Probability and Random Variables/Processes For Wireless Communications
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Lecture Number 13
Module Number 3
Transformation of Random Variables and Rayleigh Fading Wireless Channel

Hello! Welcome to another module in this massive open online course on Probability and Random Variables for Wireless Communications. So in the previous modules we have started looking at random variables and various statistical properties of random variables such as the mean, variance and other aspects and we have also looked at the applications of these random variables specifically in the context of wireless communication.

So we have looked at some of the applications in the context of wireless communication systems. Let us now continue discussion of these random variables and let us today look at a different concept, the concept of transformation of random variables, so today let us look at the transformation, the transformation of random variables.

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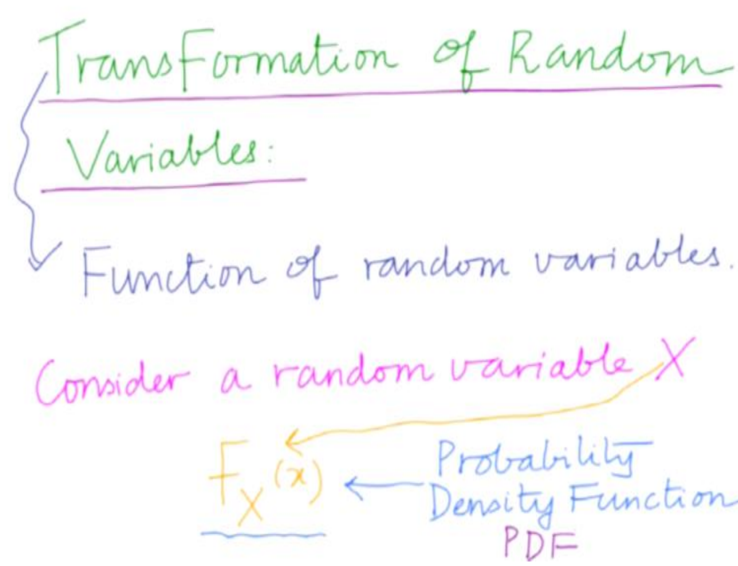

By transformation of a random variable we also mean basically a functional transformation of a random variable, so one can also think of this as basically random variables which are functions of other random variables, so frequently you can also call this as ~~Function,~~ Function of Random Variables.

Let us consider a random variable X, as we said "Every random variable is characterised by probability density function" so this random variable X is characterized by the probability density function $F_X(x)$, also abbreviated as the PDF.

So what do we have? We have this random variable capital X which is characterised as usual by the probability density function P.D.F. which is denoted by $F_X(x)$ which denotes the denotes the probability density function at the point x of the random variable capital X.

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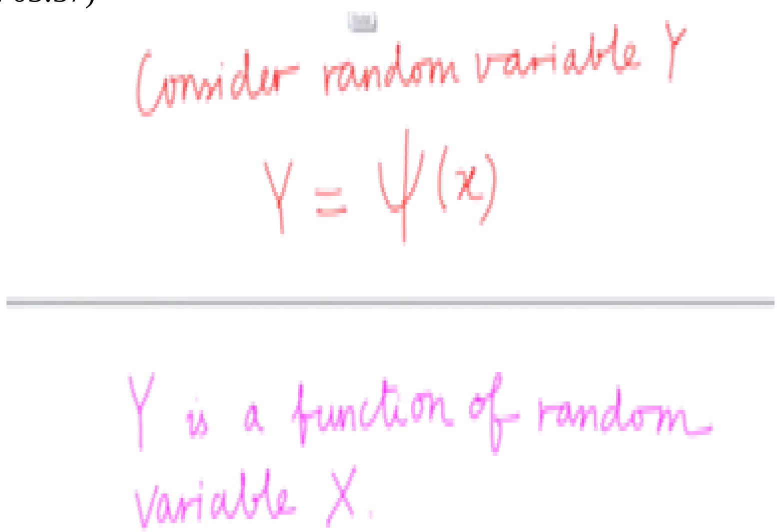

Now what we are going to do is we are going to consider another random variable Y which is given as a function of this random variable X. So Consider Y, consider random variable Y such that Y equals some function of X.

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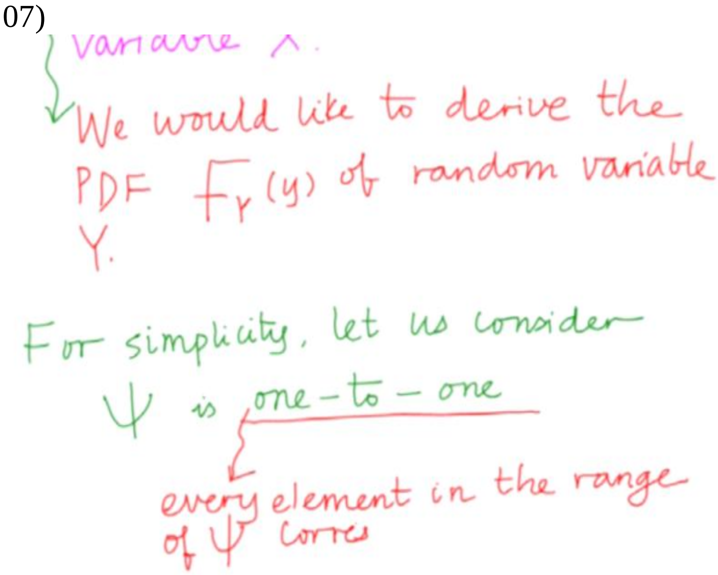

Now, let us consider a special case, let us consider, So now what we would like to do is, given the probability density function F of random variable X, we would now like to derive the probability density function of the random variable Y.

So we would like to derive the probability density function $F_Y(y)$ which is given as

$$y = \Psi(x)$$

Further, although this can be done for the general case, let us consider for the purpose of simplicity a special case when this function $\Psi(x)$ is one to one. This means that every element in the range that is, every element in the range of $\Psi(x)$ corresponds to an element in the domain.

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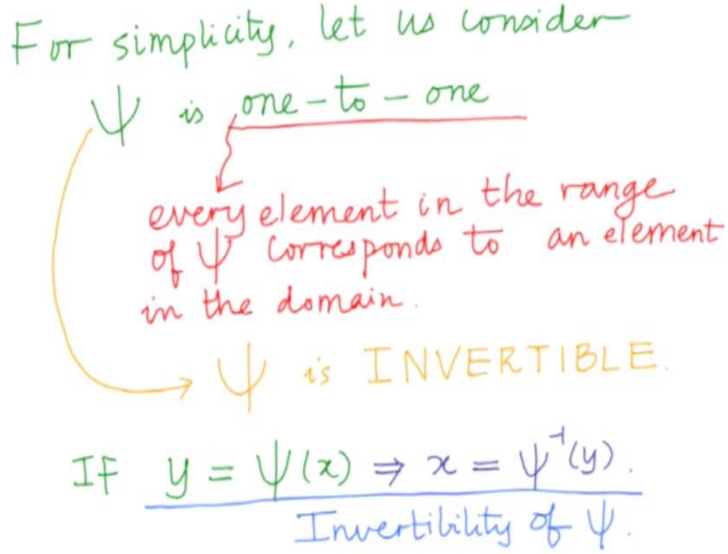

Therefore this function Si is an invertible function. This implies I can always write there exists a unique X such that

$$x = \Psi^{-1}(y)$$

i.e. X equals Ψ inverse of Y.

Now let us look, since X is equal to $\Psi^{-1}(y)$, therefore let us start with the random variable X, if this lies in an infinitesimal neighbourhood of x with that is in the interval x plus dx, this implies that the random variable Y lies in the infinitesimal neighbourhood y plus dy.

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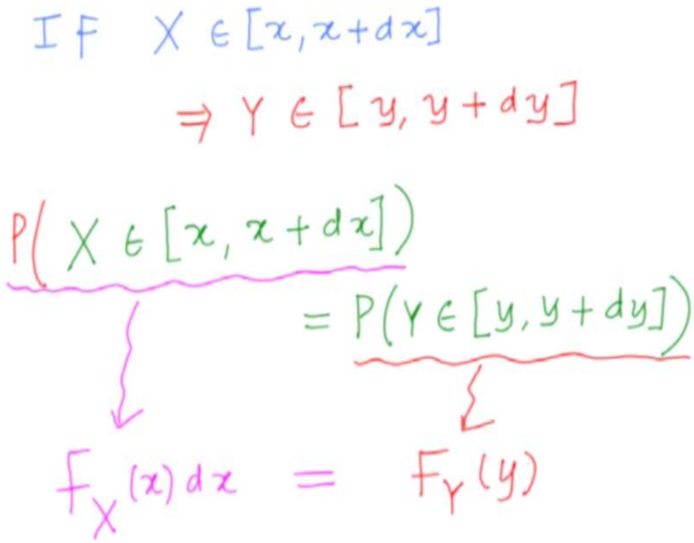

i.e.

$$\text{if} \quad X \in [x, x+dx]$$
$$\Rightarrow Y \in [y, y+dy]$$

Now therefore the probability, therefore we can say that the probability that X is element of [x, x + dx] is basically equal to the probability that Y is element of the infinitesimal neighbourhood [y, y + dy] i.e.

$$P(X \in [x, x+dx]) = P(Y \in [y, y+dy])$$

So, we are saying that these probabilities are basically equal, i.e.-

$$F_X(x)dx= F_Y(y)dy.$$

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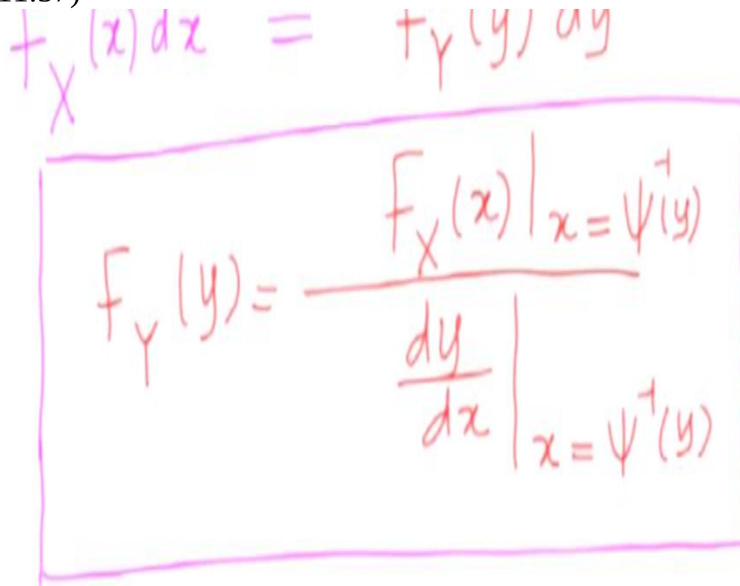

Therefore now we can write the relation the probability density function F of y of dy is equal to

$$F_Y(y) = \frac{F_X(x)}{\frac{dy}{dx}},$$

$$for, x = \Psi^{-1}(y)$$

What we are saying is basically we have equated these probabilities and we have said $F_X(x)dx$ should be equal to $F_Y(y)dy$ because the probability that random variable X lies in x to x + dx should be the same as the probability that random variable Y lies in y to y + dy.

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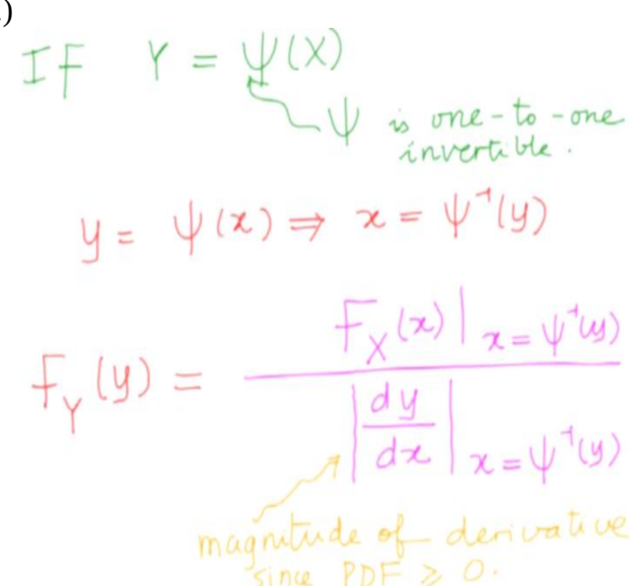

So let me write this again just to clarify this thing,

$$\text{If, } Y = \Psi(x)$$

where Ψ is basically one to one and we're considering a special case where Si is one to one and invertible.

We will consider magnitude of this derivative, so we consider the magnitude of the derivative. Since P.D.F, has to be greater than or equal to zero. Remember we said that the probability density function has to always be greater than or equal to zero, but the derivative can be also less than zero therefore to avoid this negative sign in the probability density function, we consider rather than simply dividing the derivative $\frac{dy}{dx}$, we consider the magnitude of the derivative $\frac{dy}{dx}$ at X is equal to $\Psi^{-1}(y)$. Therefore we have the formula

$$F_Y(y) = \frac{F_X(x)}{\left|\frac{dy}{dx}\right|}$$

both evaluated at the point

$$x = \Psi^{-1}(y)$$

as the function is an invertible function.

So, this is basically our expression, remember for the functional transformation of P.D.F probability density function corresponding to a functional transformation of the random variable.

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$$\boxed{f_Y(y) = \frac{f_X(x)\big|_{x=\psi^{-1}(y)}}{\left|\frac{dy}{dx}\right|_{x=\psi^{-1}(y)}}}$$

magnitude of derivative since PDF ≥ 0.

Transformation of PDF.

So this is how you derive the P.D.F. of the random variable Y which is a function of the random variable X.

Now to understand this better let us look at an example, to understand this aspect better this functional transformation of the random variables, let us look at a simple example in the context of wireless communication.

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Transformation of Random Variables:

Example from wireless comm.

If X = random variable denoting power of wireless channel.

$$f_X(x) = e^{-x}, \quad x \geq 0.$$

Remember in a wireless communication system, if we look at the power gain of the channel between the base station and the receiver as we have looked at in the previous module. We said that the channel between the transmitter and receiver is a multipath fading channel where they put the power at the receiver is fading in nature in the sense that it, it varies.

Therefore the power at the receiver in a wireless communication system is random in nature and that power of the channel can be characterised by the exponential distribution with the parameter, with the exponential parameter equal to one. Remember if X is the random variable denoting the power, denoting the power of the wireless channel then

$$F_X(x) = e^{-x}$$

Where, $x \geq 0$

Therefore the random variable which denotes the power, is distributed as an exponential, random variable

Now let us consider the amplitude of this wireless channel which is the square root of the power. Therefore now what we would like to do, we would like look at the functional transformation where

$$Y = \sqrt{X}$$

and since,

X is the power, Y equals to the amplitude of the wireless channel co-efficient. So basically we are saying that X the random variable denotes the power of the fading wireless channel.

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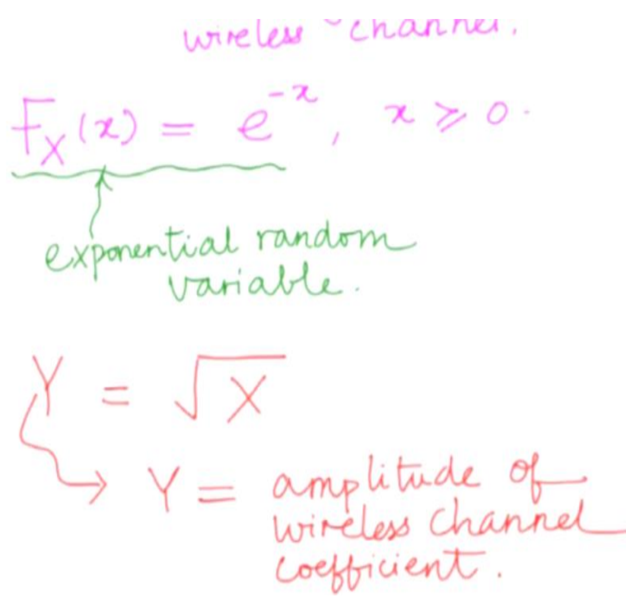

Therefore Y which is equal to square root of X denotes the amplitude of the fading wireless channel. And we’re given that this random probability density function of the power is exponential random variable

So now our question is, in this example, our question is, “What is $F_Y(y)$?”

Therefore our function Ψ as you can clearly see is

$$Y = \Psi(x) = \sqrt{x}, x \geq 0, y \geq 0$$

and we’re considering only the positive square root, the amplitude can only be positive, so we’re considering only the positive square root. So X greater than or equal to zero, Y greater than equal to zero therefore our Ψ is a one to one that is invertible function.

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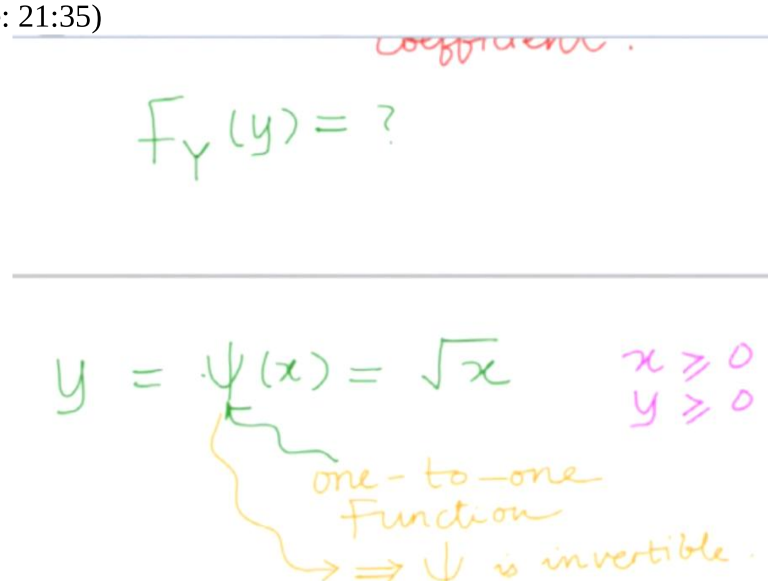

So Ψ is a one to one function implies Si is, implies that Si is invertible, therefore I can write for a given Y equal to $\Psi(x)$, I can write

$$x = \Psi^{-1}(y) = y^2$$

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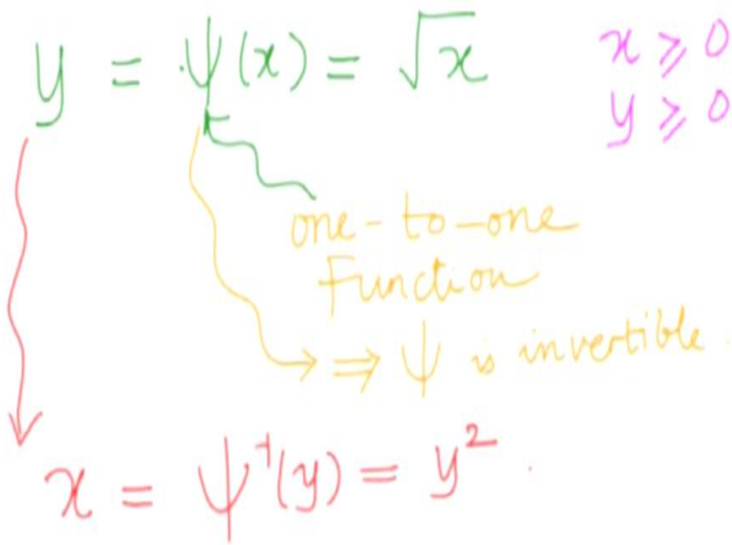

Therefore now what I have is, as Ψ is a one to one and invertible function, therefore now I can write F of Y of y that is the probability density function, $F_Y(y)$ this is equal to

$$F_Y(y) = \frac{F_X(x)}{\left|\frac{dy}{dx}\right|}$$

at $x = \Psi^{-1}(y)$

Now we have

$F_X(x) = e^{-x}$ and

$$\left|\frac{dy}{dx}\right| = \frac{1}{2\sqrt{x}},$$

Substituting, $x = y^2$, we have

$$F_Y(y) = \frac{e^{-y^2}}{\frac{1}{2y}}$$

Therefore,

$$\boxed{F_Y(y) = 2ye^{-y^2}}$$

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$$x = \psi^{-1}(y) = y^2$$

$$f_Y(y) = \frac{f_X(x)\big|_{x=\psi^{-1}(y)}}{\left|\frac{dy}{dx}\right|_{x=\psi^{-1}(y)}}$$

$$= \frac{e^{-x}\big|_{x=y^2}}{\left|\frac{1}{2\sqrt{x}}\right|_{x=y^2}}$$

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$$\left|dx\right|_{x=\psi^{-1}(y)}$$

$$= \frac{e^{-x}\big|_{x=y^2}}{\left|\frac{1}{2\sqrt{x}}\right|_{x=y^2}}$$

$$f_Y(y) = \frac{e^{-y^2}}{\frac{1}{2\sqrt{y^2}}} = 2ye^{-y^2}$$

So this is the probability density function of the, probability density function of the amplitude of the fading, this is the probability density function of the amplitude of the fading wireless channel. And this is, we are saying that this probability density function of the amplitude of the fading wireless channel is given as $F_Y(y)$ which is equal to $2ye^{-y^2}$ and this is a very important probability density function in the context of wireless communications.

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$$f_Y(y) = 2ye^{-y^2}, \quad y \geq 0$$

Probability density function of the amplitude of the fading wireless channel.

Rayleigh probability density function.

Some of you might already be familiar with it. This is known as the Rayleigh probability density function, that is when the power of the channel co-efficient is distributed as the exponential random variable, the amplitude of the fading wireless channel co-efficient is distributed as a Rayleigh random variable. So this is basically, known as the Rayleigh probability density function.

This is known as the Rayleigh probability density function, and the channel, the channel in which the amplitude follows the Rayleigh probability ~~(den)~~ density function that is known as the Rayleigh fading wireless channel. So the channel with the Rayleigh probability density for the amplitude is known as the Rayleigh fading wireless channel. This is one of the most important models for wireless, one of the most important models or one of the most important probability density functions that (you) that is used to model the behaviour of the random fading wireless channel.

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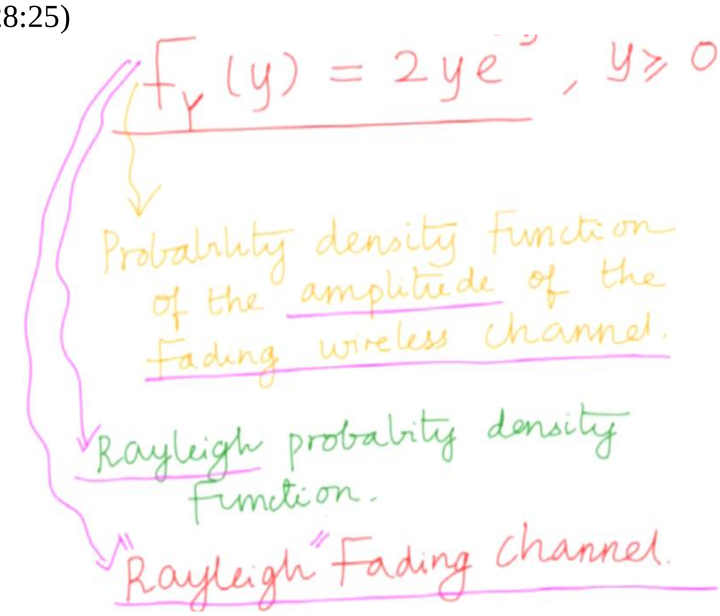

So in this module what we have seen is we have seen very important concept of basically transformation or functional transformation of random variables, we have derived the, when a random variable Y is given as a one to one that is invertible function of the random variable X we have derived the probability density function for the random variable Y in terms of the probability density function of the random variable X and finally we have seen an very interesting example in the context of wireless communication that is given the probability density function of the power of the wireless channel co-efficient.

We have derived the equivalent probability density function of the amplitude and demonstrated it is given by the Rayleigh probability density function and such a channel is known as the Rayleigh fading wireless channel. So we will stop this module here and we will look at other aspects in the subsequent lecture. Thank you very much.