Principles of Modern CDMA/MIMO/OFDM Wireless Communications
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Lecture – 44
SVD Based MIMO Transmission

Hello, welcome to another module in this massive open online course on the Principles of the CDMA, MIMO, OFDM by Wireless Communication System. So, in the previous module we have looked at SVD or Singular Value Decomposition based MIMO wireless transmission, we have looked at how the Singular Value decomposed, that first what is the singular value decomposition? How it can be applied in the context of MIMO transmission? That is the post processing at the receiver, the pre coding at the transmitter, the special multiplexing or the decoupled special channels and finally, the Optimal Power Allocation across the various sub channels to maximize the some rate or achieve the Shannon capacity of the MIMO wireless channel.

Now, to better understand all these aspects, let us look at a simple example which will summarize these aspects related to SVD or singular Value Decomposition based MIMO channel.

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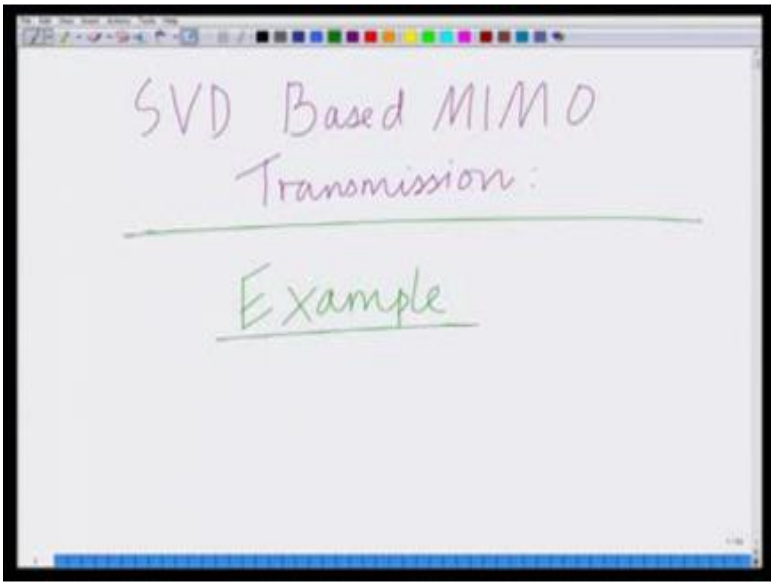

So, today we are going to look at SVD Based MIMO transmission, but we are going to consider a simple example to help understand these aspects comprehensively.

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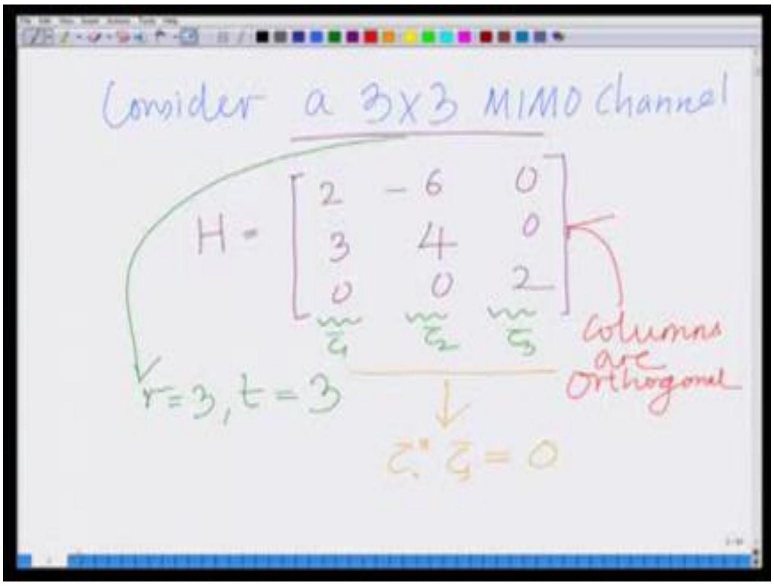

So, let us say we are with 3 cross 3 MIMO channel, that is the MIMO channel matrix is H which is given as

$$H = \begin{bmatrix} 2 & -6 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which means that basically my r = 3, my t = 3 that is number of receive antennas is equal to 3, number of transmit antennas is equal to 3, my channel matrix h is this 3 x 3 matrix. Now let us denotes these columns by $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$.

Now, you can see that in this MIMO channel matrix, in this 3 x 3 MIMO channel matrix which has 3 columns, $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$, these 3 columns are orthogonal. You can see that, if I take any 2 columns out of these 3 columns, I will have $\bar{c}_i^H \bar{c}_j = 0$. On this matrix the columns are orthogonal, for instance let us look at $\bar{c}_1$, $\bar{c}_2$.

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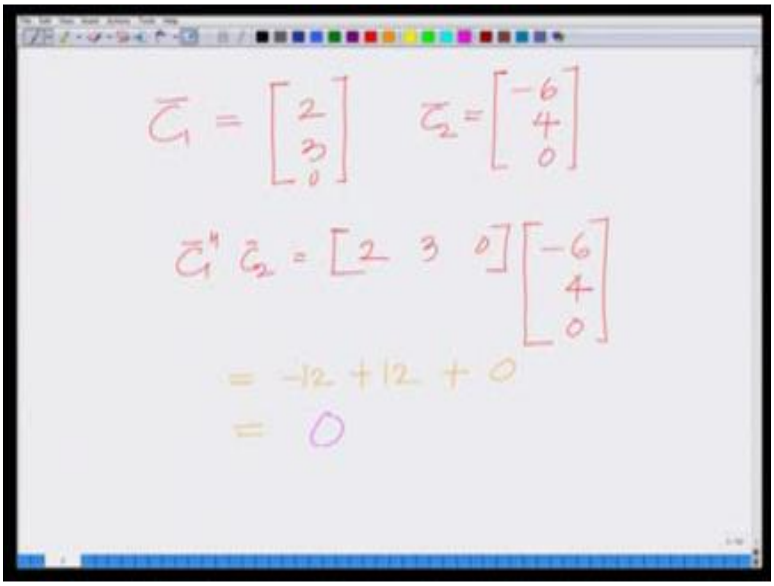

So, if I look

$$\bar{c}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \bar{c}_2 = \begin{bmatrix} -6 \\ 4 \\ 0 \end{bmatrix}$$

$$\bar{c}_1{}^H \bar{c}_2 = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -6 \\ 4 \\ 0 \end{bmatrix}$$

$$= -12 + 12 + 0 = 0$$

All the columns of this 3 x 3 channel matrix are orthogonal. How do we do the singular value decomposition? I am going to illustrate now how to perform the singular value decomposition.

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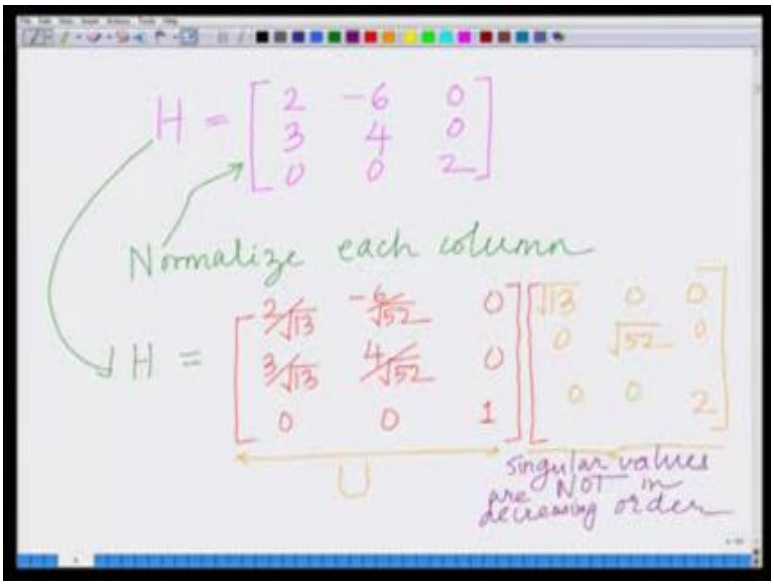

Let us look at, so the SVD can be found as follows

$$H = \begin{bmatrix} 2 & -6 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

So, let us look at this channel matrix, now you can see I can normalize each column that is

$$H = \begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{-6}{\sqrt{13}} & 0 \\ \frac{3}{\sqrt{13}} & \frac{4}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{13} & 0 & 0 \\ 0 & \sqrt{52} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H = \begin{bmatrix} \frac{-6}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{4}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{52} & 0 \\ \sqrt{13} & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H = \begin{bmatrix} \frac{-6}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{4}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{52} & 0 & 0 \\ 0 & \sqrt{13} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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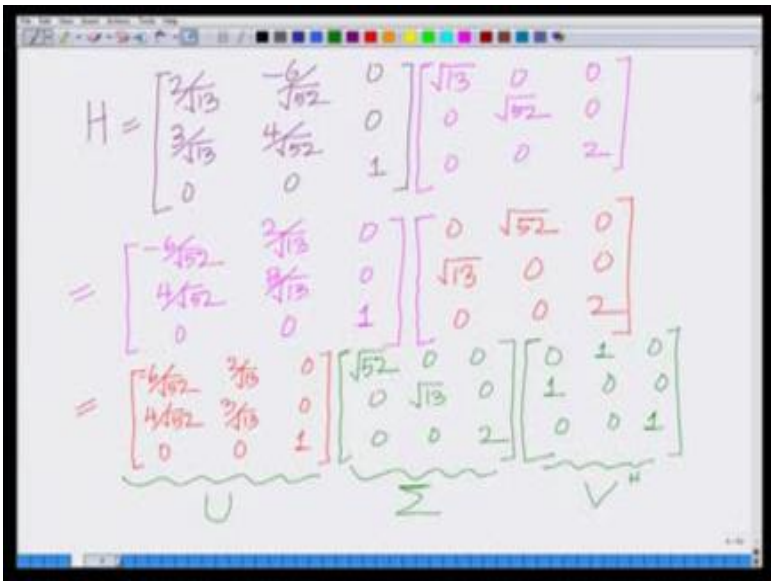

So, now you can see I have my matrix U, I have my diagonal matrix Σ, which has the non negative singular values in decreasing orders, and I have my matrix V^H and now if you can verify this matrixes U, Σ and V satisfy the properties.

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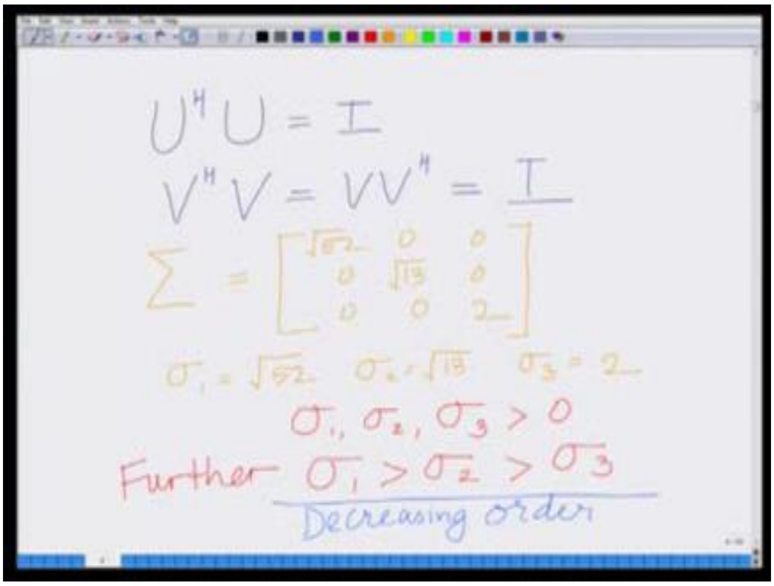

The matrixes U, Σ, V, I have hermition,

$U^H U = I$

$V^H V = V V^H = \mathrm{I}$

$\sigma_1 = \sqrt{52}$, $\sigma_2 = \sqrt{13}$, $\sigma_3 = 2$

$\sigma_1 > \sigma_2 > \sigma_3$

This forms a valid singular value decomposition for the matrix H, that is the 3 x 3 channel matrix H, that has been described in the beginning of this SVD Based MIMO transmission example. So, now, let us proceed with the post processing and preprocessing. So, now, we have the matrixes U, Σ, V^H.

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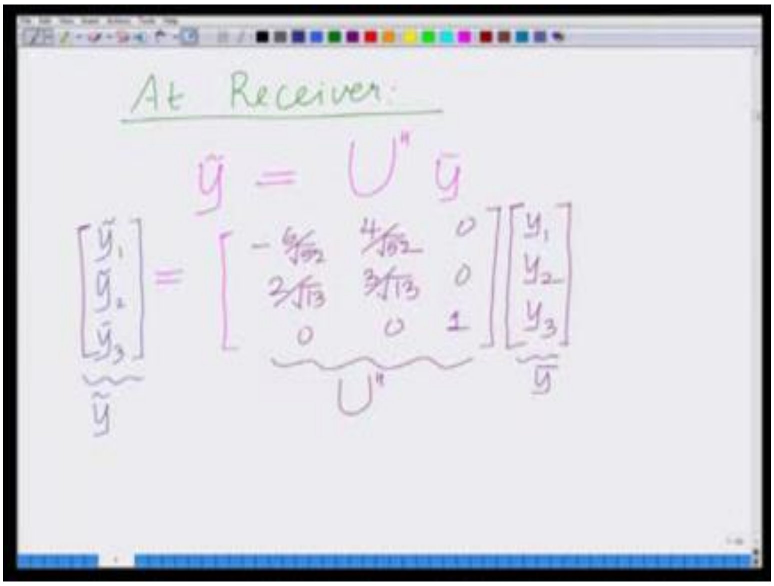

As we said at receiver we perform

$\tilde{y} = U^H \bar{y}$

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} = \begin{bmatrix} \frac{-6}{\sqrt{13}} & \frac{4}{\sqrt{13}} & 0 \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

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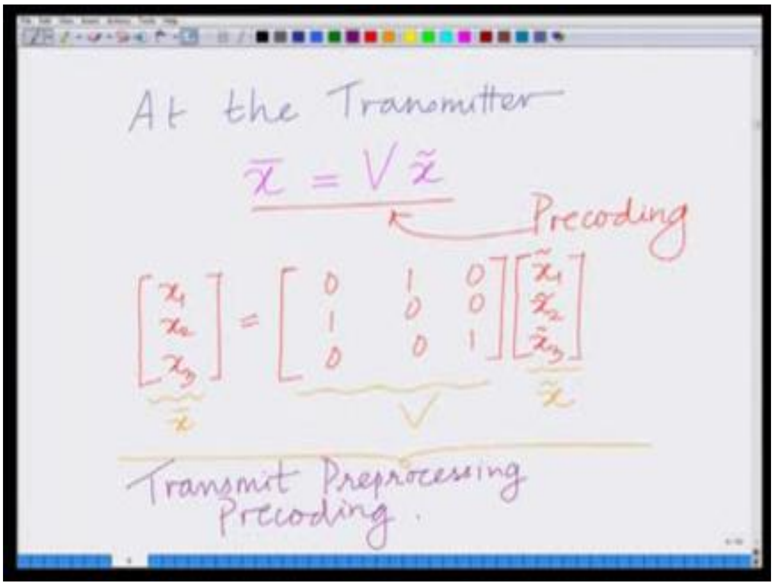

Now, let us look at the preprocessing at the transmitter, at the transmitter remember I perform $\bar{x}$ equals,

$$\bar{x} = V\tilde{x}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_t \end{bmatrix}$$

This step is also known as "Transmit Preprocessing or Pre Coding".

So, this step is also known as the Transmit Pre processing step in SVD Based MIMO transmission or basically pre coding. Now after this what I am got at the receiver as we already seen this in the previous module, I am going to have $\tilde{y} = \Sigma\tilde{x} + \tilde{W}$

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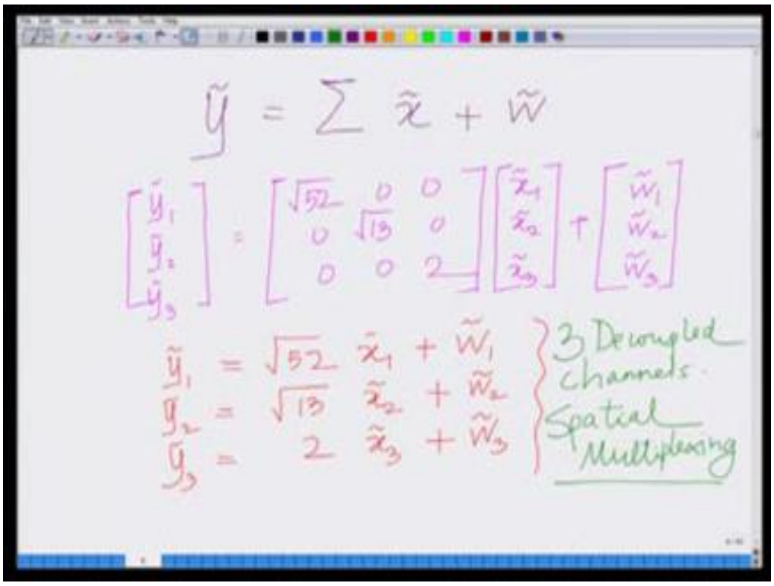

So, what I am going to have at the receiver is I am going to have

$$\tilde{y} = \Sigma\, \tilde{x} + \tilde{W}$$

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} = \begin{bmatrix} \sqrt{52} & 0 & 0 \\ 0 & \sqrt{13} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \tilde{W}_t \end{bmatrix}$$

So, what do I have as a result I have 3 Decoupled channels and this is basically this we said Spatial Multiplexing, Why is this Spatial Multiplexing? Because now I am able to transmit 3 symbols, what are the 3 symbols $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ on the same frequency, at the same time. Because I am using the spatial dimension in the 3 cross MIMO channel I am using the 3 transmit and 3 receive antenna to simultaneously multiplex 3 symbols $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ across the MIMO wireless channel, this is also known as “Spatial Multiplexing”, that is what we have seen in the previous module this is known as Spatial Multiplex.

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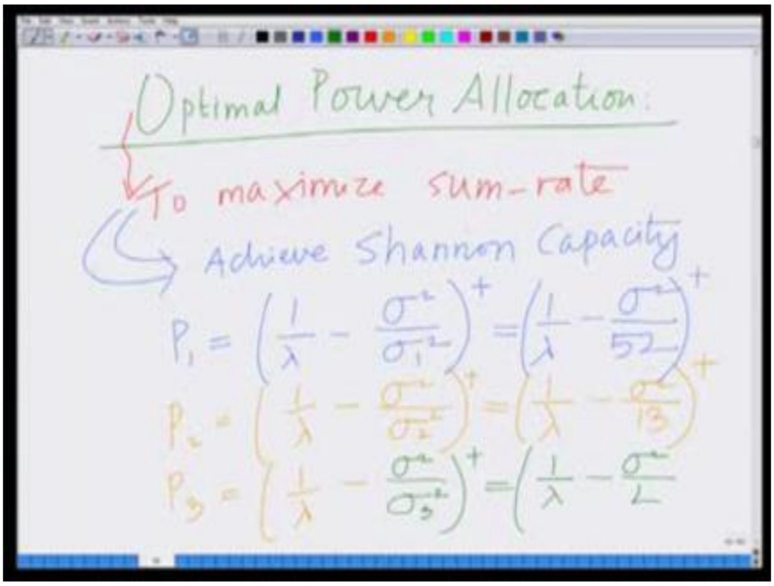

Now, let us look at Optimal Power Allocation, in this Optimal Power Allocation that is if the transmit power is P, Optimal Power Allocation for what? Optimal Power Allocation to maximize some rate that is basically, which implies maximizing the some rate, this automatically implies to achieve capacity or Shannon capacity I have

$$P_1 = \left(\frac{1}{\lambda} - \frac{\sigma^2}{{\sigma_1}^2}\right)^+ = \left(\frac{1}{\lambda} - \frac{\sigma^2}{52}\right)^+$$

$$P_2 = \left(\frac{1}{\lambda} - \frac{\sigma^2}{{\sigma_2}^2}\right)^+ = \left(\frac{1}{\lambda} - \frac{\sigma^2}{13}\right)^+$$

$$P_3 = \left(\frac{1}{\lambda} - \frac{\sigma^2}{{\sigma_3}^2}\right)^+ = \left(\frac{1}{\lambda} - \frac{\sigma^2}{4}\right)^+$$

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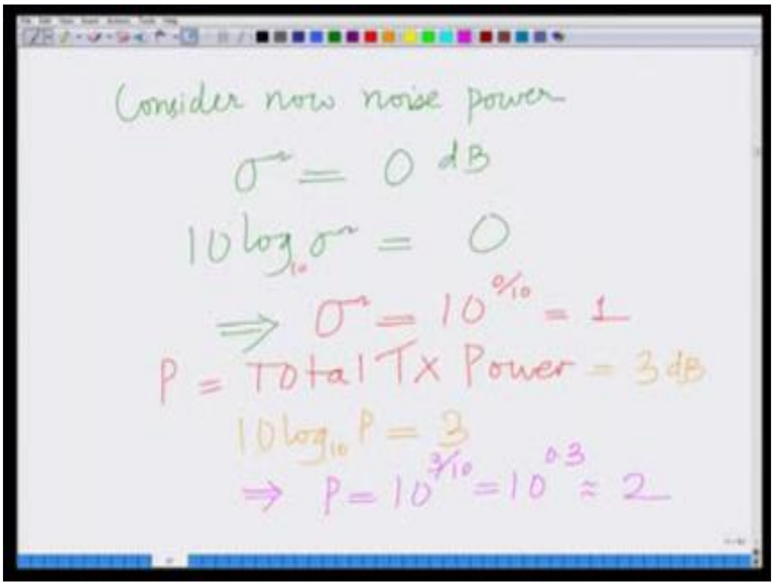

Now, to complete this example let us consider a noise power $\sigma^2 = 0$ dB, which implies

$$10 \log_{10} \sigma^2 = 0$$

$$\sigma^2 = 1$$

So, the noise power is 1, let us consider P equals the total power, that the total transmit power $P = 3$ dB, which means

$$10 \log_{10} P = 3$$

$$P = 10^{0.3} = 2$$

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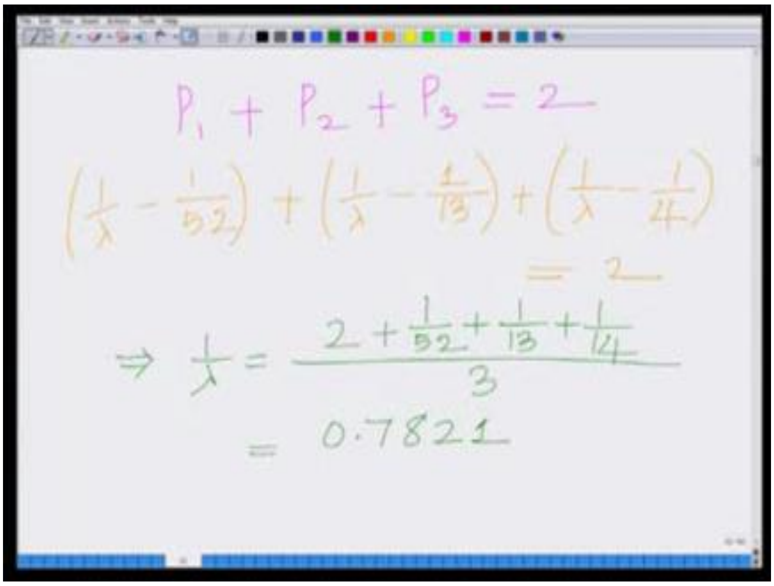

Therefore we must have

$$P_1 + P_2 + P_3 = 2$$

which means if I now substitute

$$\frac{1}{\lambda} - \frac{1}{52} + \frac{1}{\lambda} - \frac{1}{13} + \frac{1}{\lambda} - \frac{1}{4} = 2$$

$$\frac{1}{\lambda} = 0.7821$$

So, what I am doing I am finding the power, each of the power Optimal Power Allocation, the sum of the power must be equal to the total transmit power that is 3 dB or 2, from this equation what I am finding this quantity 1 over lambda.

One over lambda where lambda is the Lagrange multiplier of the optimization problem, remember that. So, I found out $\frac{1}{\lambda}$ is 0.7821, I do not need to find λ because really all I need is $\frac{1}{\lambda}$ to find each of the powers. Now substituting this $\frac{1}{\lambda}$, I can find each of the powers so power 1.

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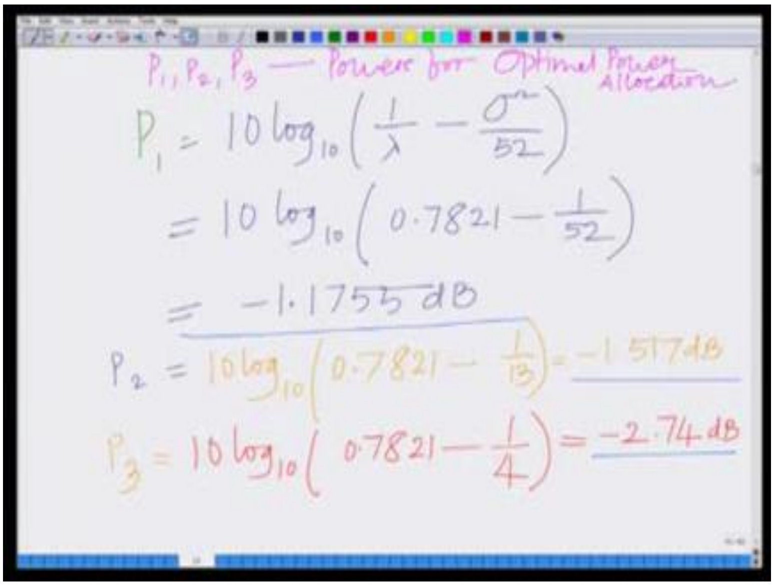

So, if I look at this.

$$P_1 = 10 \log_{10} \left(0.7821 - \frac{\sigma^2}{52} \right) = -1.1755\text{dB}$$

$$P_2 = 10 \log_{10} \left(0.7821 - \frac{\sigma^2}{13} \right) = -1.517\text{dB}$$

$$P_3 = 10 \log_{10} \left(0.7821 - \frac{\sigma^2}{4} \right) = -2.74\text{dB}$$

These are the powers for Optimal Power Allocation, what we have said is these are the power is that maximize the some rate of the MIMO wireless channel or basically achieve the Shannon capacity of the MIMO wireless channel and if you observe closely you will also observe something interesting that the $P_1 =$ - 1.17 dB, $P_2 =$ - 1.51 dB, $P_3 =$- 2.74 dB.

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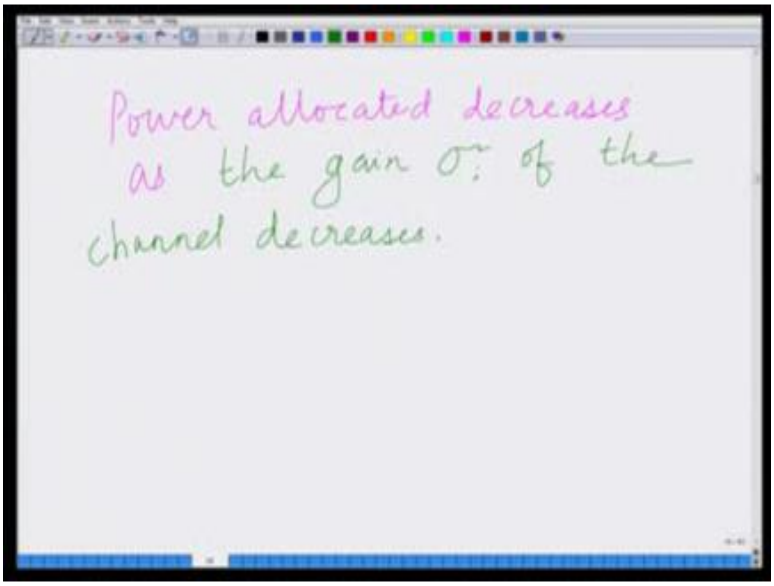

You will observe that the power is decreasing, Power allocated decreases as the gain σ_i^2, you will observe interestingly that is, for various channels i, which have a higher gain, that is higher value of σ_i^2, have a higher power, the channel which have a lower value of σ_i^2,, that is lower gain, have lower power.

So, the power allocated in the Optimal Power Allocation decreases as the gain of the channel progressively decreases, which means very interestingly you are allocating lesser power to poorer channel and your allocating more power to stronger channel and that helps you optimize the power allocation for MIMO wireless system towards maximizing the transmitted rate between the transmitter and receiver, we also said this algorithm known as the "Water Filling Power Allocation".

The Water Filling Power Allocation basically states that, in a Weasel where the weasel height is high, the water level or the height of the water level is the basically the net water level, which is basically level of water minus as the height of the weasel is lower. That is the depth of the water is greater, where the depth of the water level is. So, this is water filling allocation, it basically translates into the fact that in the MIMO power allocation more power is allocated to the stronger channels, less power is allocated to the weaker channel.

Therefore in this module, we are comprehensively seen of an example of SVD Based MIMO transmission, in which we have considered a simple 3 x 3 MIMO wireless channel, we have looked at the a Singular Value Decomposition of this MIMO wireless channel, the post processing at the transmitter, the post processing at the receiver, the preprocessing or pre coding at the transmitter, the diagonalization into the decoupled MIMO channels, spatial multiplexing and then finally, Optimal Power Allocation. So, we will conclude this module here.

Thank you very much.