**Principles of Modern CDMA/MIMO/OFDM Wireless Communications**
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**Lecture - 50**
**BER Performance of OFDM Systems**

Hello, welcome to another module in this massive open online course on the principles of CDMA, MIMO, OFDM wireless communication systems. So, in the previous modules we have looked at the operation over the schematic, the operations at the transmitter the operation in the receiver and OFDM wireless communication systems, the system model for an OFDM sub wireless system. So, today let us look at the Bit Error Rate performance of an OFDM system.

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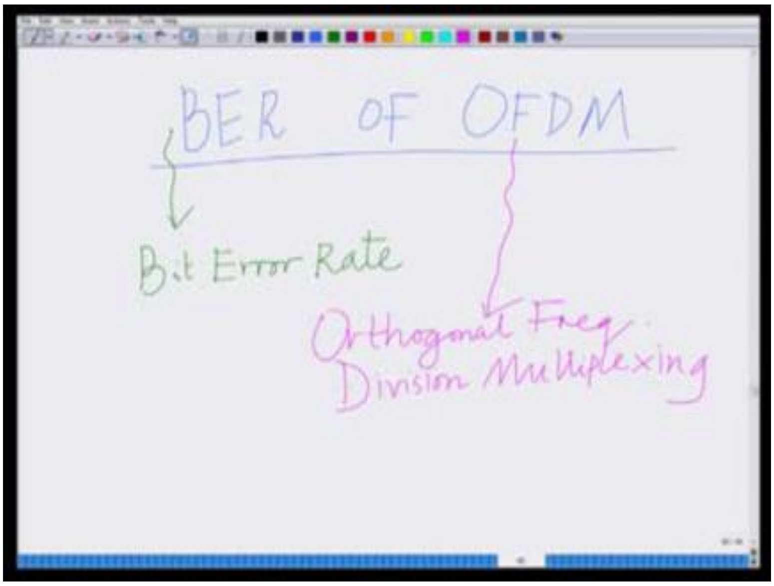


So, to characterize the performance of an OFDM system, similar to what we have done before, let us look at the Bit Error Rate of an OFDM system, I hope everyone remembers this BER stands for the bit error, this stand for the Bit Error Rate and OFDM is orthogonal frequency division multiplexing. So, we are looking at the bit error rate of an OFDM system, and we have seen that the OFDM system.

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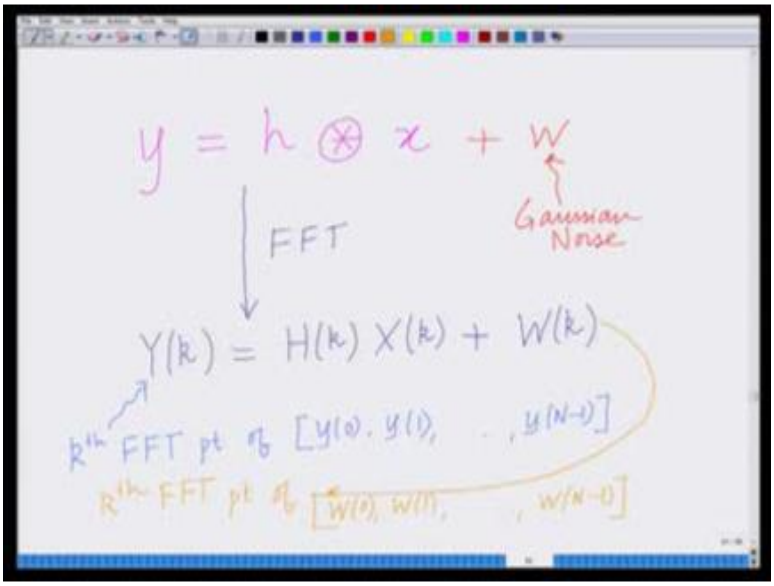


The system model can be expressed as the received signal

$$y = h \oplus x$$

Y equals the channel H circularly convolved with the input x and previously we have ignore the effect of noise.

Let us now add the noise, so W is Gaussian noise. Now we are adding Gaussian noise to this and now once we take the FFT at the receiver after remember in OFDM, we have

$$y = h \oplus x + w$$

$$Y(k) = H(k) . X(k) + W(k)$$

where $X(k)$ is the symbol on the k sub carrier plus, so is the FFT operation is a linear operation plus $W(k)$ where $W(k)$ is the FFT of the noise, so we have $Y(k)$ is the k-th FFT point of the received signal $y(0)$, $y(1)$, $y(N-1)$, this is the k-th FFT point $W(k)$ this is the k-th FFT point of the noise samples, $w(0)$, $w(1)$, $w(N-1)$, so what we are saying is very straight forward, we are saying that in time domain, we have the received signal y is the channel H circularly convolved with the input signal or the transmit signal X plus the noise w at the receiver.

Now, after taking the FFT at the receiver we have $Y(k)$ where $Y(k)$ is the k-th FFT point of the received samples $y(0)$, $y(1)$, $y(N-1)$, equals $H(k)$ where $H(k)$ is the k-th channel coefficient, that is the FFT of the 0 padded channel coefficients $h(0)$, $h(1)$, $h(N-1)$ followed by (N-1) times $X(k)$, where $X(k)$ is the symbol transmitted on the k-th subcarrier plus $W(k)$ where $W(k)$ is the noise is the k-th FFT point of the noise samples $w(0)$, $w(1)$, $w(N-1)$.

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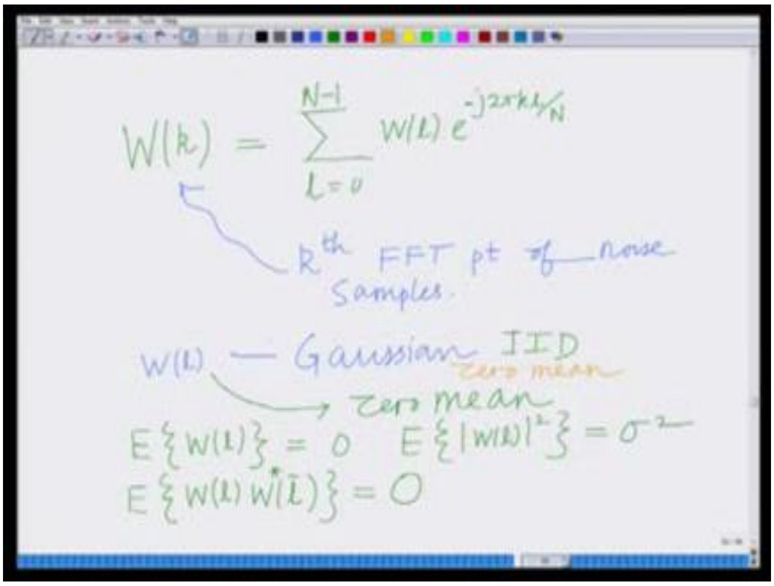


In other words, the noise $W(k)$ is given as I have

$$W(k) = \sum_{l=0}^{N-1} w(l)\, e^{-j2\pi k\frac{l}{N}}$$

So, capital $W(k)$ is the k-th FFT point, this is the k-th FFT point of the noise sample and what we are going to assume about the noise samples? Is each $w(l)$ is IID

Gaussian is independent identical Gaussian, so the $w(l)$ are IID Gaussian and also 0 mean, which means

$E\{ w(l) \} = 0$

$E\{ |w(l)|^2 \} = \sigma^2$

$E\{ w(l)\, w(\tilde{l})^* \} = 0$ because these noise samples are independent.

So, what we are going to assume about the noise samples are $w(l)$ are Gaussian, these are these are Gaussian, these are 0 mean; that means, to say $E\{ w(l) \} = 0$, expected noise for a sigma square, which means $E\{ |w(l)|^2 \} = \sigma^2$, further the noise is IID independent, which means $E\{ w(l)\, w(\tilde{l})^* \} = 0$, that is 2 noise samples are $l$ and $\tilde{l}$ are independent are uncorrelated. So, the noise samples are Gaussian, they are IID independent identically distributed and also the noise samples are the noise sample. So, the noise samples $w(0)$, $w(1)$, $w(N-1)$ are IID, that is independent identically distributed Gaussian and they are 0 mean.

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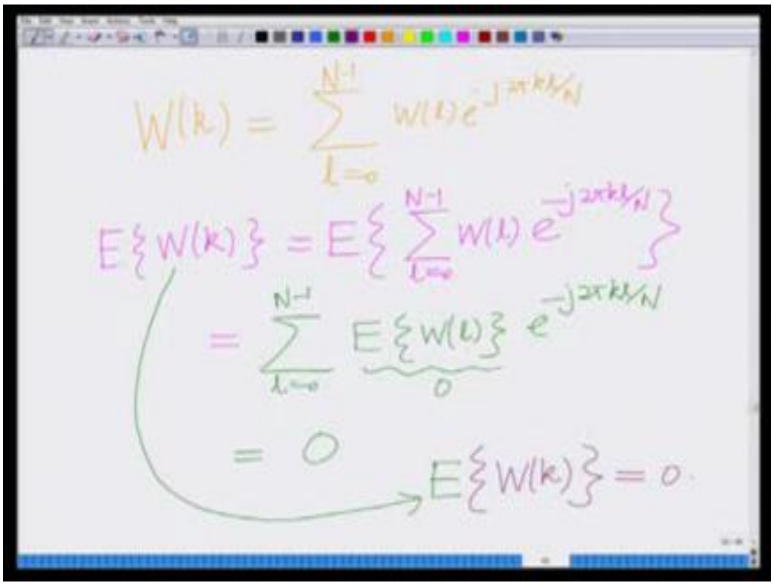


Now, let us look at the properties of $W(k)$. If I look at $W(k)$, which is

$W(k) = \sum_{l=0}^{N-1} w(l)\, e^{-j2\pi k\frac{l}{N}}$

$E\{\ w(k)\ \} = E\{\ \sum_{l=0}^{N-1} w(l)\, e^{-j2\pi k\frac{l}{N}}\ \}$

$= \sum_{l=0}^{N-1} E\{\ w(l)\ \} e^{-j2\pi k\frac{l}{N}} = 0$

as $E\{\ w(l)\ \} = 0$

that is since the FFT is a linear combination the noise samples the at the input of the noise samples each noise sample is 0 mean and since the FFT is a linear operations, naturally at the output of the FFT that is each W $w(k)$k, the mean of the each noise sample $w(k)$ at the output of the FFT is also 0, so no surprises there. So, each noise sample in a $w(k)$ on each subcarrier is also 0.

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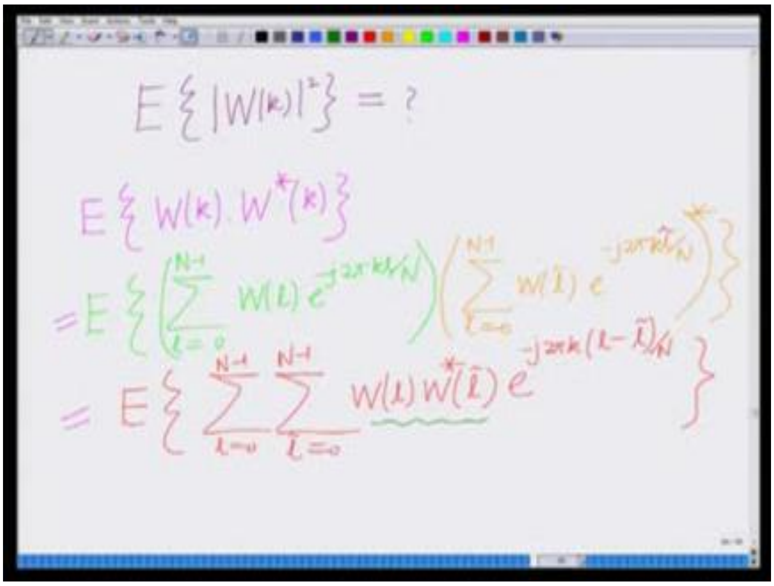


Now, let us look at the variance of each noise sample, let us look at $E\{\ |w(l)|^2\ \}$ and let us ask this question what is this?

$E\{\ |w(k)|^2\ \} = E\{\ w(k)\ w(\tilde{k})^*\ \}$

$$= E\{( \sum_{l=0}^{N-1} w(l)\, e^{-j2\pi k\frac{l}{N}} ) ( \sum_{\tilde{l}=0}^{N-1} w(\tilde{l})\, e^{-j2\pi k\frac{\tilde{l}}{N}} )^* \}$$

$$= E\{ \sum_{l=0}^{N-1} \sum_{\tilde{l}=0}^{N-1} w(l)w(\tilde{l})\, e^{-j2\pi k\frac{(l-\tilde{l})}{N}} )^* \}$$

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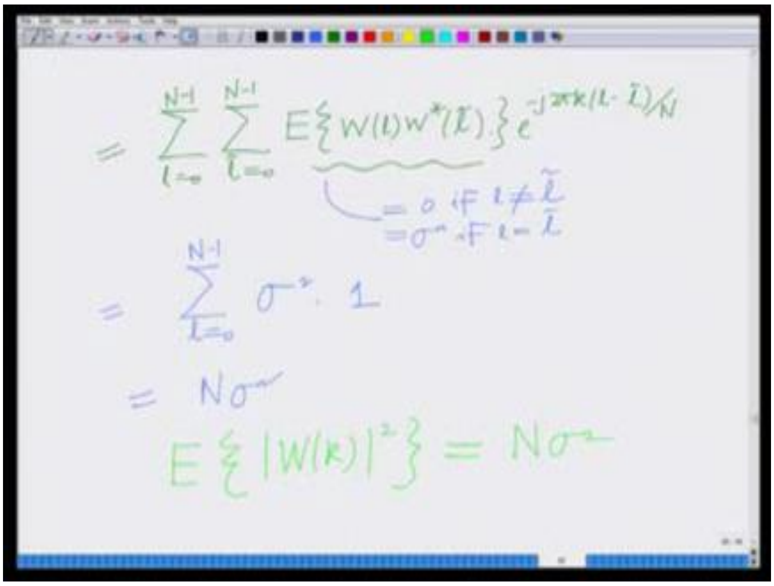


Therefore I can simplify this as

$$\sum_{l=0}^{N-1} \sum_{\tilde{l}=0}^{N-1} E\{ w(l)w(\tilde{l})^* \}\, e^{-j2\pi k\frac{(l-\tilde{l})}{N}} )$$

$$E\{ w(l)w(\tilde{l})^* \quad = \sigma^2 \text{ if } l = \tilde{l}$$

$$= 0 \text{ if } l \neq \tilde{l}$$

$$= \sum_{l=0}^{N-1} \sigma^2$$

$$= N\sigma^2$$

$$E\{ |w(k)|^2 \} = N\sigma^2$$

So, what we have is at the output of the FFT the noise $w(k)$ is 0 mean and it has a variance of $N\ \sigma^2$ further since each W k is generated as a linear combination of the small $w(k)$, that it is a linear combination of Gaussian input noise samples.

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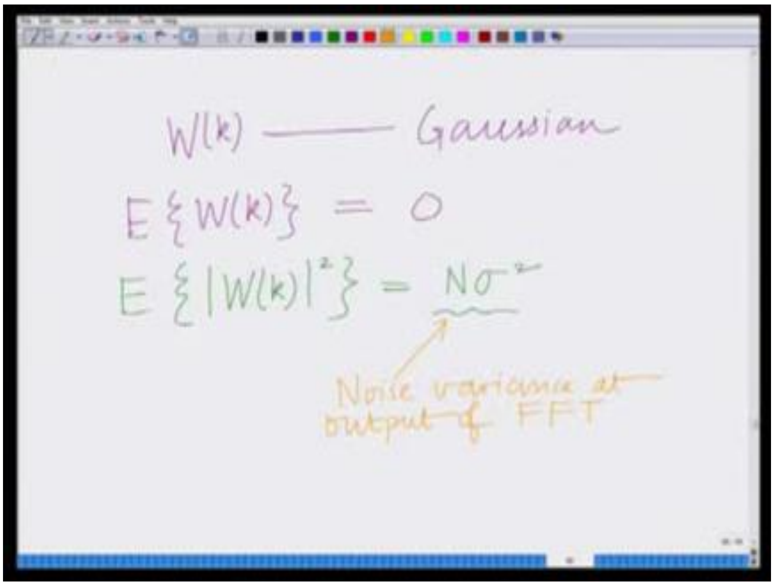


Therefore, each $w(k)$ is also Gaussian therefore, as a result the noise can be characterized as follows, the noise at the output of the FFT each $w(k)$, this is Gaussian $E\{\ w(k)\ \} =\ 0$ the variance of the noise, that is $E\{\ |w(k)|^2\ \} = N\ \sigma^2$.

So, we have to characterized and this is the noise variance at output of FFT, what is this is the noise variance at the output of the FFT operation at the receiver. So, we have characterized the noise the properties of the noise at the output of FFT. Now let us characterize the property of the channel coefficient H of k remember, H of k is given by the FFT of the 0 padded channel coefficients $h(0)$, $h(1)$, $h(L-1)$ followed by $N - L$ 0s.

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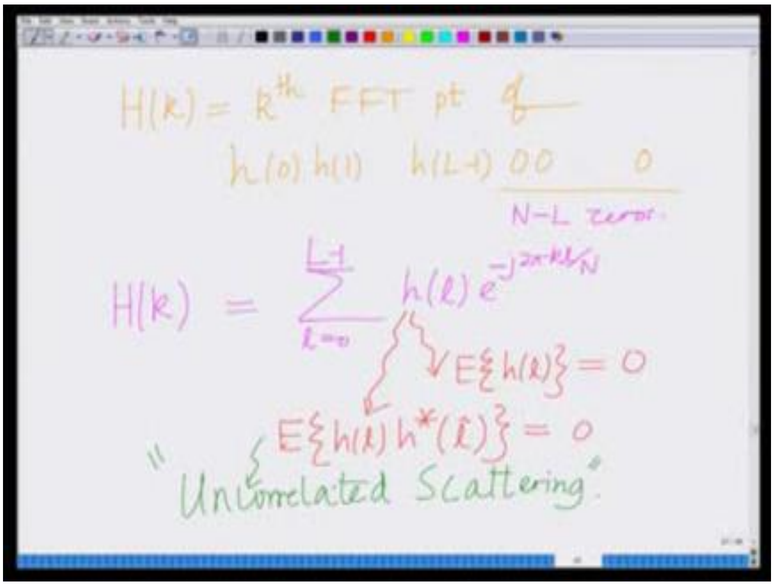


So, H k equal k-th FFT point of the 0 padded sequence, $h(0)$, $h(1)$, $h(L-1)$, 0 and what is this these are N - L 0's.

Hence, if I write the expression for $H(k)$, naturally $H(k)$ is given as

$$H(k) = \sum_{l=0}^{L-1} h(l)\, e^{-j2\pi k\frac{l}{N}}\,)$$

$$E\{\, h(l)\, \} = 0$$

$$E\{\, h(l)\, h(\tilde{l})^{\,*}\, \} = 0$$

that is expression for the k-th FFT point, what we are going to, now assume that each $h(l)$ is 0 mean, that is each $h(l)$ is a Rayleigh fading channel coefficient, which means each $h(l)$ is a complex Gaussian, which means each $h(l)$ is 0, together we are going to assume that expected value of $h(l)$ times and this an important assumption expected value of $h(\tilde{l})^{\,*}$ this is equal to 0 that is the noise that is the channel types $l$ and $\tilde{l}$ are uncorrelated, this is termed as the uncorrelated scattering assumption.

What are we assuming, we are assuming that each $h(l)$ each channel coefficient $h(l)$ is the Rayleigh fading channel coefficient between each $h(l)$ is complex Gaussian, we are assuming that each $h(l)$ is 0 mean, $E\{ h(l) \} = 0$. Expected value of 2 different channel coefficients, that is the correlation between 2 different channel coefficients. $E\{ h(l) \, h(\tilde{l})^* \} = 0$ which means that the coefficient and uncorrelated this is known as uncorrelated scattering this is known as uncorrelated scattering environment.

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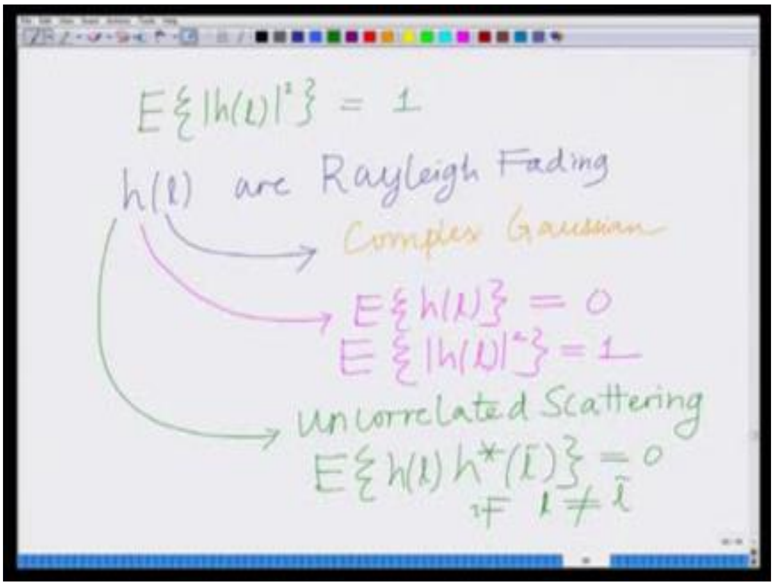


And also we are going to assume that the power of each h l that is the standard Rayleigh fading coefficient, $E\{ |h(l)|^2 \}$ is identity.

So, what are we assuming ? we are assuming the various $h(l)$'s are Rayleigh fading, which means each $h(l)$ is complex Gaussian,

$$E\{ h(l) \} = 0$$

$E\{ |h(l)|^2 \} = 1$ that is average power 1 and also we are assuming the uncorrelated scattering, which means

$E\{ h(l)\, h(\tilde{l})^* \} = 0$

So, we are assuming the uncorrelated scattering example.

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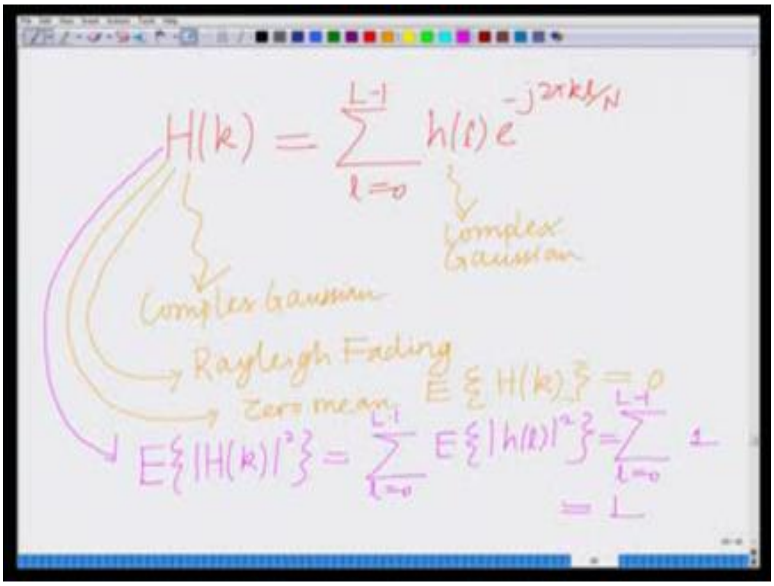


And now again, I am not going to the go through the details again but, if I look at the expression $H(k)$ is

$$H(k) = \sum_{l=0}^{L-1} h(l)\, e^{-j2\pi k\frac{l}{N}})$$

$$E\{ |H(k)|^2 \} = \sum_{l=0}^{L-1} E\{ |h(l)|^2 \} = \sum_{l=0}^{L-1} 1 = L$$

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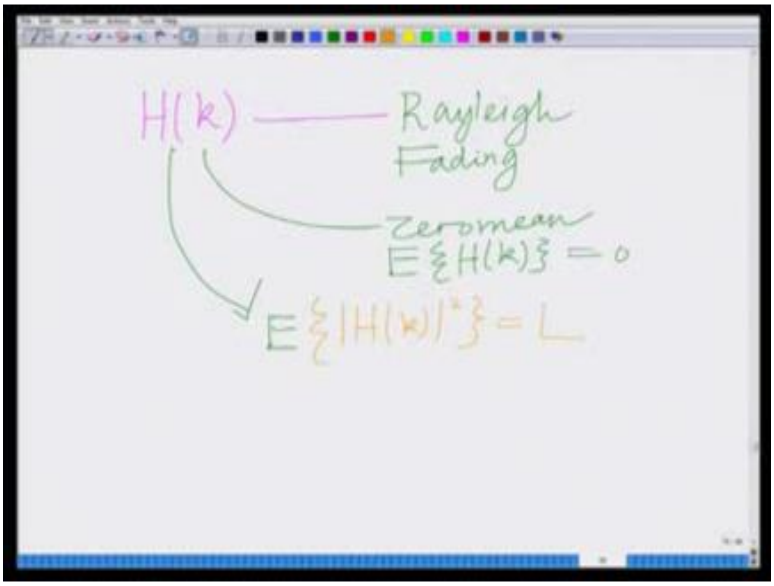


So, what we have at the end is that each $H(k)$ is intern Rayleigh fading coefficient, each $H(k)$ is a Rayleigh fading coefficient with 0 mean, that is

$$E\{ H(k) \} = 0$$

$$E\{ |H(k)|^2 \} = L$$

that is k-th FFT point of the 0 faded channel coefficients, $h(0)$, $h(1)$, $h(L-1)$, these are 0 mean $H(k)$ is Rayleigh fading in nature it is 0 mean, it has average power L.

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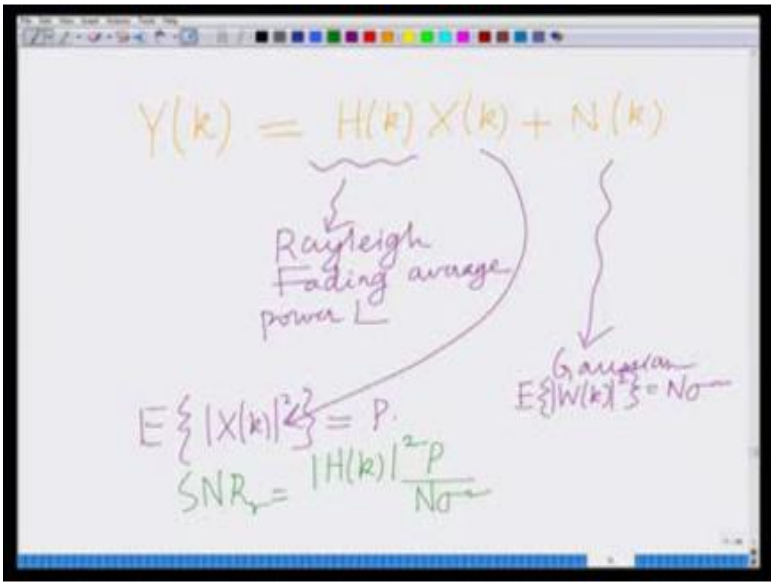


So, now if I look at my system model across each sub carriers I have

$$Y(k) = H(k) \cdot X(k) + W(k)$$

$$E\{\,|X(k)|^2\,\} = P$$

$$SNR_r = \frac{|H(k)|^2 P}{N\,\sigma^2}$$

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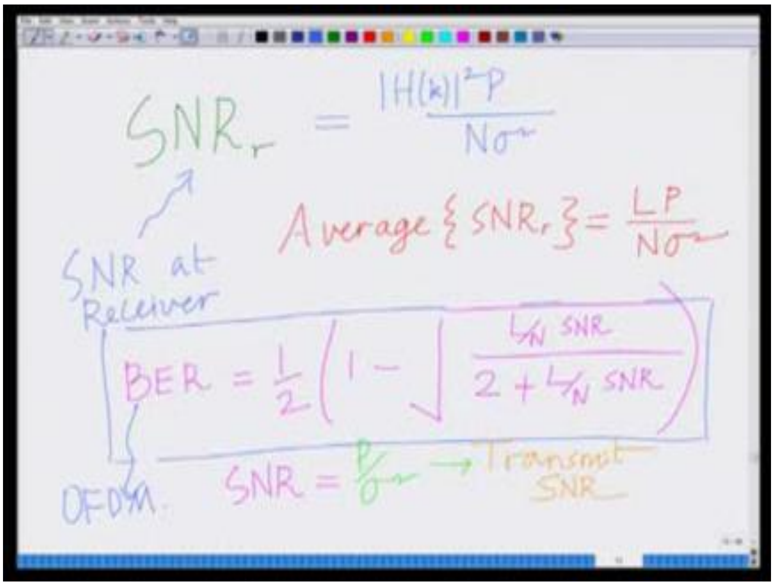

So, what we are saying is the following thing the SNR at the receiver, this SNR at receiver SNR at equals

$$\mathrm{SNR_r} = \frac{|H(k)|^2 P}{N\,\sigma^2}$$

$$\text{Average } \mathrm{SNR_r} = \frac{L\,P}{N\,\sigma^2}$$

So, the way to talk about this is the average of the received SNR is $\frac{L\,P}{N\,\sigma^2}$, therefore, since the channel is Rayleigh fading the Bit Error Rate for BPSK transmission is

$$\mathrm{BER} = \frac{1}{2}\left(1 - \sqrt{\frac{\frac{L}{N}SNR}{2 + \frac{L}{N}SNR}}\right)$$

$$\mathrm{SNR} = \frac{P}{\sigma^2}$$

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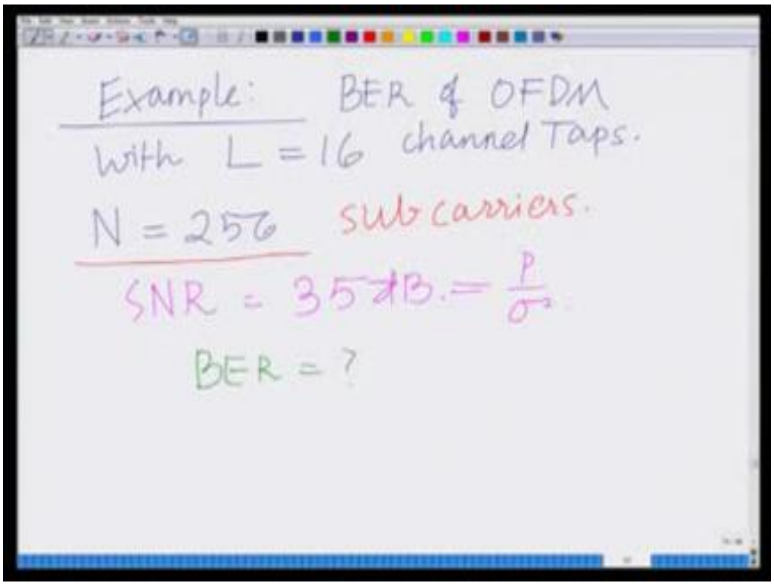


Let us do a simple example to understand this Bit Error Rate. So, what I am doing is I am trying to do a simple example to understand the Bit Error Rate of OFDM system. So, we want to calculate the Bit Error Rate of OFDM system with L = 16 channel taps, N = 256 sub carriers and my SNR = 35 d B, which is basically my $\frac{P}{\sigma^2}$ and my question is for this system.

What is the Bit Error Rate this OFDM system ?

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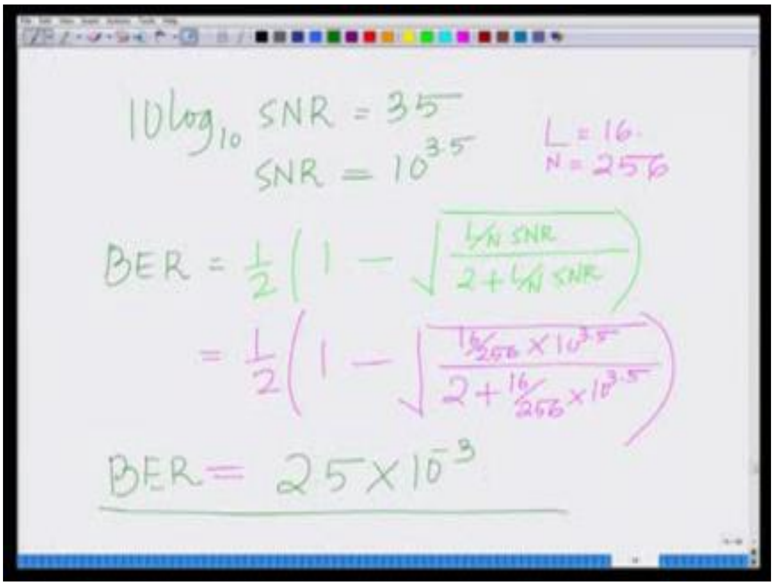


And as we know, we have

$$SNR_{dB} = 10 \log_{10} SNR = 35$$

$$SNR = 10^{3.5}$$

$$BER = \frac{1}{2}\left(1 - \sqrt{\frac{\frac{L}{N}SNR}{2 + \frac{L}{N}SNR}}\right)$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{\frac{16}{256}SNR}{2 + \frac{16}{256}SNR}}\right)$$

$$= 2.5 \times 10^{-3}$$

So, this is the bit error rate of the OFDM system.

So for this example, we have been using the relation that we have derived for the bit error rate, we have derived the fact that the Bit Error Rate of this system is $2.5 \times 10^{-3}$. So, we have been able to show you considered an OFDM system right we have look we have added noise, so now we start we considered in this module we have considered the noise at the receiver, which we have previously ignored then we looked at the effect of the noise in particular, we have derived the property the statistical properties of the noise that is the mean and the noise power we have also derive characterize the channel coefficient of approaching sub carrier. Hence we have been able to derived the bit error rate performance of this OFDM system approaching sub carrier and we have also seen a simple example to calculate the bit error rate performance of this OFDM system with this we will conclude this module.

Thank you very much.