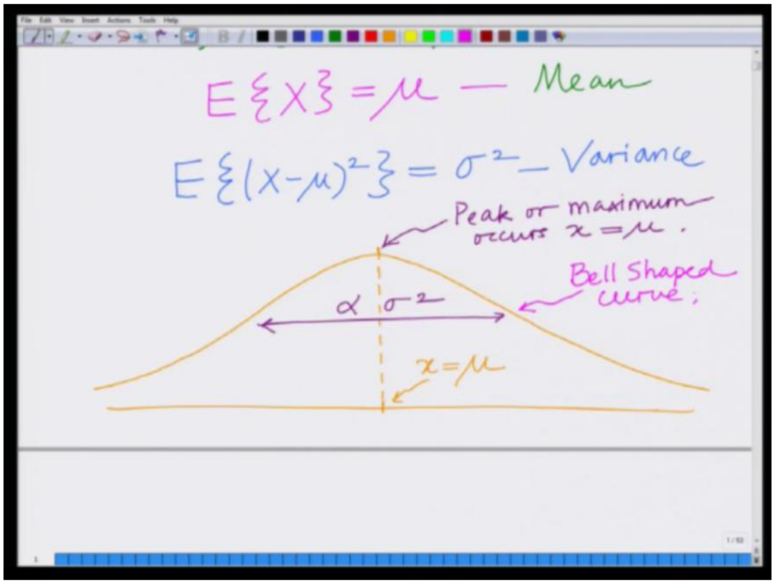

If the variance is very small, then the Gaussian probability density function is much more peaky, that is it's width is much more smaller. Now let us look at another very interesting property, a very interesting and very important property of the probability density function.

Gaussian probability density function is very important in the context of communication. For instance, all communication systems suffer from noise degradation, that is noise at the receiver, and very frequently, the noise process at the receiver is modeled as a Gaussian random process.

So this is one of the most important application in the context of communication system that noise is modeled, as a Gaussian random variable or a Gaussian process. As we're going to see in the next couple of lectures that this noise is a Gaussian random variable which is a function of time, which is a random variable at each instant of time that becomes a Gaussian random process.

Further, specifically in the context of a wireless communication system, the fading channel co-efficient is also modeled as a complex Gaussian random variable, that is a Gaussian, a complex quantity with the real part, and the imaginary part distributed as independent Gaussian random variables, that is known as a 'Rayleigh channel fading co-efficient'.

So specifically in the context of a wireless communication channel, it is modeled as a complex Gaussian random variable and this leads to what is known as a 'Rayleigh complex Gaussian random variable'.

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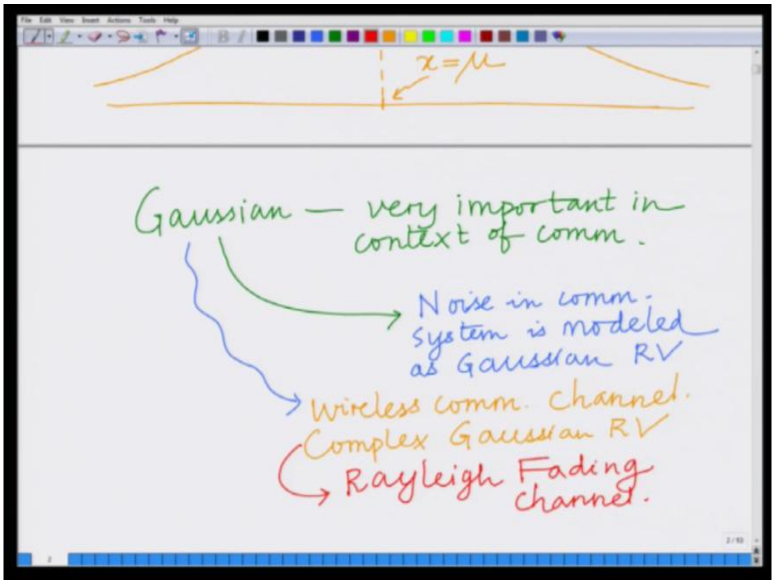

So Gaussian is very important in the context of both conventional communication systems, because the noise process at the receiver is modeled as a Gaussian as well as in wireless communication system, because typically the fading channel co-efficient is modeled as a complex Gaussian random variable.

Further, Gaussian with mean 'μ' and variance 'σ^2' is also denoted as.

$$N(\mu, \sigma^2)$$

Let us now look at another very interesting and practically very important property of the Gaussian random variable, that is, if I look at a linear combination of Gaussian random variables, this results in another Gaussian random variable. That is whenever I take a group of Gaussian random variables and consider a linear combination, a linear transformation of this Gaussian random variables, the resulting random variable is another Gaussian random variable.

Therefore, linear transformation of a bunch of Gaussian random variables results in another Gaussian random variables. So this is a very important property of the Gaussian random variable,

So let us illustrate this mathematically. So what do we mean? So let's say we have a group of Gauss, we have a group of random variables, so let

$\{X_1, X_2, \ldots, X_L\}$ be L Gaussian Random Variables

So these are L Gaussian random variables such that

Mean $= E(X_i) = \mu_i$ and

$$variance = \sigma_{X_i}^2 = E\{(x_i - \mu_i)^2\}$$

Therefore I can denote the Gaussian random variable X_i as

$$X_i = N(\mu_i, \sigma_i^2)$$

And further, let us define another new quantity that is known as the co-variance, i.e. if I look at

$$E\{(x_i - \mu_i)(x_j - \mu_j)\}, where, i \neq j$$

this is equal to σ_{ij}, a quantity is also known as, this is known as the co-variance of this Gaussian random variable X .

That is, $\sigma_{ij} = E\{(x_i - \mu_i)(x_j - \mu_j)\}, where, i \neq j$

Now let us consider the linear combination of these random variables. Consider, consider X, defined as a linear combination of $\{X_1, X_2, \ldots, X_L\}$ that is, we have a linear combination of these Gaussian random variables, that is,

$$x = a_1x_1 + a_2x_2 + \ldots + a_Lx_L$$

So what we're doing is we're linearly combining these Gaussian random variables x_1, x_2 upto x_L. So X is a linear combination of L Gaussian RV.

And using the previous property, we've said that the linear combination of a group of random variables results in another Gaussian random variable, so capital X which is a linear combination of x_1 , x_2 up to x_L, is another Gaussian random variable.

Further, let us find now the mean and the variance of this new Gaussian random variable X which has been generated as a linear combination of x_1, x_2 upto x_L.

So let us start by finding out the mean.

The mean of this Gaussian random variable is given as

$$\mu = E\{X\}$$

But X is given as

$$X = a_1X_1 + a_2X_2 + \ldots + a_LX_L$$

$$\mu = E\{X\} = E\{a_1X_1 + a_2X_2 + \ldots + a_LX_L\}$$

now the mean i.e. expectation is a linear operator so I can take the expected operator inside, so this is

$$\mu = a_1E\{X_1\} + a_2E\{X_2\} + \ldots + a_LE\{X_L\}$$

And therefore, now I can write this as, we know expected value of each X_i is μ_i, therefore this is equal to

$$\mu = E\{X\} = a_1\mu_1 + a_2\mu_2 + \ldots + a_L\mu_L$$

So our mean of this random variable is

$$\mu = E\{X\} = \sum_{i=1}^{L} a_i\mu_i$$

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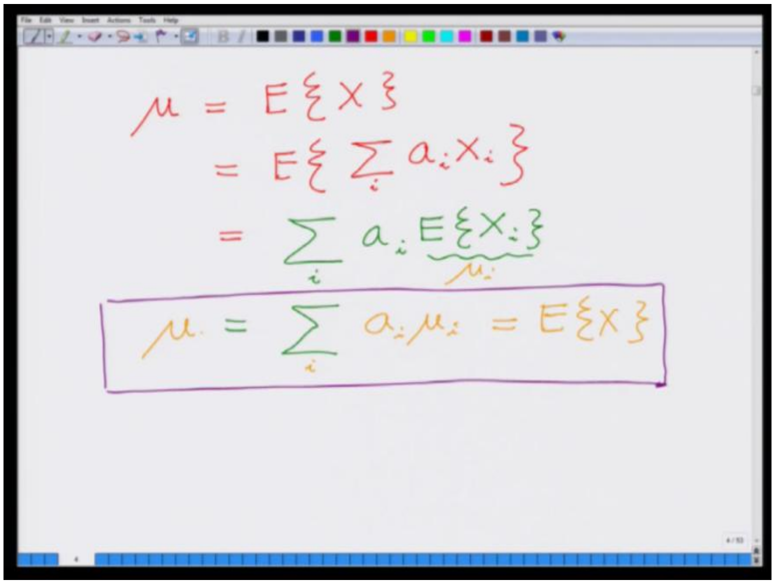

This is also the expected value of this random variable X. So we've derived that the expected value of this random variable X.

Now let us compute the variance which is much more tricky.

The variance is

$$\sigma_X^2 = E\{(X - \mu_X)^2\}$$

Now, let us substitute the expression for both X. Expression for X is

$$X = a_1X_1 + a_2X_2 + \ldots + a_LX_L = \sum_{i=1}^{L} a_iX_i$$

and,

$$\mu = E\{X\} = \sum_{i=1}^{L} a_i\mu_i$$

thus,

$$\sigma_X^2 = E\{(\sum_{i=1}^{L} a_iX_i - \sum_{i=1}^{L} a_i\mu_i)^2\}$$

$$= \left\{ (\sum_{i=1}^{L} a_i E[X_i - \mu_i])^2 \right\}$$

And this can be written as-

$$E \left\{ \left(\sum_{i=1}^{L} a_i [X_i - \mu_i] \right) \left(\sum_{j=1}^{L} a_j [X_j - \mu_j] \right) \right\}$$

This is basically,

$$E \{ \Sigma \Sigma a_i a_j (X_i - \mu_i)(X_j - \mu_j) \}$$

Now, we consider the summation in the following two cases-

Case 1: when i=j,

Then, $\sigma_X^2 = \sum_i a_i^2 E\{X_i - \mu_i\}^2$

Case 2: when i≠j,

Then, $\sigma_X^2 = E \{ \Sigma \Sigma a_i a_j (X_i - \mu_i)(X_j - \mu_j) \}$

Combining both cases, we can write the Variance for the given X as-

$$\sigma_X^2 = \sum_i a_i^2 E\{X_i - \mu_i\}^2 + E \{ \Sigma \Sigma ai\ aj\ (Xi - \mu i)(Xj - \mu j)\}$$

as expectation is a linear transformation, we have,

$$\sigma_X^2 = \sum_i a_i^2 E\{X_i - \mu_i\}^2 + \Sigma \Sigma aiaj\ E(Xi - \mu i)(Xj - \mu j)$$

And, as $E(X_i - \mu_i)(X_j - \mu_j) = \sigma_{ij} = Covariance$

thus,

$$\sigma_X^2 = \sum_i a_i^2 E\{X_i - \mu_i\}^2 + \sum \sum a_i a_j \sigma_{ij}$$

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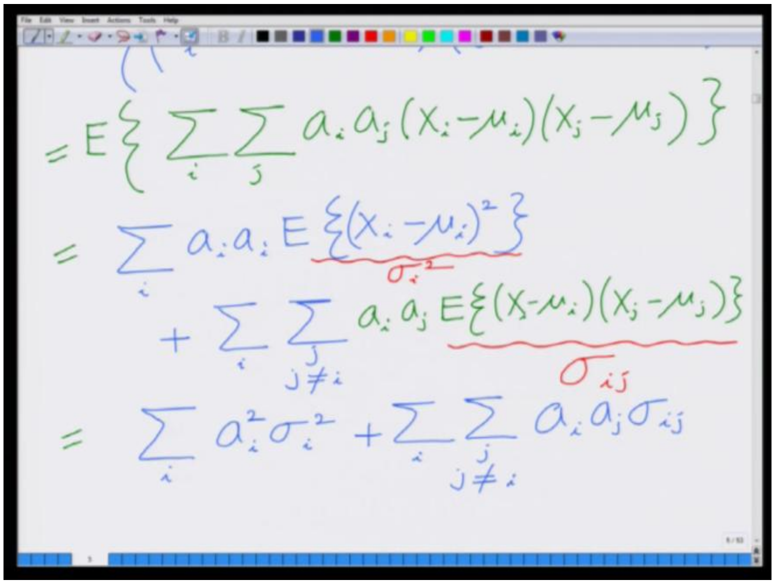

Therefore, the Gaussian random variable X can now be represented as

$$X = N(\mu_X, \sigma_X^2)$$

So this is the Gaussian random variable X, which is generated as a linear combination of this Gaussian random variables, x_1 , x_2 upto x_L.

So what we have done is we've considered this Gaussian random variable which is basically a linear combination of the Gaussian random variables, x_1 , x_2 upto x_L, and we've calculated the mean and variance of this new Gaussian random variable x, in terms of the means, the variances and the co-variances of the Gaussian random variables x_1 , x_2 upto x_L.

And this result is a very useful result especially in the context of communications and wireless communications, which one has to invoke frequently, because the noise process in the receiver is Gaussian, and also several interesting things, for instance, just, the Rayleigh fading channel co-efficient can be modeled as a complex Gaussian random variable with real part Gaussian and the imaginary part also Gaussian. So this comprehensively clarifies the probability density function and various key properties of the Gaussian random variable.

So we'll stop this module here and explore other aspects in the subsequent modules. Thank you very much.