Principles of Modern CDMA/MIMO/OFDM Wireless Communications
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Lecture – 41
Singular Value Decomposition (SVD)

Hello, welcome to another module in this massive open online course on the principles of CDMA/ MIMO/ OFDM communication systems. So, far we have seen various schemes for MIMO transmission and reception we have seen the Zero Forcing Receiver, we have seen transmit Beam Forming Alamouti Code. Let us now look at a slightly different perspective or MIMO, which is first let us look up look at a unique decomposition of the MIMO wireless channel this is known as the SVD or the Singular Value Decomposition.

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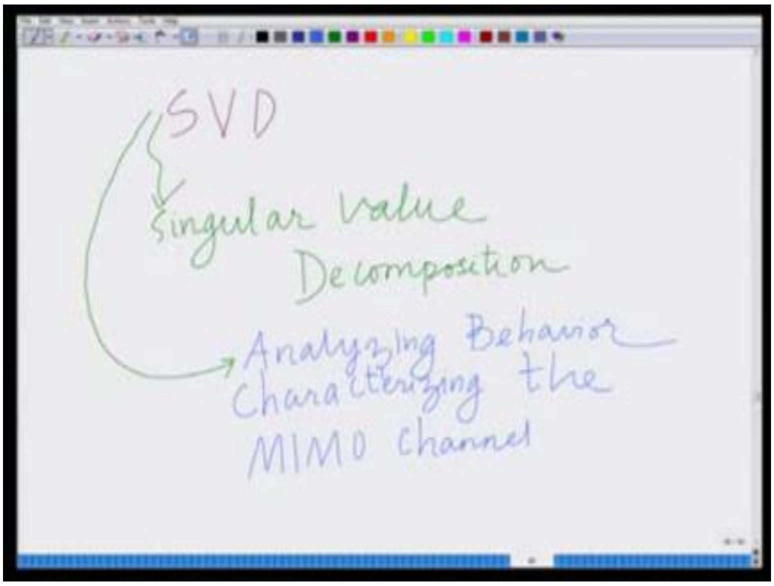

So, what we are going to look at starting in this module is what is known as the SVD, and this stands for the Singular Value Decomposition and what this is? This is very useful in Analysing Behaviour or Characterising the MIMO channel. And what is this? This is a decomposition the Singular Value Decomposition is a decomposition of the channel matrix, just like, we have the Eigen value decomposition this is another decomposition for the MIMO channel matrix, but Eigen value decomposition exist only for square matrices, but the Singular Value Decomposition is a general decomposition and exist also for non-square matrices and therefore, this can be used gain valuable insides in to the properties and behaviour of the appropriate underlying MIMO channel.

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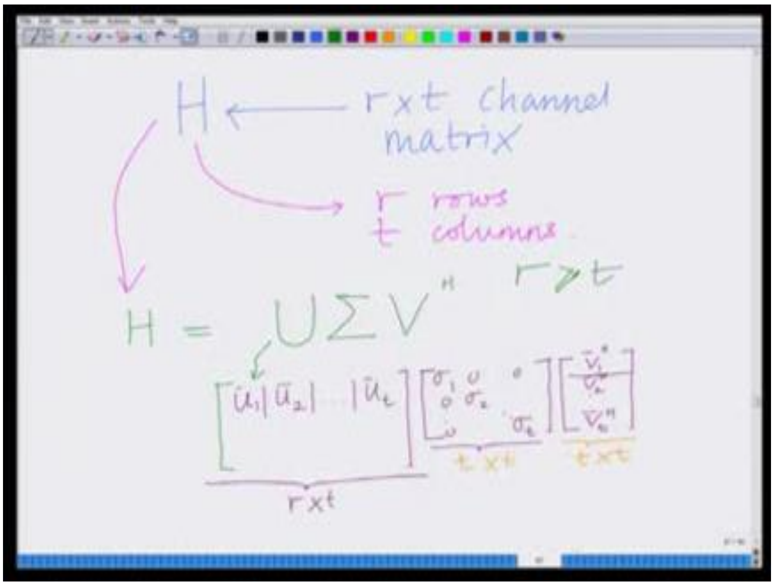

So, let us start with a brief illustration of the Singular Value Decomposition. So, let us consider the channel matrix H which is r x t channel matrix and this has r rows, this has r rows and t columns and this channel matrix H can be decomposed as

$$H = U \Sigma V^{H}$$

$$= \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_t \end{bmatrix} \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_t \end{pmatrix} \begin{bmatrix} \bar{v}_1^{H} \\ \bar{v}_2^{H} \\ \bar{v}_t^{H} \end{bmatrix}$$

And the different matrix is the properties of the different matrix are as follows.

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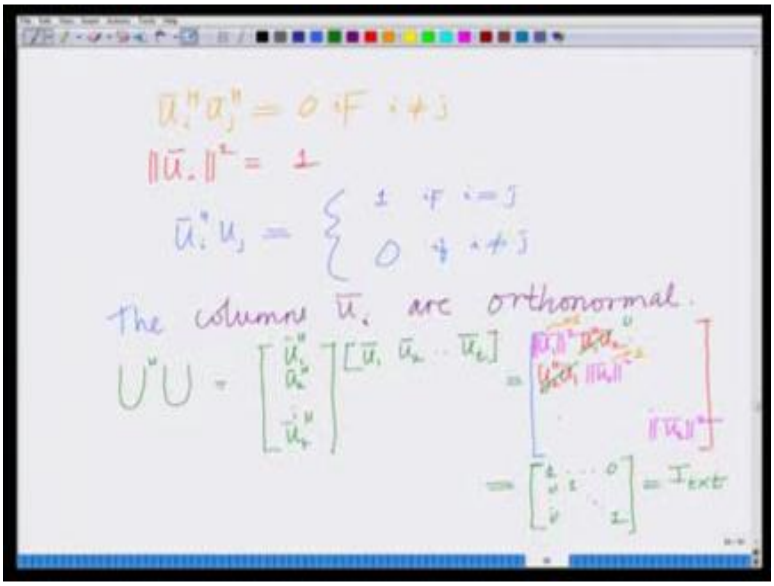

So, I have

$$\bar{u}_i{}^H \bar{u}_j{}^H = 0 \text{ if } i \neq j$$

$$\| \bar{u}_i \|^{2 = 1}$$

This property is designated as stated as follows the columns $\bar{u}_i$ are orthonormal. So, these columns are orthonormal.

Now; that means, if I look at this matrix

$$U^H U = \begin{bmatrix} \bar{u}_1{}^H \\ \bar{u}_2{}^H \\ \bar{u}_t{}^H \end{bmatrix} [\bar{u}_1 \quad \bar{u}_2 \quad \bar{u}_t\ ] = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = \mathrm{I}_{3x3}$$

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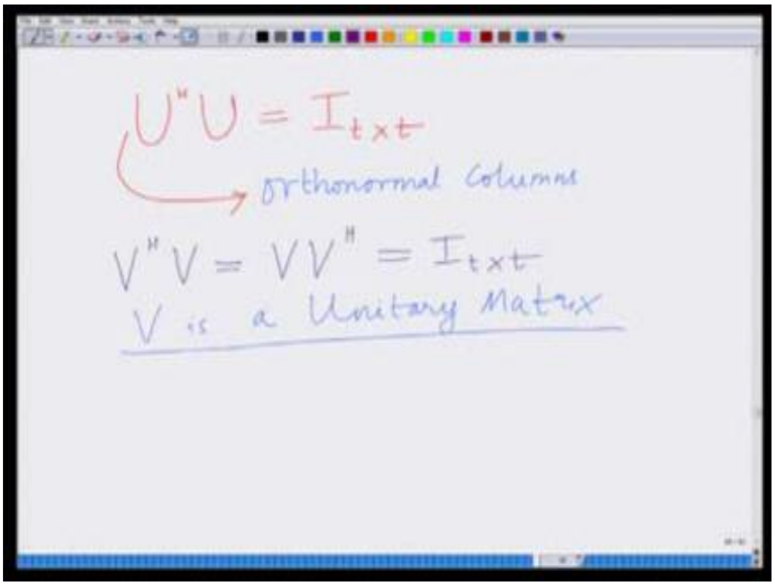

So, we have the first property that is

$U^H U = \mathrm{I}_{txt}$

Now, let us look at the matrix V, the matrix V also be similar property. So,

$V^H V = V V^H = \mathrm{I}_{txt}$

and this matrix V therefore, this is also known as a Unitary matrix, this matrix V is known as a Unitary matrix. Now let us look at the matrix Σ, the matrix Σ which is the matrix of singular values, remember Σ is the diagonal matrix of singular values.

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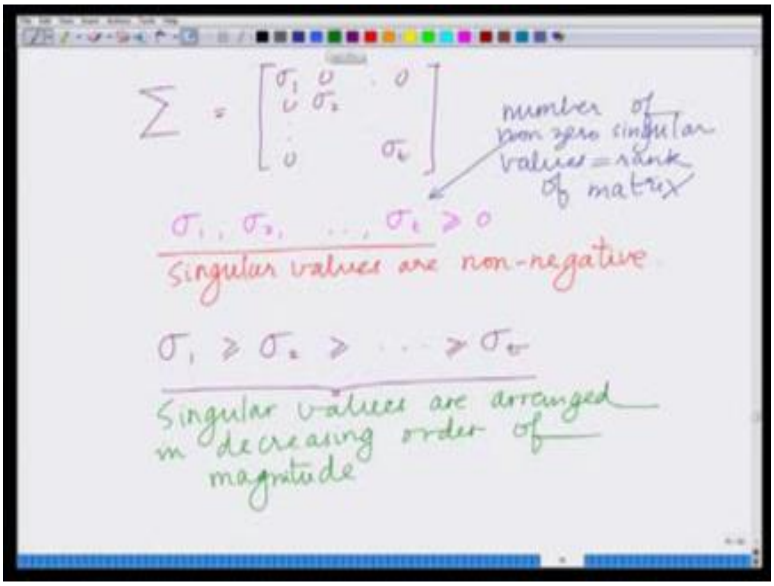

This matrix is such that all the singular values are greater than 0 that is they are non-negative. So, singular values are non-negative and also importantly the singular values are arranged in decreasing order of magnitude. So, basically this says that, the singular values are arranged in decreasing order of magnitude. So, to summarise basically for the Singular Value Decomposition what are the properties of this Singular Value Decomposition you have to summarise them in a Hermitian as, see channel matrix r x t channel matrix H with r greater than or equal to t, can be decomposed as the product of team 3 matrices that is $U \Sigma V^H$, where U is a matrix is an r x t matrix which contains orthonormal columns such that, $U^H U = I_{txt}$; V is a Unitary matrix such that $V^H V = V V^H = I_{txt}$ and Σ is a diagonal matrix of singular values which are non-negative and the singular values are arranged in decreasing order also an interesting property of the singular value is that the number of non-negative singular values denotes the rank of the matrix.

So, number of non-zero singular values is equal to the rank of the matrix and also; let me clearly write this again here that, this Singular Value Decomposition is valid for r greater than or equal to t that is the singular matrix decomposition, that we have just described is valid for r is greater than or equals to t.

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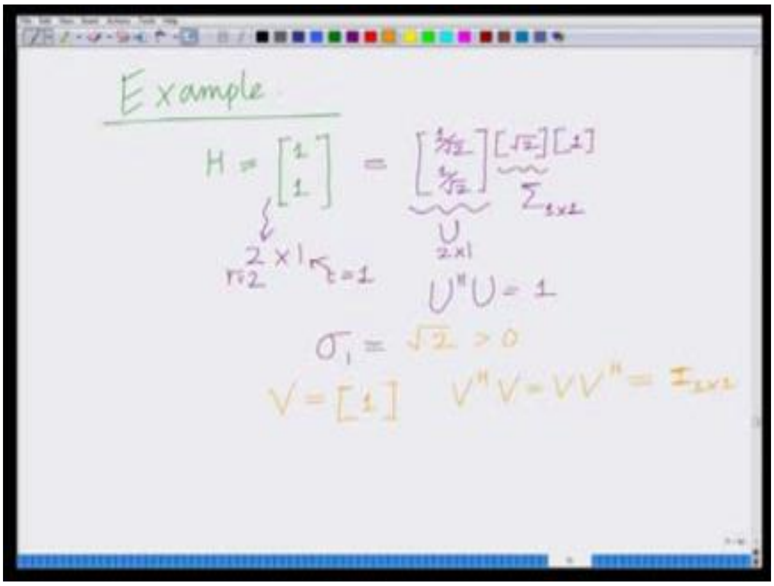

Let us look at some simple examples to understand these things, so, examples of SVD or the Singular Value Decomposition example. So, let us look at lets starts with a very simple example the first case, let us start with this simple example; H is equal to 2 x 1 this is a basically, 2 x 1 MIMO channel. In fact, this is a Single Input Multiple Output as my or a SIMO channel basically, we have r = 2 and t = 1 correct. I can write this now as basically,

$$H = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} [\sqrt{2}\ ]\ [1\ ]$$

So, now look at this I have my matrix U which is 2 x 1 which obeys the property $U^H U = I$. In fact, 1 which is the 1 x 1 identity I have the diagonal matrix of singular value Σ which is a 1 x 1 and root 2, $\sigma_1 = \sqrt{2}$ which is greater than 0. Therefore, $V^H V = V V^H = I_{txt}$ that is 1 x 1, it is the 1 x 1 identity which is equal to which is equals to 1. So, we have looked at a simple case of a Singular Value Decomposition of a Single Input Multiple Output channel in which, both the coefficients are 1 I have said that this can be decomposed as the matrix U which is the 2 x 1 matrix containing the elements $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ naturally its column is orthonormal in this case, there is only 1 column.

So, it is a normal that is normalised to unit norm the singular value is $\sqrt{2}$, which is greater than 0 times, the matrix V which is again the trivial matrix 1 for this scenario. So, that explains that is the simple example of Singular Value Decomposition.

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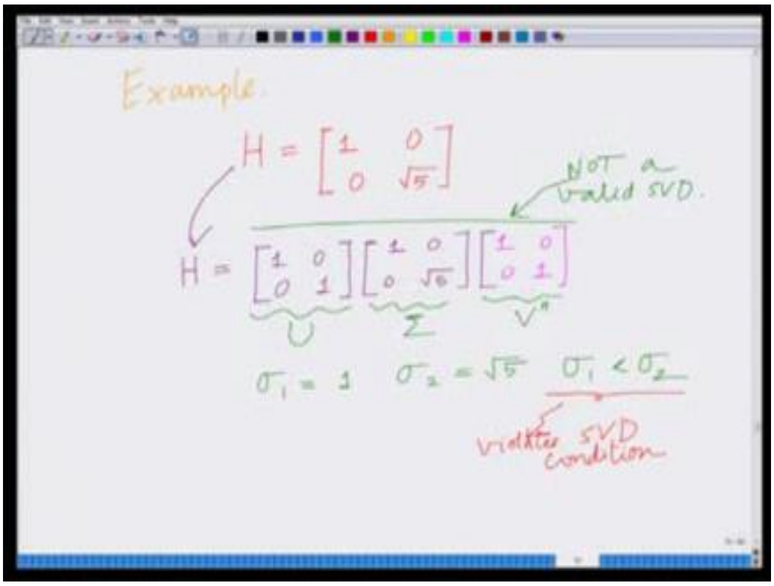

Let us look at a slightly more interesting example; let us look at a slightly more interesting example of the Singular Value Decomposition. I have the channel matrix H which is

$$H = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is a Singular Value Decomposition this is my matrix U, this is my matrix Σ, this is my matrix V^H.

It satisfies all the properties of Singular Value Decomposition except now look at this I have $\sigma_1 = 1$, $\sigma_2 = \sqrt{5}$ therefore, σ_1 is less than σ_2 and this condition violates the SVD condition or the SVD property because, remember the singular values have to arranged in descending order of magnitude or in other words, σ_1 has to be greater than

σ_2 ; however, here σ_1 is less than σ_2 correct, all the other properties are satisfied that is I have U Hermitian U is identity because, U is the Identity matrix $U^H U = V V^H$ is also identity the singular values are positive except the singular values are not arranged in the descending order therefore, we have to rectify this since this is not a valid SVD. So, what I want to say is that, this is not a valid SVD, then how do we decompose the Singular Value Decomposition.

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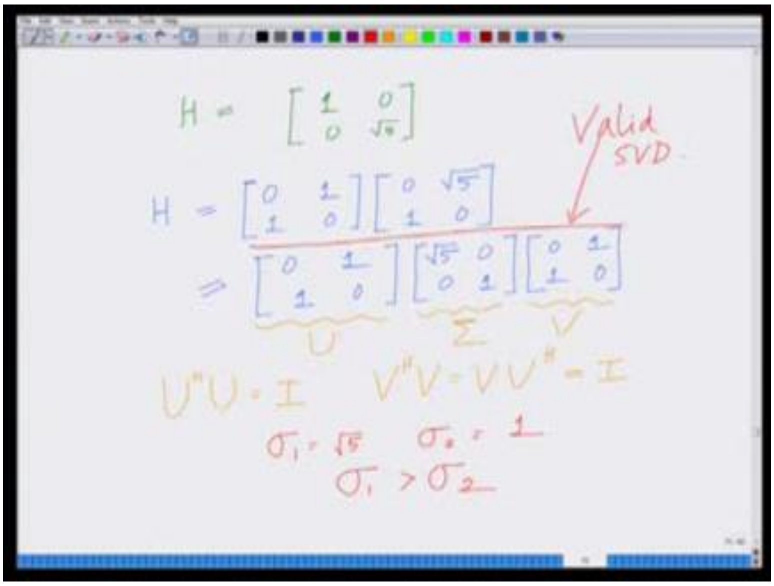

I have H is equal to

$$H = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{5} \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{5} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This is the Flip Identity Matrix and you can see that flips the rows of H. The 2 singular values are $\sigma_1 = \sqrt{5}$, $\sigma_2 = 1$ and I have $\sigma_1 > \sigma_2$ therefore, yes this is a valid. So, this is valid Singular Value Decomposition.

So, what we have done is, we have done a simple manipulation we have pre multiplied and post multiplied this matrix which is the Flip Identity Matrix that basically shuffles the singular value $\sqrt{5}$ and 1 to the correct order. So, that will now we $\sigma_1 > \sigma_2$, both the singular values are positive and therefore, this is the valid Singular Value Decomposition. So, now let us look at a slightly more refined example, let us look at a third example of another Singular Value of Decomposition.

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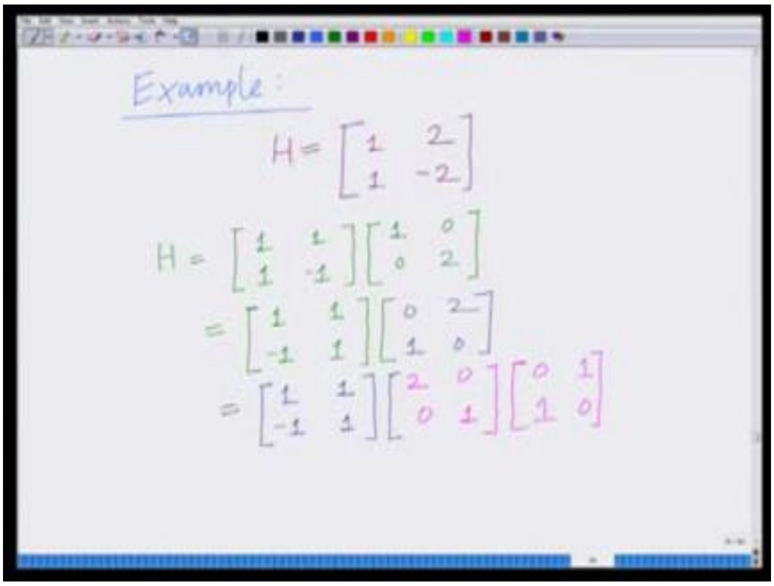

This is a slightly more advanced example of a Singular Value Decomposition that is; I have H is equal to

$$H = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

And now, you can see I have now we can see I still do not satisfy my property of Singular Value Decomposition because, this u matrix does not contain orthonormal rows correct, orthonormal columns. So, for this purpose what I am going to do is I am going to normalise the columns of this U matrix.

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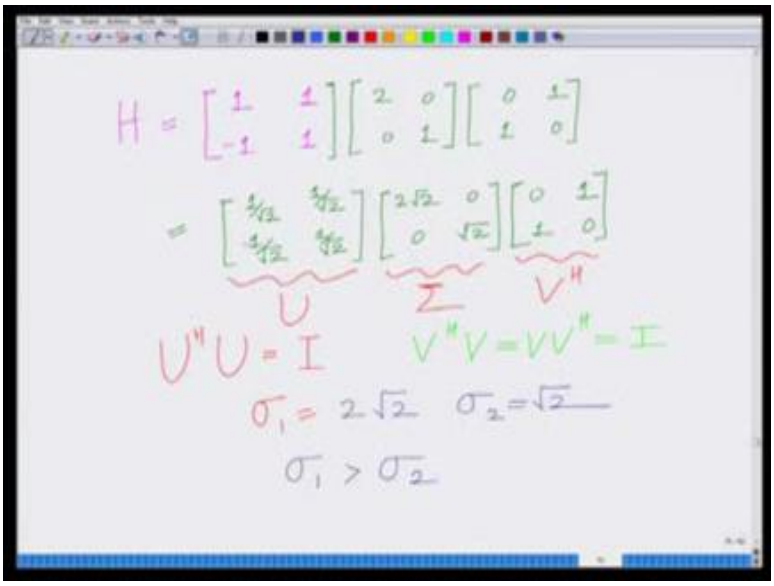

So, I have so, far I have H is equal to

$$H = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

You can see I have well now you can see, I have the matrix U, I have the matrix sigma, I have the matrix U equals identity, $U^H U = V V^H = I,$ the 2 singular values are $\sigma_1 = 2\sqrt{2}$, $\sigma_2 = \sqrt{2}$ and I have $\sigma_1 > \sigma_2$ and both the singular values are greater than 0.

And I have $\sigma_1 > \sigma_2$ that is the 2 singular values are arranged in descending order therefore, once again this satisfies all the properties of the Singular Value Decomposition therefore, this is a valid Singular Value Decomposition. So, since what we have done in this module is, we have defined a new decomposition for the MIMO channel matrix that is a Singular Value Decomposition we have defined the various components or the various component matrices of the Singular Value Decomposition namely, the matrix is U sigma V and we have illustrated the Singular Value Decomposition using some simple examples and that comprehensively clarifies this concept of Singular Value Decomposition. Now the application of Singular Value Decomposition is something that we are going to look at in the subsequent module. Let us conclude this module here.

Thank you very much.