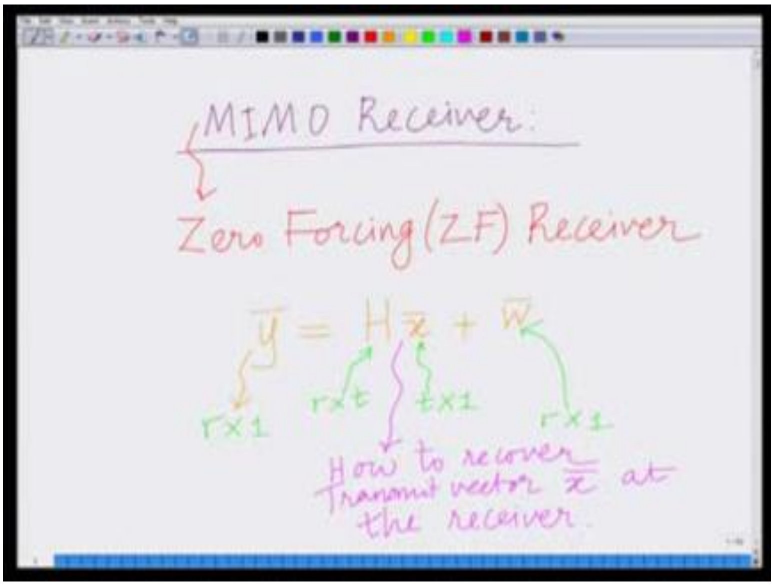
Principles of Modern CDMA/MIMO/OFDM Wireless Communications
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 36
MIMO Receivers

Hello, welcome to another module. In this massive open online course in the principles of CDMA, MIMO, OFDM wireless communication systems. In the previous modules, we have introduced the paradigm or the model for MIMO wireless communication system. We have looked at the matrix channel model of MIMO wireless communication system, and we have also looked at the various special cases of MIMO wireless communication system. Let us not delve deeper into the theory of MIMO communication systems. So, today what we are going to discuss is the receiver in the MIMO wireless system.

(Refer Slide Time: 00:46)



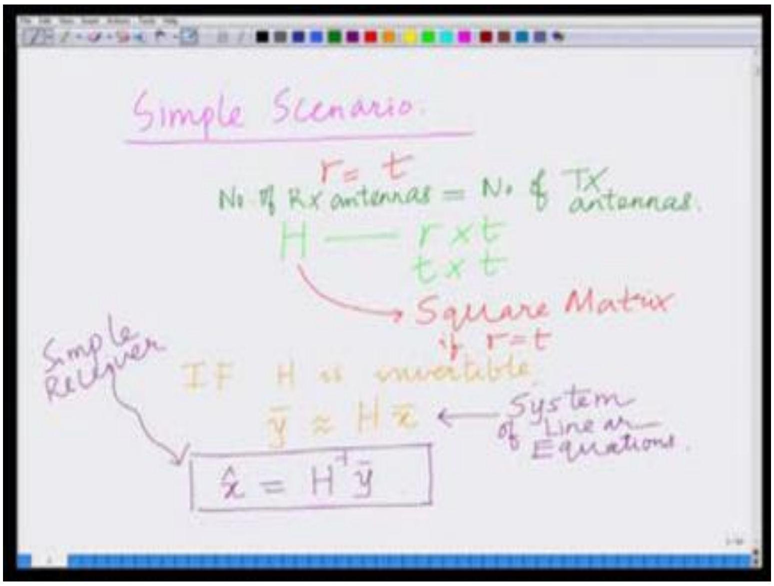
We are going to focus on MIMO receiver that is what is the signal processing that has to be carried out at the receiver. In particular we will look at what is known as the zero forcing or ZF Receiver. Now remember, we have our MIMO system model. Our MIMO system model is given as

$$\bar{y} = H\bar{x} + \bar{w}$$

Let me refresh your memory $\bar{y}$, this is r x 1 vector, $\bar{x}$ is the t x 1 transmit vector and H is the r x t transmit channel matrix $\bar{w}$ is the r x 1 noise vector.

Now the question is at the receiver how do we recover this transmitted vector $\bar{x}$. So, we are observing y bar that is $\bar{y}$ is the received vector, now from $\bar{y}$ how do we recover $\bar{x}$, that is how do we estimate $\bar{x}$. This is the problem of reception in the MIMO system. So, how the question we want to ask is how to recover $\bar{x}$? How to recover the transmitted symbols, how to recover transmit vector $\bar{x}$ at the receiver. Now, that is how to recover $\bar{x}$ from $\bar{y}$.

(Refer Slide Time: 02:52)



Now, let us look at a simple scenario. If for instance, let us look at a simple scenario. For this simple scenario, let us assume that r = t, if r = t remember our matrix H which is an r x t matrix is basically it is a t x t matrix or in other words, H is a square matrix, if r = t. Therefore, what we are saying is our matrix, that is if r = t that is the number of received antenna's is equal to number of transmit antenna's, then the matrix H which is r cross t is essentially a t x t matrix since r = t therefore, it is a square matrix . Now, therefore, if H is invertible that is not all square matrices are invertible some square matrices also need not be invertible.

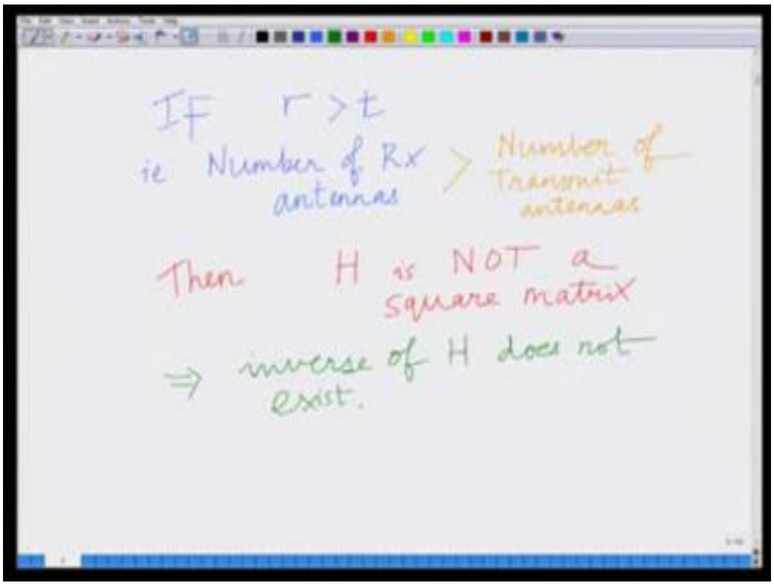
So if H is invertible then let us consider, if H is invertible, then a simple receiver for this scenario is to use the system of linear equations $\bar{y} \approx H\bar{x}$, this is our system of liner equations, that is we are ignoring the noise. So, this is our system of linear, this is our system of linear equations and since H is invertible I can recover $\hat{x}$ as

$$\hat{x} = H^{-1}y$$

This is our simple receiver structure. If r = t that is number of received antennas, is equal to number of transmitting antennas. What we are saying is basically, if r is equal to t that is the number of received antennas is equal to number of transmit antennas, then the channel matrix H is square matrix and therefore, if H is invertible at the receiver, I can simply perform $H^{-1}y$ and from $H^{-1}y$ I can obtain x. So, I can use the equation $\bar{y} \approx H\bar{x}$, I can treat this as a system of linear equations and I can solve for the estimate $\hat{x}$ as $\hat{x} = H^{-1}y$, that is the simple MIMO receiver.

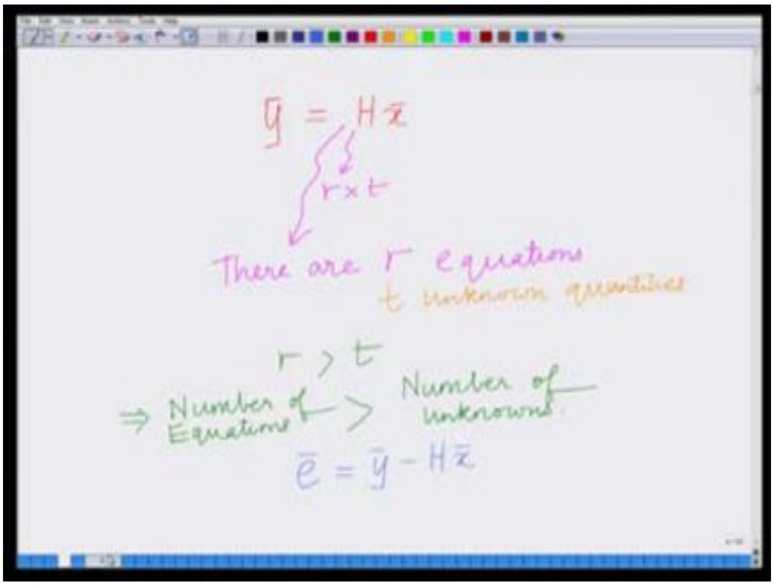
But remember as we said, as we qualified this is always do not always possible. Why is this not always possible? Because H even though $r = t$ that is H is a square matrix H need not be invertible. Further the most important and interesting scenario occurs, when r is not equal to t that is when $r > t$. Then the matrix is not a square matrix, so while we can use this if $r = t$ and H is invertible.

(Refer Slide Time: 06:42)



If r > t, that is strictly greater than t, that is number of received antennas is strictly greater than number of transmit antennas, then H is not a square matrix. Which basically implies that H that is inverse of H, does not exist there is nothing known as an inverse of a non square matrix. Since H is a non square matrix, the inverse of H does not exist and in such a scenario how do we reconstruct or how do we recover the transmitted symbol vector x from the observed symbol vector y. For instance, if the number of receive antennas is 3 number of transmit antennas is 2, then the matrix H is a 3 x 2 matrix, that is nothing like a inverse of 3 x 2 matrix right. So, in this scenario how do you recover the transmitted symbol vector x from the received symbol vector y and that is why we need a special concept or we need a specialized architecture for this MIMO receiver which we are going to discuss below.

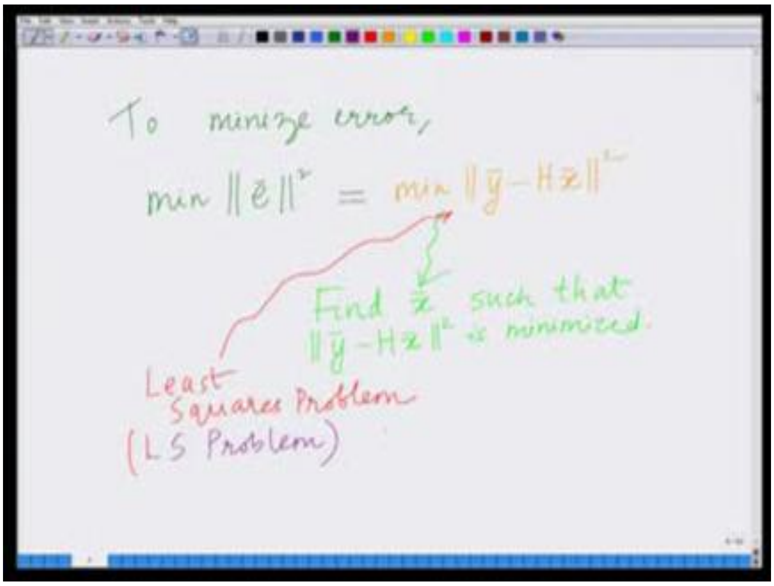
(Refer Slide Time: 08:47)



Now, let us look at what is happening, when $r > t$. Let us look at this system of equations. Let us look at $\bar{y} = H\bar{x}$ now H is $r \times t$, which implies this systems of equations, there are basically r equations, and what are the unknowns? The unknowns are basically, the elements of the vector x, so there are t unknown quantities. So, basically $r > t$, if r is greater than t this implies that the number of equations is greater than the number of unknowns. Therefore, this is an inconsistent system of equation that is we have a scenario where we have more equations than unknown, therefore, we cannot exactly solve for this vector x from this system of linear equation. Therefore, what is the best that we can do in this scenario, the best that we can do in this scenario is the following. Since we cannot exactly solve for x, what we cannot do is we can denote this error as

$\bar{e} = \bar{y} - H\bar{x}$.

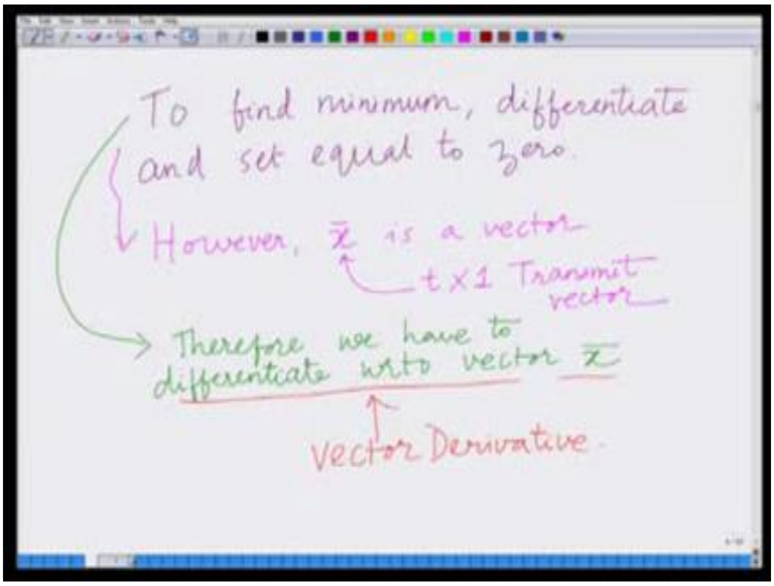
(Refer Slide Time: 10:36)



And the best we can do that is to minimize the error, minimize $||\bar{e}||^2$, which is basically minimizing $||\bar{y} - H\bar{x}||^2$, that is what do we want to do? We want to find the $\bar{x}$ such that the error is minimized; that means, such that $||\bar{y} - H\bar{x}||^2$ is minimized. So, what we want to do? Now you can look at this structure, the structure of this problem is such that it is a square of the norm and we are minimizing the squares, this is known as least squares problem this is also abbreviated as the LS.

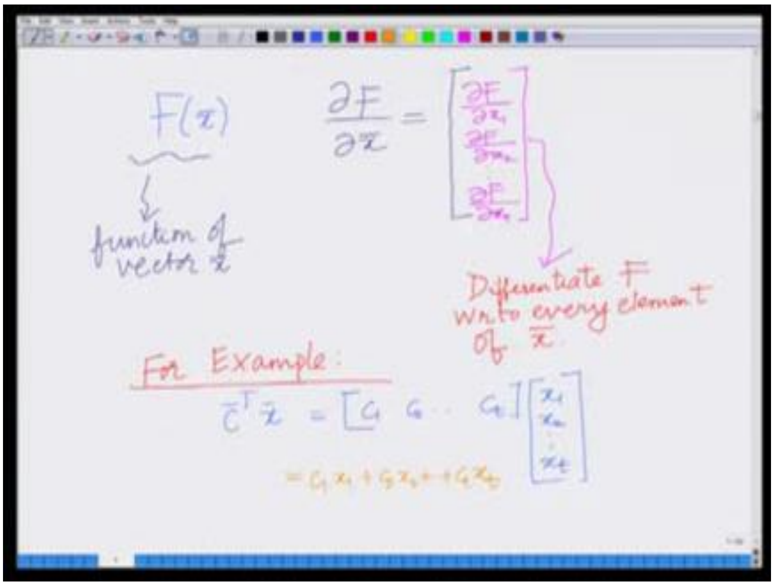
This minimization of $||\bar{y} - H\bar{x}||^2$ is also known as least squares problem or it is also known abbreviated as the LS problem. Therefore, we have to find the minimum of this least squares cost function, which is $||\bar{y} - H\bar{x}||^2$, and how do we find the minimum of a given cost function? To find the minimum of a given cost function, we have to differentiate it and set it equal to zero and find that x where it is equal to zero that gives the minimum, is not it. So, we have to differentiate it and set it equal to zero. The only problem here is that x is a vector. So, we have a vector $\bar{x}$. So, we have to differentiate it with respect to a vector.

(Refer Slide Time: 13:04)



So therefore to find the minimum we have to differentiate and set equal to zero; however, $\bar{x}$ is a vector remember, more specifically $\bar{x}$ is a t x 1 transmit vector. Therefore, we have to differentiate with respect to vector $\bar{x}$, which means we need the concept of what is known as a vector. Many of you must be familiar with the scalar derivative. So, we need now the concept of a vector derivative. Since, we need to differentiate with respective to the vector $\bar{x}$ which is a t dimension vector and this vector derivative can be defined as follows consider a function F of the vector x, so this is any function. This is a function of vector $\bar{x}$.

(Refer Slide Time: 14:49)



Let us say, I have $F(\bar{x})$ which is a function of vector $\bar{x}$, its derivate with respect to $\bar{x}$is simply defined as

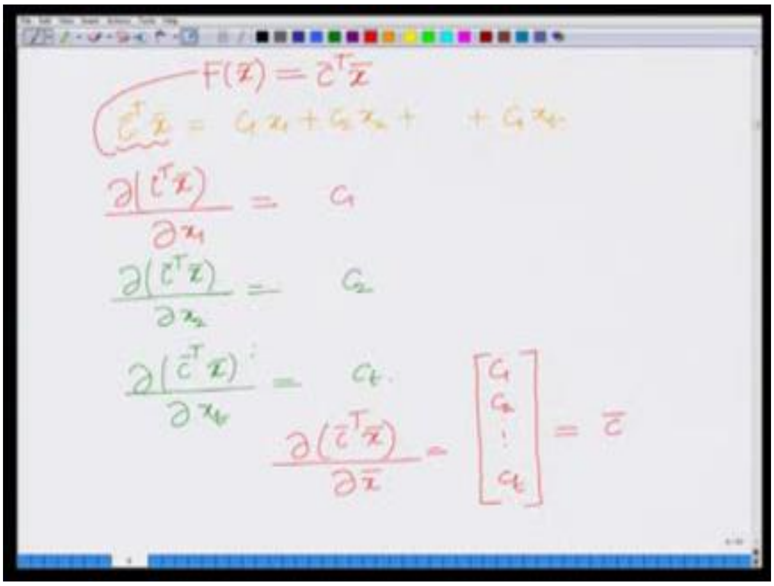
$$\frac{\partial F}{\partial \bar{x}} = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \\ \frac{\partial F}{\partial x_t} \end{bmatrix}$$

So, let us say I have a function F, which is a function of the vector $\bar{x}$, its derivative with respect to $\bar{x}$ is basically differentiate F with respect to every element of the vector x and therefore, it will be a t dimensional vector. So, what is the derivative? That is differentiate F with respect to every element of $\bar{x}$. Now, let us look at a simple example to understand this. For example, let us consider this simple example, which is

$$\bar{c}^{\,T} \bar{x} = \begin{bmatrix} c_1 & c_2 & c_t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_t \end{bmatrix}$$

$$= c_1 x_1 + c_2 x_2 + \ldots + c_t x_t$$

(Refer Slide Time: 17:17)



Now if you look at this, let us look at this,

$$\bar{c}^T\,\bar{x} = c_1x_1 + c_2x_2 + \ldots + c_tx_t$$

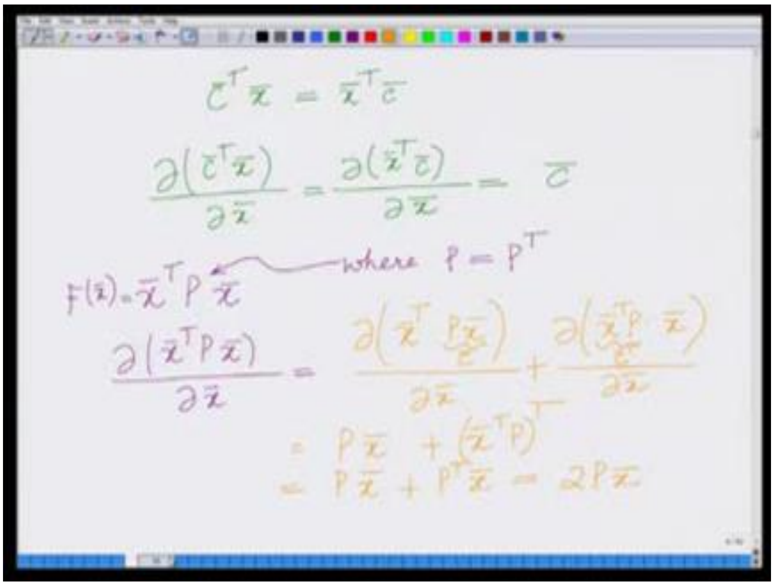
$$\frac{\partial\,(\bar{c}^T\,\bar{x})}{\partial\,x_1} = c_1$$

$$\frac{\partial\,(\bar{c}^T\,\bar{x})}{\partial\,x_2} = c_2$$

$$\frac{\partial\,(\bar{c}^T\,\bar{x})}{\partial\,x_t} = c_t$$

$$\frac{\partial\,(\bar{c}^T\,\bar{x})}{\partial\,\bar{x}} = \begin{bmatrix} c_1 \\ c_2 \\ c_t \end{bmatrix} = \bar{c}$$

(Refer Slide Time: 18:53)



Similarly, let us look at another act, another aspect. Now, we know that

$$\bar{c}^T \bar{x} = \bar{x}^T \bar{c}$$

$$\frac{\partial (\bar{c}^T \bar{x})}{\partial \bar{x}} = \frac{\partial (\bar{x}^T \bar{c})}{\partial \bar{x}} = \bar{c}$$

Now, let us look at another interesting function, let us look at what is known as quadratic form? That is $\bar{x}^T P \bar{x}$, where $P = P^T$, that is P is a symmetric matrix, that is this is also known as a quadratic form, this is our $F(\bar{x})$. Now, to differentiate this with respect to $\bar{x}$, I am going to use the product rule. So, I am going to write this as the derivative of $\bar{x}^T P \bar{x}$

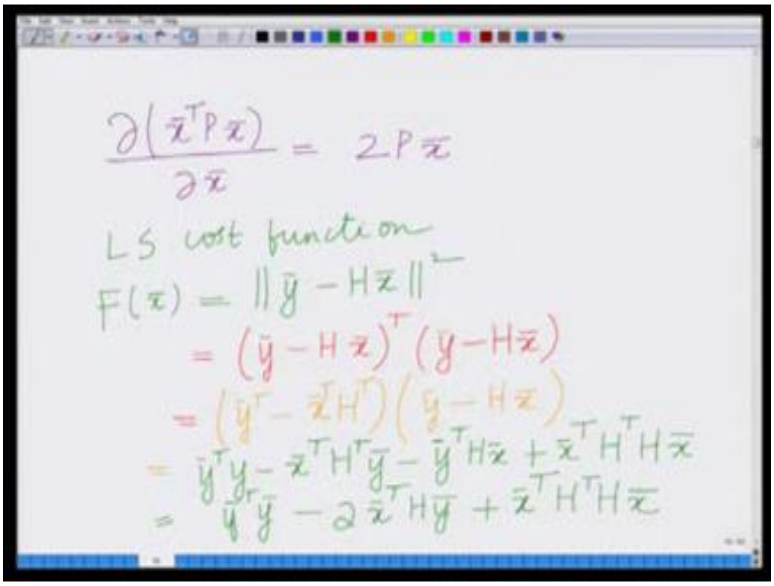
$$\frac{\partial (\bar{x}^T P \bar{x})}{\partial \bar{x}} = \frac{\partial (\bar{x}^T \; P \bar{x})}{\partial \bar{x}} + \frac{\partial (\bar{x}^T P \;\; \bar{x})}{\partial \bar{x}}$$

$$= P \bar{x} + (\bar{x}^T P)^T$$

$$= P \bar{x} + P^T \bar{x}$$

$$= 2 P \bar{x}$$

(Refer Slide Time: 21:23)



So,

$$\frac{\partial\,(\bar{x}^T P\,\bar{x})}{\partial\,\bar{x}} = 2\,P\,\bar{x}$$

We have for the vector derivative where we are differentiating with respect to the vector $\bar{x}$.

Now, let us go back to our cost function that is $||\bar{y} - H\,\bar{x}||^2$, which we are usually concerned with and now let us differentiate this with respect to $\bar{x}$. So, our original cost function, least square cost function or LS cost function that is

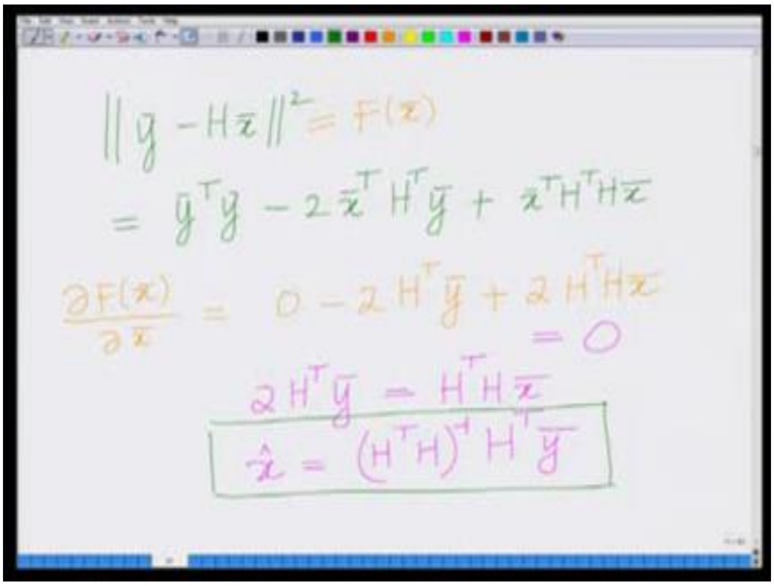
$$F(\bar{x}) = ||\bar{y} - H\,\bar{x}||^2$$

$$= (\bar{y} - H\,\bar{x})^T(\bar{y} - H\,\bar{x})$$

$$= (\bar{y}^T - \bar{x}^T H^T)\,(\bar{y} - H\,\bar{x})$$

$$= \bar{y}^T\bar{y} + \bar{x}^T H^T \bar{x} - 2\bar{x}^T H\bar{y}$$

(Refer Slide Time: 23:51)



Therefore I can write,

$$F(\bar{x}) = \left\|\bar{y} - H\bar{x}\right\|^2 = \bar{y}^T\bar{y} + \bar{x}^T H^T \bar{x} - 2\bar{x}^T H\bar{y}$$

Now differentiating this $F(\bar{x})$ with respect to $\bar{x}$ I have

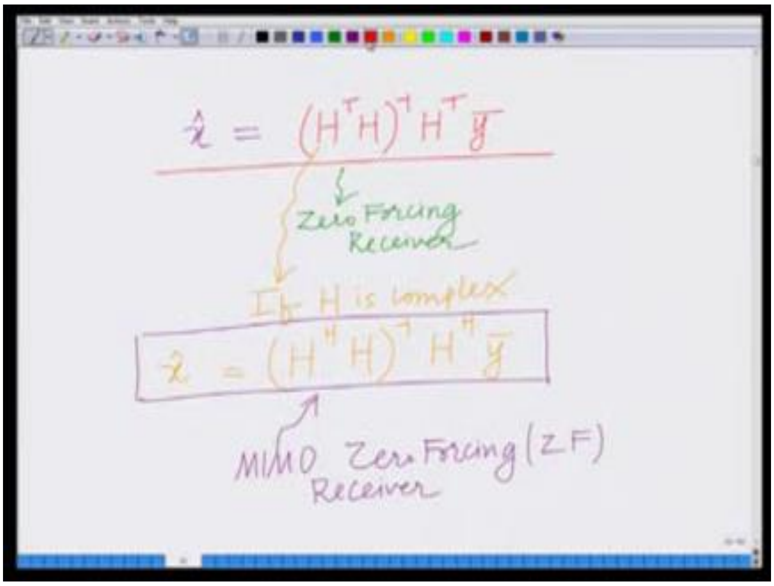
$$\frac{\partial\,(F(\bar{x}))}{\partial\,\bar{x}} = 0 - 2H^T\bar{y} + 2H^T H\bar{x} = 0$$

$$2H^T\bar{y} = 2H^T H\bar{x}$$

$$\hat{x} = (H^T H)^{-1} H^T\bar{y}$$

Therefore, what we have obtained after this procedure, what we have obtained is basically the fact, that the estimate $\hat{x}$, which minimizes the square of the error that it the least squares cost function $\left\|\bar{y} - H\bar{x}\right\|^2$ is $\hat{x} = (H^T H)^{-1} H^T\bar{y}$ which is also known as the least squares solution, which is also basically known as the zero forcing MIMO receiver. This is basically our zero forcing MIMO receiver.

(Refer Slide Time: 26:29)

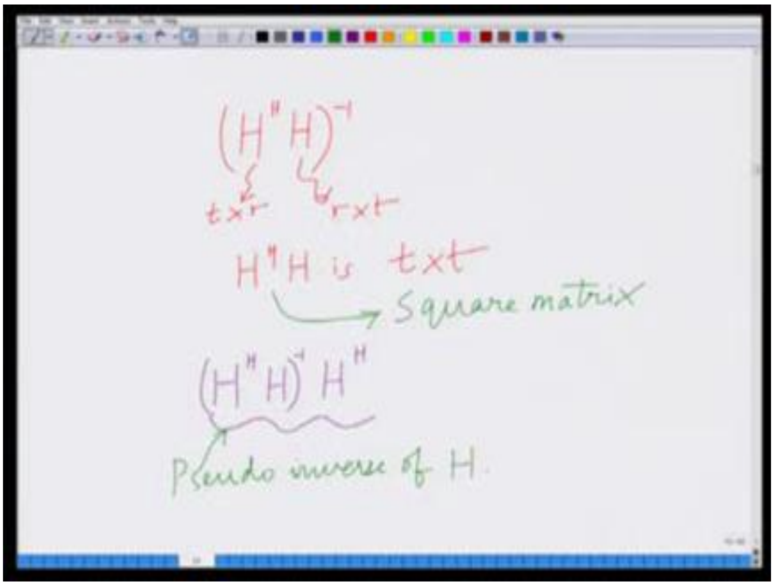


If the channel matrix H is complex, then $\hat{x}$ is going to be I am going to simply replace the transpose by Hermitian.

$$\hat{x} = (H^H H)^{-1} H^H \bar{y}$$

This is the MIMO zero forcing receiver also known as the ZF receiver, this is the MIMO zero forcing receiver.

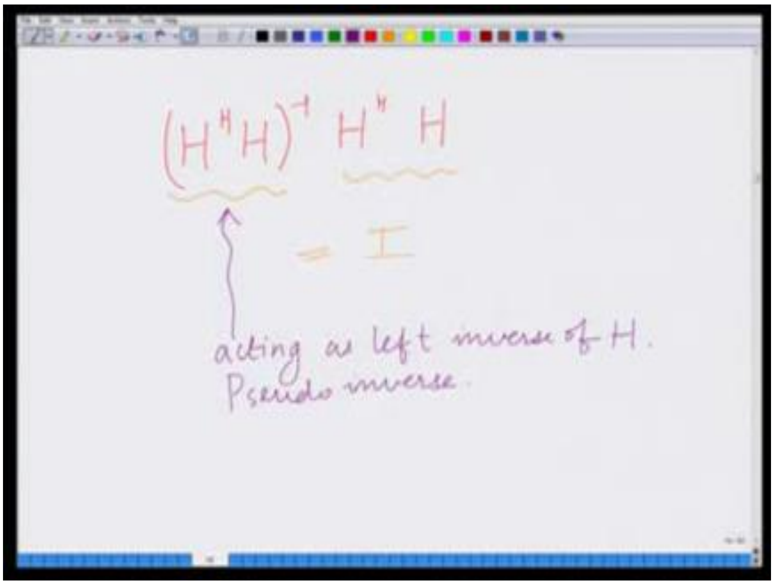
(Refer Slide Time: 27:53)



And you can see this quantity let us take a look at this quantity, let us take a look at $(H^H H)^{-1}$. Now if H is r x t, H^H is t x r. So, $H^H H$ is the product of a t x r matrix and r x t matrix it is the t x t matrix therefore, $H^H H$ is the square matrix, so even if H is a non square matrix with r and r > t, $H^H H$ is a square matrix, so $H^H H$ can be invertible therefore, zero forcing receiver exists even when r is greater than t because we are not taking the inverse of H rather, we are considering the inverse of $H^H H$, which exists even for non square matrices. Therefore, this is a general MIMO receiver not just for the case, where the number of receive antenna is equal to the number of transmit antenna that this is also valid, when the number of received antennas is greater than the number of transmit antennas.

Remember, this is the motivation with which we initially set out to find the MIMO receiver. And if you look at this quantity, this matrix $(H^H H)^{-1} H^H$ this is called the pseudo inverse or left inverse of H.

(Refer Slide Time: 29:40)



This is called the pseudo inverse of H because if you multiply this by H I have

$$(H^H H)^{-1} H^H H = I$$

Therefore, this is acting as a left inverse of H therefore, this is a pseudo inverse of H. Because H is a non square matrix, but when you look at this matrix $(\mathrm{H}^{H}H)^{-1}\mathrm{H}^{H}$, when I multiply this by H it gives me identity therefore, this is acting as an inverse of H. This is a pseudo inverse of H, so we use this pseudo inverse of H for the MIMO zero forcing receiver.

So, for the MIMO zero forcing receiver we multiply the receive vector y on the left by the pseudo inverse of this matrix H, not the inverse of the matrix H because inverse does not exist if H is not square, we multiply it rather by the pseudo inverse of this matrix H to get the estimate or to get the reconstructed vector x hat in the MIMO system, this is known as the MIMO zero forcing receiver, and a MIMO zero forcing receiver minimizes this reconstruction error, that is $||\bar{\mathrm{y}} - \mathrm{H}\,\bar{\mathrm{x}}||^2$ or the norm of the error square therefore, it is basically minimizing the least squares cost function.

So, in this module we have seen how to derive, how to construct the MIMO zero forcing receiver in a MIMO wireless communication system, so let us stop this module here.

Thank you.