Advanced 3G and 4G Wireless Communication
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Lecture - 23
SVD Based Optimal MIMO Transmission and Capacity

Welcome to another lecture in the course on 3G, 4G wireless communication systems.

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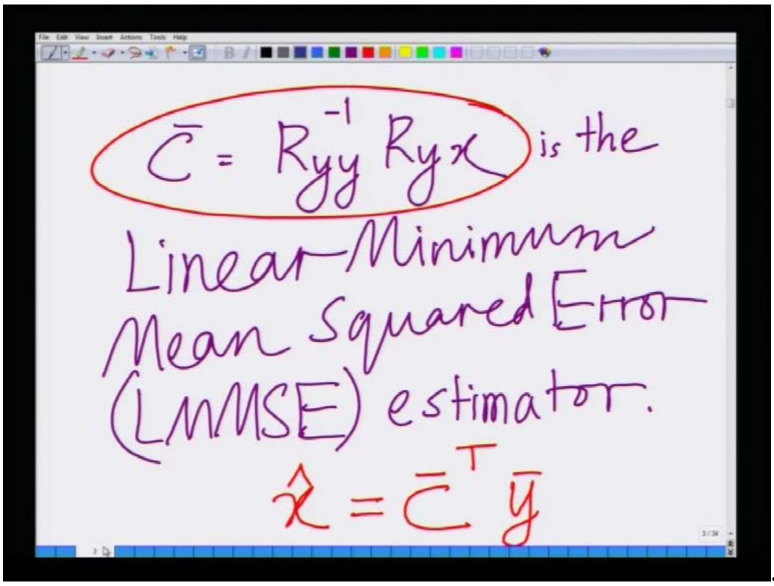

In the last lecture, we had gone over the MMSE receiver. In fact, beside that, is the LMMSE receiver, that is, the linear minimum mean squared error receiver for a wireless communication system. We said that, the beam forming vector for the LMMSE receiver is given as Ryy inverse Ryx; where, Ryx is the cross correlation between y and x. Ryy is the correlation between y and y, that is, the correlation of the vector y.

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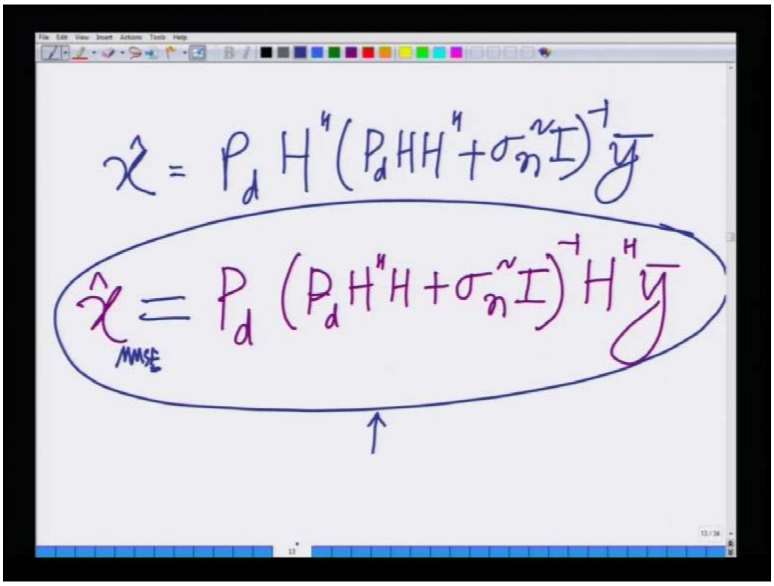

We said, the LMMSE estimator for the context of MIMO wireless communication can be derived as P d times P d H hermitian H plus sigma n squared I inverse in H hermitian y, that is, the MMSE or the LMMSE estimator. In fact, x hat MMSE is the LMMSE estimate at the receiver. So, this can be used to decode the received symbols; that is, you apply the LMMSE estimator on the received vector y. And on that, x hat LMMSE is essentially you perform your hard decisions, that is, you perform whatever you do in a normal communication system; that is, map these then to that transmitted consolation symbol; that is, if it is BPSK, map it to plus 1 of minus 1 and so on.

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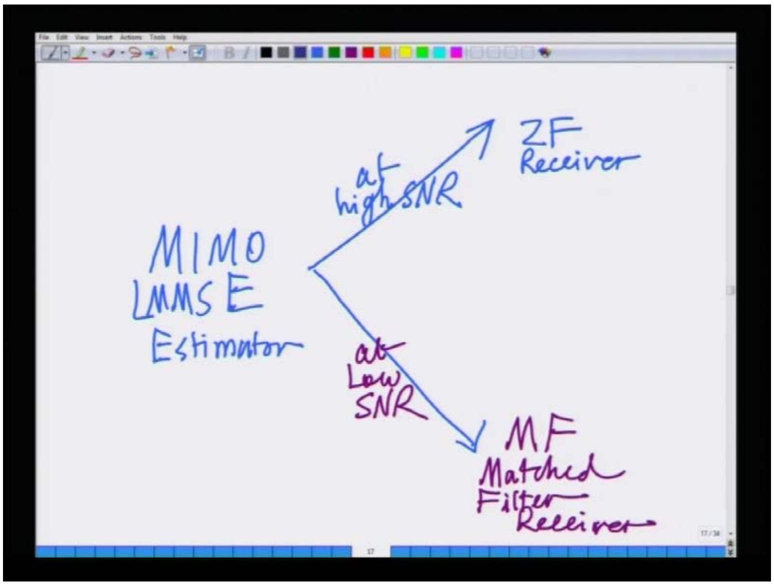

We had also started looking at the singular (( )) So, said that, the LMMSE estimator at high SNR reduces to the zero forcing receiver; and at low SNR, it reduces to the matched filter. And we also said, the LMMSE estimator is robust to noise; it does not result in noise enhancement or noise amplification as the zero forcing receiver enhances better compared to the zero forcing receiver.

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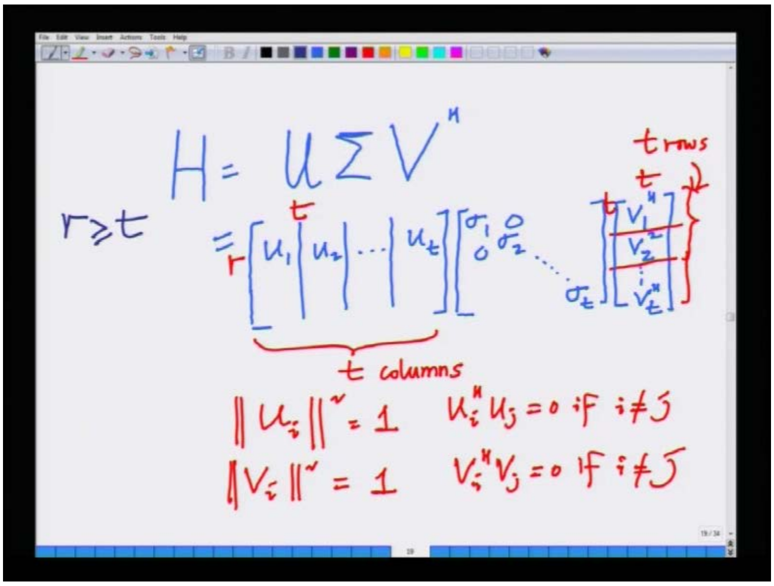

We also started looking at the singular value decomposition of a MIMO channel matrix H. We said, a MIMO channel matrix H can be expressed as U sigma V hermitian; where, the matrix U is such that, its columns are unit norm… H can be expressed as U sigma hermitian; where the matrix, where U is such that, its columns are unit norm; that is, norm U i square is 1 and U i hermitian U j, that is, U 1 hermitian U 2, U 2 hermitian U 3, so on is 0.

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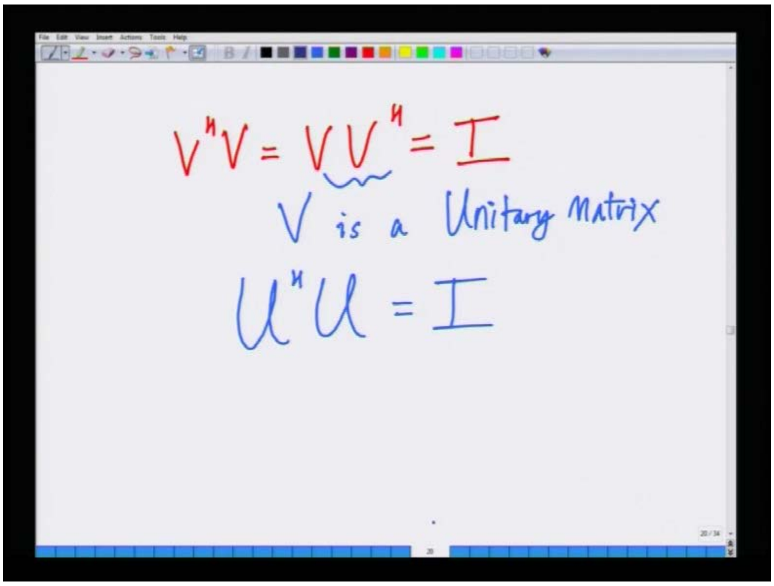

Similarly, V is also unit norm and the columns are V, are also unit norm and they are also orthogonal to each other. Further, we said that, V is a matrix – unitary matrix in the case r is greater than or equal to t. V hermitian V – V V hermitian equals identity. However, since r (( )) Assuming that the number of received antenna is greater than or equal to number of transmit antennas, U hermitian U is identity. However, U U hermitian is only identity if r is exactly equal to t.

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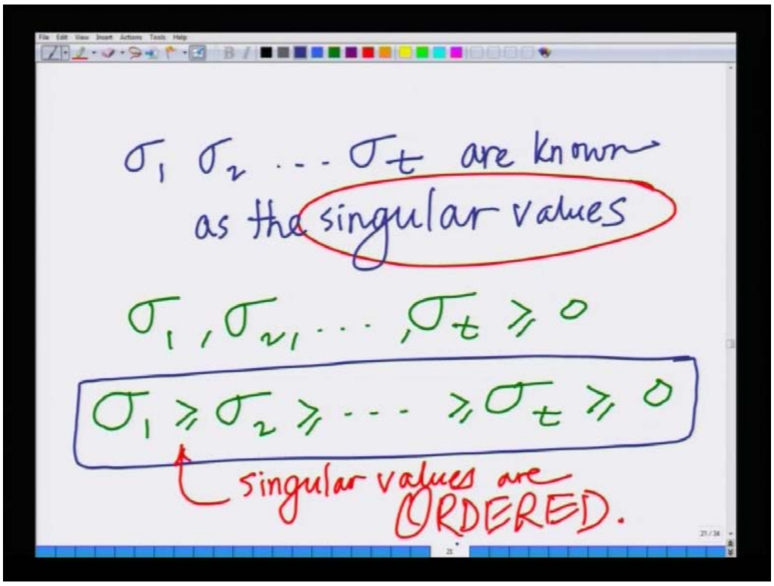

And, further, we have also said the singular values are important part of the singular value decomposition. They are non-negative. They can be 0 or positive. And the singular values are ordered; that is, sigma 1 greater than equal to sigma 2 greater than equal to so on up to sigma t; that is, they are arranged in decreasing order. And that is an important condition of the singular value decomposition.

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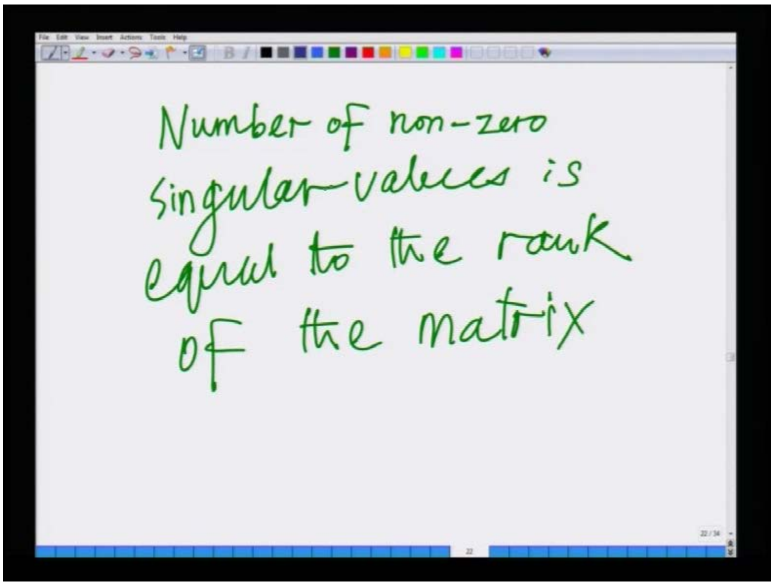

And, more importantly, the number of nonzero singular values is equal to the rank of the matrix H; that is, if it has two singular values, then the rank of the matrix is 2. So… And we also said, singular value decomposition is more general unlike the eigenvalue decomposition. For instance, eigenvalue decomposition exists only for square matrices; however, the singular value decomposition has no such restriction and exists for any general matrix; that is, non-square matrices also.

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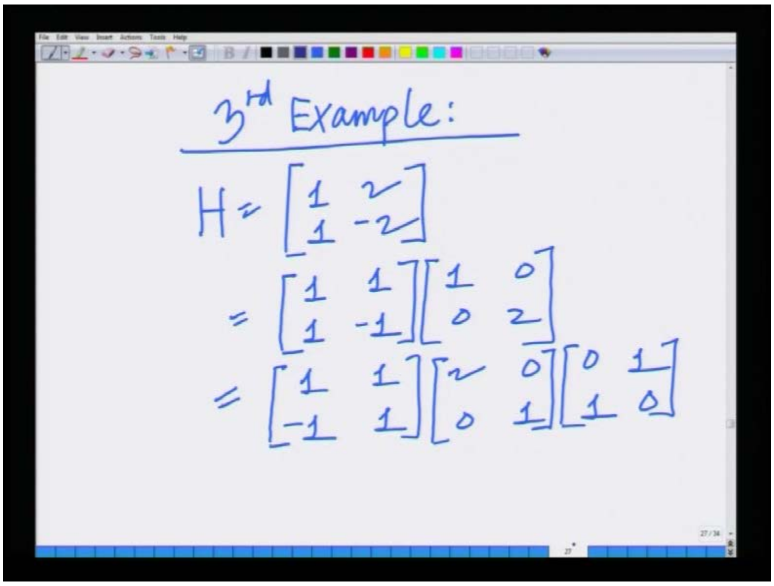

We looked at some examples of the singular value decomposition. For instance, we looked at this example of the matrix 1, 1, 2, minus 2.

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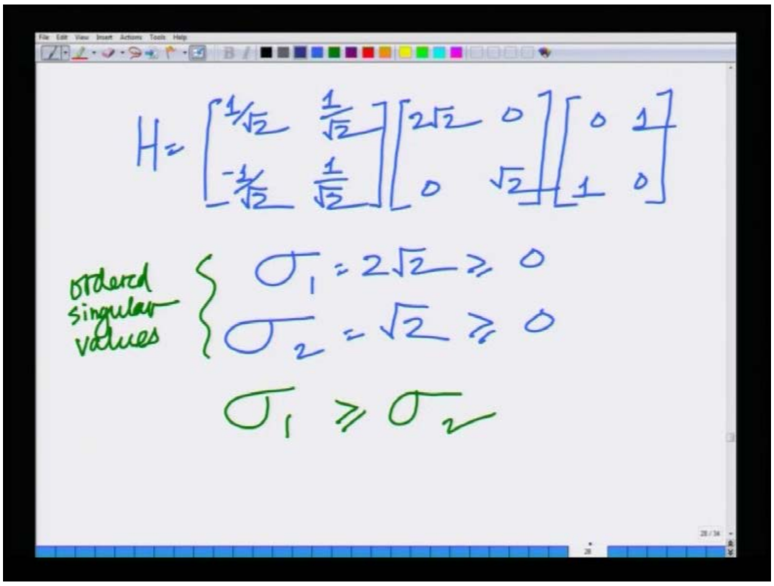

And, we said that, its singular value decomposition is given as follows as shown and the singular values are sigma 1 equals 2 root 2 and sigma 2 equals 2 and so on. This is the point, where we left at last time and let us continue with the lecture today. What we want to do today is we want to use the singular value decomposition to understand how to do manipulations or how to perform transmission and reception in a MIMO wireless communication system.

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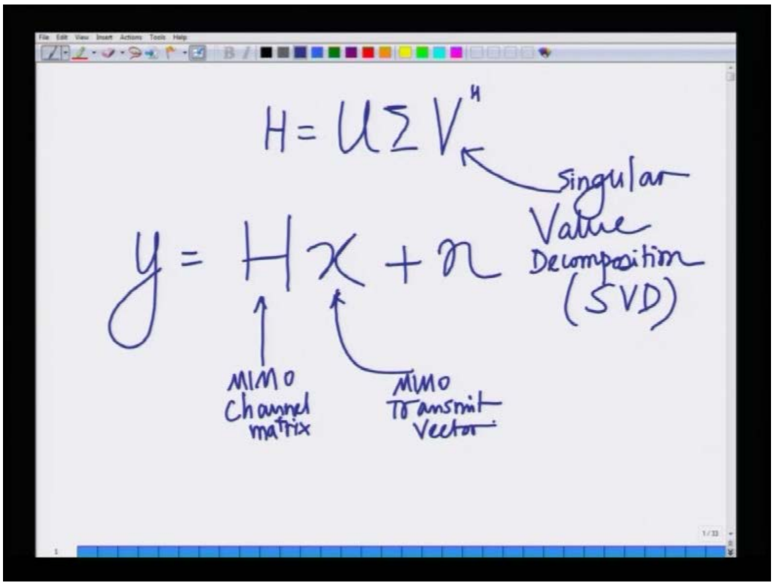

So, remember any MIMO channel matrix can be (( )) equals U sigma V hermitian. This is from the singular value decomposition or the SVD. This is also abbreviated as the SVD as we already saw. Now, we also know that, the MIMO system model equals y, is given as y equals Hx plus n. This is the channel matrix. This is the MIMO channel matrix. This is the MIMO transmit vector. Now, what I am going to do is, I am going to employ the singular value decomposition to perform manipulations on this. What I am going to do is I am going to substitute here that, H equals U sigma V hermitian; that is, I am going to substitute the singular value decomposition.

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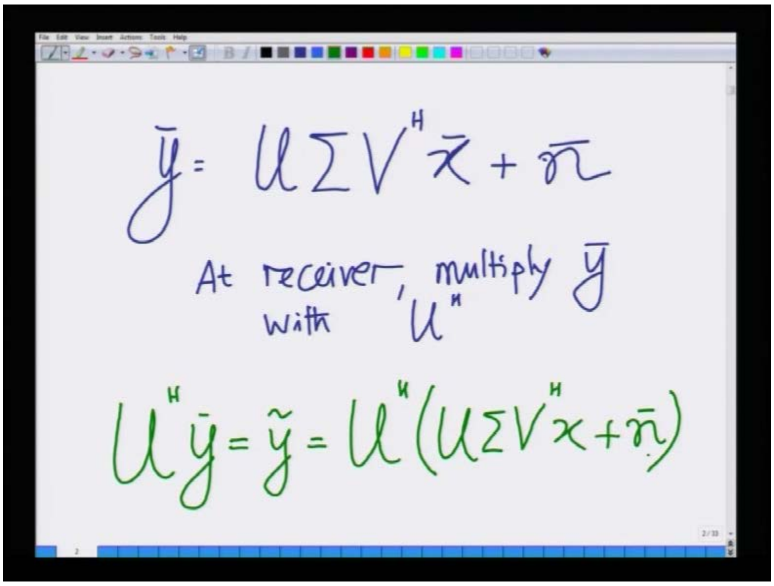

Hence, I can write this as y equals H – I will replace by U sigma V hermitian x plus n. At the receiver, what I can do is, I can multiply the received vector y by U hermitian. This is nothing but received beam forming; you have multiple beam formers in the matrix u. So, at receiver, multiply y bar – the vector; in fact, these are vectors with U bar U hermitian – the matrix U hermitian. Now, once you perform that, it becomes clear that, U hermitian y. Let me denote this by y tilde. This is equal to U hermitian U sigma V hermitian x plus n bar; that is, I have not done anything so far; I am just taking the expression as it is replacing h by U sigma V hermitian and multiplying this by U hermitian.

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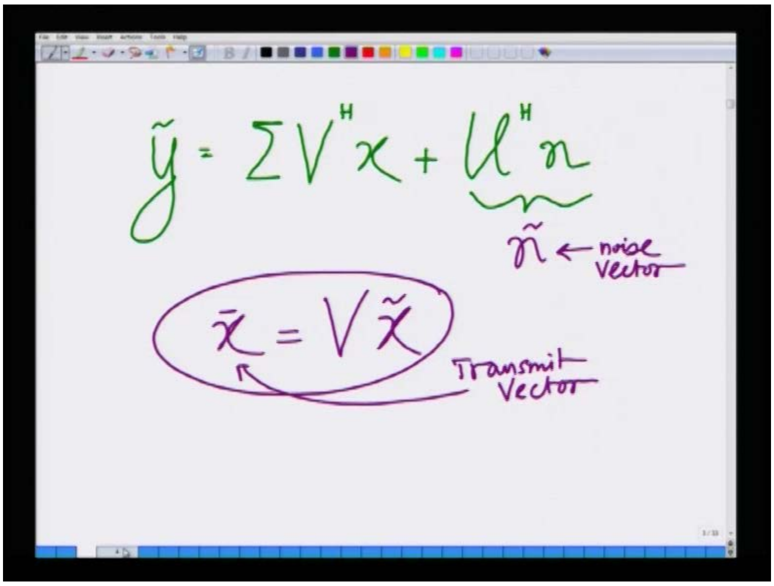

Hence, we get y tilde equals… Now, if you look at this expression here; we have already said that, U hermitian U is identity matrix. Hence, this term here vanishes; you simply have U hermitian U is identity; identity times any matrix or vector is simply that vector. So, you will have y tilde equals sigma V hermitian x plus U hermitian n. Hence, the matrix U has now disappeared from the expression. This U hermitian n can be denoted by some n tilde. This is the noise vector. This is some modified or effective noise vector after multiplying by U hermitian.

Now, I will do… So, this manipulation of multiplying by U hermitian has to be done at the receiver. Now, I will do some manipulation at the transmitter. In fact, this is known as pre-coding, because remember, it has to be done before transmission of x on the channel. Hence, it is known as pre-coding. What I am going to do is, I am going to denote the transmit vector x such that x bar equals matrix V times x tilde. That is what I am going to do. I am going to take a vector x tilde of transmit symbols; multiply it by V; and this is the vector i am going to transmit. So, the transmit vector… So, this is the transmit vector – x bar. And it is given as V times x tilde. This is the vector I am going to transmit. So, let us see how that affects the MIMO communication systems.

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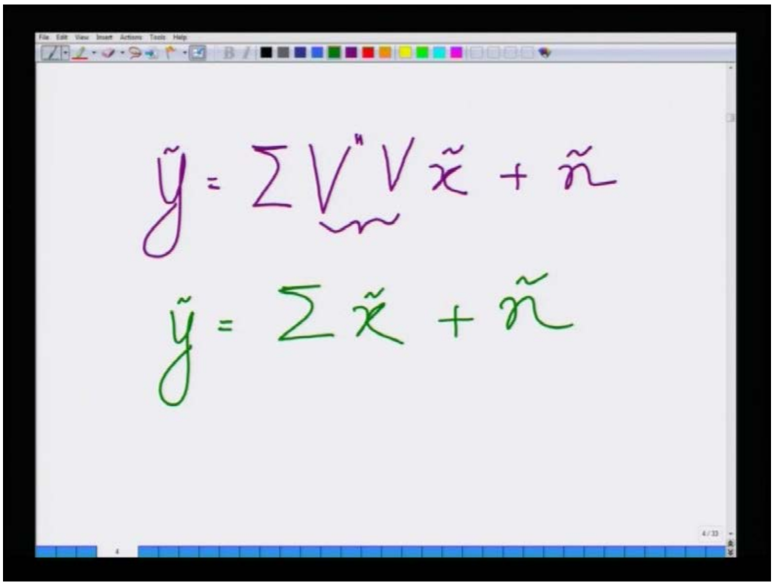

So, I will have y tilde equals sigma V hermitian times – now, x is V times x tilde plus n tilde. Now, observe that, V hermitian V is identity. Remember, we said earlier that, V V hermitian V hermitian V is identity. Hence, this now reduces to something very simple. This is sigma times x tilde plus n tilde. Hence, as a result of performing replacing H by singular value decomposition and performing the manipulations; that is, at the receiver, we multiply by U hermitian; at the transmitter, we pre-code using the matrix V. What has happened now is that, now I have a system model y equals sigma, that is, the matrix sigma times x tilde plus n; however, we know that, sigma is a diagonal matrix. Hence, I can… In fact, it will be clear, if I write this out elaborately.

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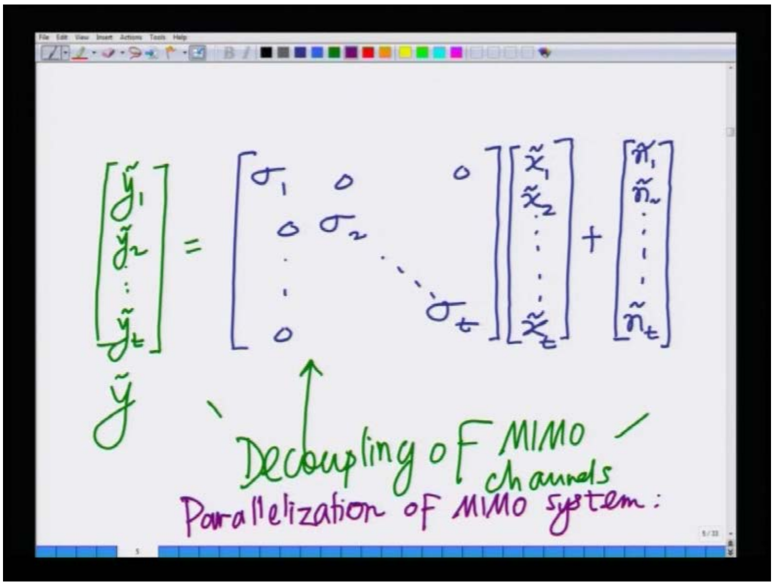

Let me write it out. I have y 1 tilde, y 2 tilde. You can verify that vector y tilde is of dimension t is nothing but sigma 1, sigma 2, sigma t times x 1 tilde, x 2 tilde, so on up to x t tilde plus n 1 tilde n 2 tilde, so on up to n t tilde. In fact, you can observe here now, that, because this is a diagonal matrix, I have y 1 tilde equals sigma 1 x 1 tilde plus some noise; y 2 tilde equals sigma 2 x 2 tilde plus some noise; so on, so forth. y t tilde equals sigma t x t tilde plus some noise. Remember, now, earlier, if you remember the MIMO lecture, we said, all the symbols interfere at every received antenna; that is, we had y 1 equals H 1 1 x 1 plus H 1 2 x 2 plus plus so on; that is, all the transmitted symbols are interfering with at every received antenna; which means there is interference or there is simply a superposition of all these transmitted symbols.

However, now, if you look at it, I have y 1 tilde equals sigma 1 x 1 tilde; y 2 tilde equals sigma 2 x 2 tilde; that is, each x 1 x i tilde appears only at the receiver antenna – y i tilde. There is no interference between these symbols in this transformed domain. This is also known as decoupled; or, this is known as the different channels are decoupled from each other. So, this is known as decoupling of MIMO. This is also known as parallelization of the MIMO system. This can also be said as parallelization of… This is known as decoupling of the MIMO system. This can also be said as parallelization of MIMO system. So, we have the net…

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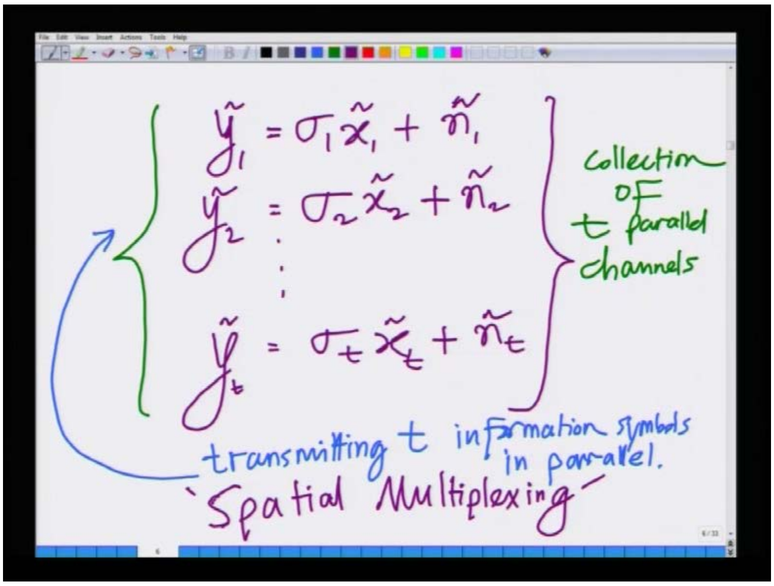

It can be written as y 1 tilde equals sigma 1 x 1 tilde plus n 1 tilde; y 2 tilde equals sigma 2 x 2 tilde plus n 2 tilde; y t tilde equals sigma t x t tilde plus n t tilde; each it is a separate collection; looks as if it appears as if it is a separate collection of t communication channels. In the first channel, you are transmitting x 1 tilde, receiving y 1 tilde; second channel – you are transmitting x 2, receiving y 2 tilde. In the t-th channel, you are transmitting x t tilde, receiving y t tilde. This is a collection of t parallel channels. Looks like a collection of t parallel channels. In fact, the gain of channel 1 is sigma 1; gain of channel 2 is sigma 2, so on; the gain of channel t is sigma t.

And that is possible because of the MIMO singular value decomposition. Now, looking at y 1 tilde, you can decode x 1 tilde; looking at y 2 tilde. you can decode x 2 tilde; looking at y t tilde, you can decode x t tilde. That is the advantage of using this singular value decomposition based MIMO beam forming at the receiver and pre-coding at the transmitter, you can decouple the MIMO system into t dependent.

Now, you can also see where the spatial multiplexing of MIMO is coming from. Look at this; each MIMO channel is a combination of x 1 tilde, x 2 tilde, x t tilde. So, you are transmitting t information symbols. So, let me write that also. So, you are transmitting t information symbols in parallel. Hence, this is also known as spatial multiplexing. Spatial multiplexing is nothing but transmitting multiple streams to the same channel. Here in fact, we are using the same channel; we are not using different frequency over different time; same time, same frequency.

Using the properties of MIMO spatial channels, we are transmitting different t information symbols. Hence, you can in fact transmit t parallel streams of information. This is nothing but spatial multiplex. And this is spatial multiplexing of t parallel streams of information. So, this is nothing but spatial multiplexing. As we said, in one of the first lectures on MIMO, spatial multiplexing is nothing but multiplexing several information streams to space; that is, using the same space to parallelly transmit several streams of information.

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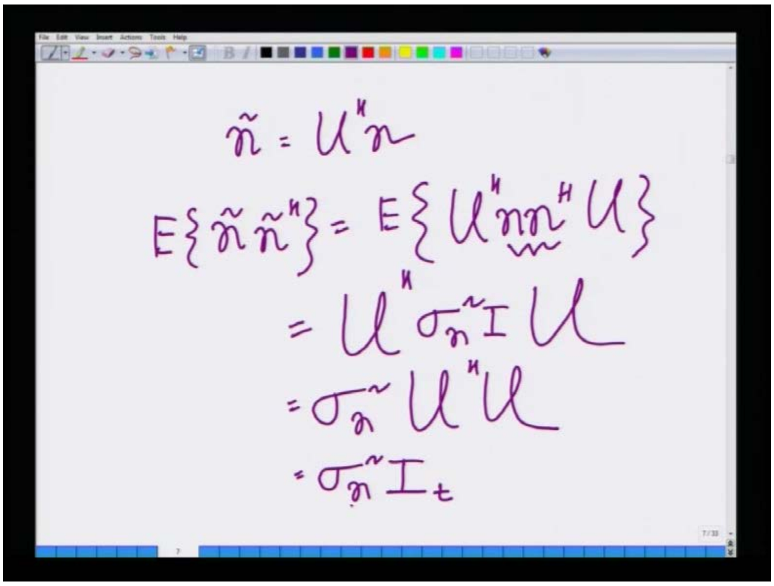

Let me say a small word of about this noise n tilde; and that is fairly straightforward. What is the variance of this noise tilde – in noise n tilde? In fact, the vector n tilde equals U hermitian n. If I look at the covariance of n tilde, that is, expected n tilde n tilde hermitian, which is expected U hermitian n n hermitian U; which is… If you look at expected n n hermitian, that is nothing but sigma n squared identity. So, this is U hermitian sigma n squared identity U; which is sigma n squared U hermitian U. Again, we said U hermitian U is nothing but the identity matrices. This is also sigma n squared identity. In fact, this is identity of matrix t. Hence, you can also see that, the different elements of n tilde have variance sigma n squared; so noise power of sigma n squared. And also, they are uncorrelated across the different antennas; that is, n 1 tilde is uncorrelated to n 2 tilde. That is what covariance equals sigma n squared I t means.

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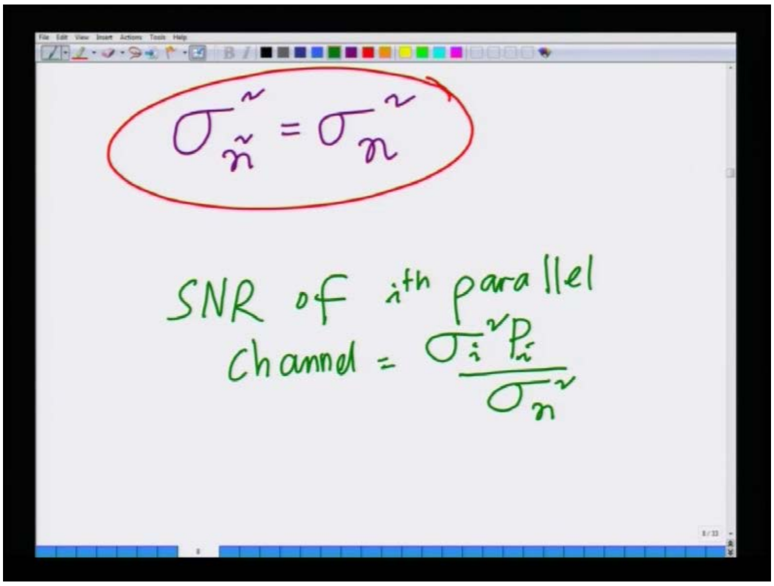

Hence, we said we have sigma n tilde square power of noise is nothing but sigma n square; that is, the power of the noise before beam forming. So, the variance – the power of the noise before beam forming is nothing but the power of the noise after beam forming. Hence, we have parallelization in these separate channels. And this is nothing but spatial multiplexing. As I said, from this structure, we can see that, it is nothing but… sigma 1 is nothing but gain of channel 1 and so on.

So, in fact, the SNR of this channel is nothing but sigma 1 squared times the power allocated to channel 1 divided by sigma n squared noise power… SNR of the channel 2 is sigma 2 square power P 2 of channel 2 divided sigma n squared and so on. So, the power, the SNR… Let me write that also here; SNR of i-th parallel equals sigma i square P i divided by sigma n square; that is, sigma i square gain singular value square of this singular value times p i, that is, power allocated to the i-th channel divided by sigma n squared.

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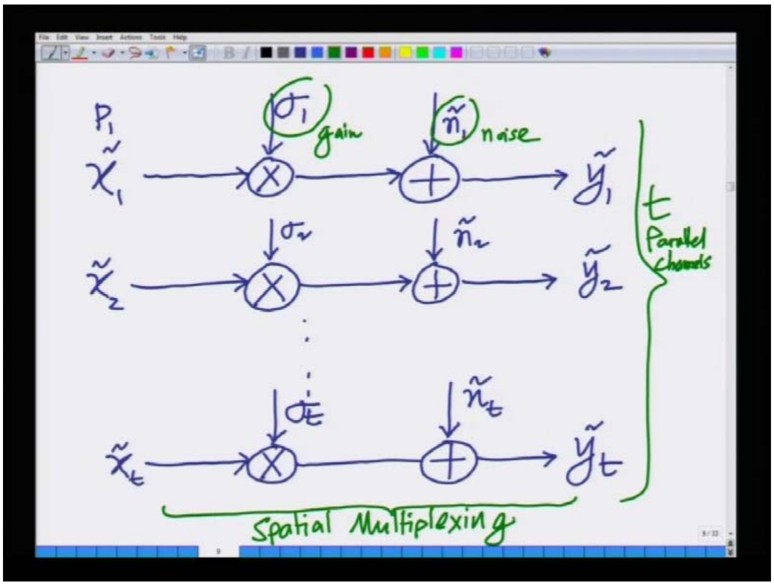

Let me represent this same thing schematically, so that the understanding becomes clearer. I have x 1 tilde. I am transmitting this of power P 1; I am transmitting this through a channel of gain sigma 1. And at the receiver, there is addition of noise n 1 tilde; and I obtain y 1 tilde; that this is sigma 1; this is the gain; this is the noise. Similarly, x 2 tilde of power P 2; I transmit through a channel of gain sigma 2. At the receiver, there is addition of noise – n 2 tilde, y 2 tilde is received at the receiver, so on and so forth. x t tilde is transmitted through this channel of gain sigma t. It has received; there is noise n t tilde and is received; received symbol is y t tilde. This is nothing… As we said, this is nothing but t parallel channels.

This is a schematic diagram of t parallel channels. And this is nothing but spatial multiplexing. This is nothing but spatial multiplexing of the wireless communication channels, because I have t streams of information transmitted in parallel. This is the spatial multiplexing property of the MIMO wireless communication system. Since the SVD helps us neatly decouple this MIMO system from an interference super imposition based system into something that is decoupled, where you can look at it as if, if you have t independent pipes for transmission of information and you are transmitting independent streams of information across each of these streams.

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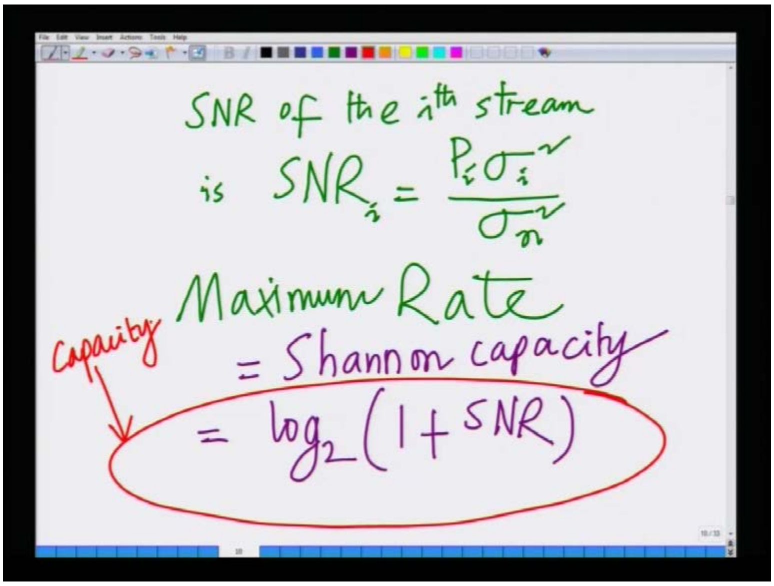

In fact, if you want to characterize the maximum rate at which information can be transmitted in this MIMO system, the SNR of the i-th stream is SNR i equals P i sigma i square divided by sigma n squared; that is, transmitted power into sigma i square divided by sigma n squared. And we know from a resultant communication theory that, the maximum rate is given by the Shannon capacity; maximum rate equals the Shannon capacity, which is log twice 1 plus SNR. Hence, the maximum rate – it is given by the Shannon capacity. This is nothing but the capacity of the channel. This is the capacity. This is nothing but the capacity of the channel in fact.

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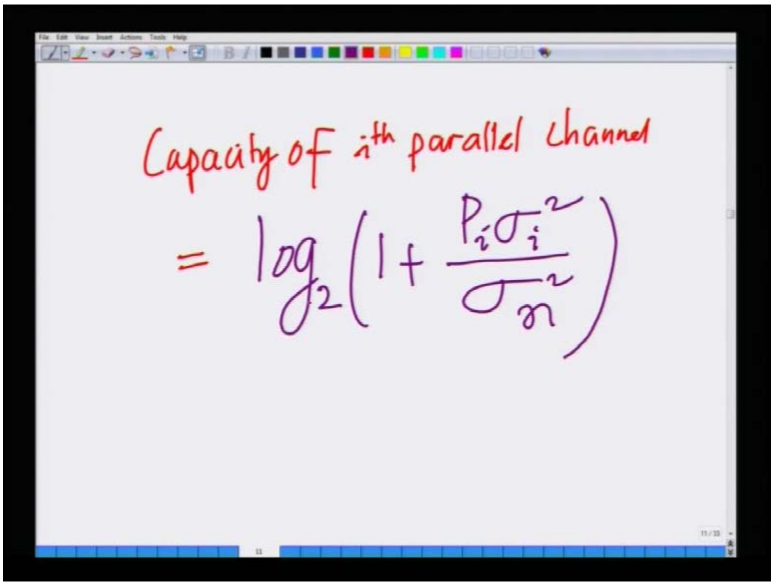

Hence, the capacity of the i-th parallel channel equals log 2 1 plus SNR – 1 plus P i sigma i square divided by sigma n squared. In fact, this is log to the base 2; this is the capacity of the i-th channel.

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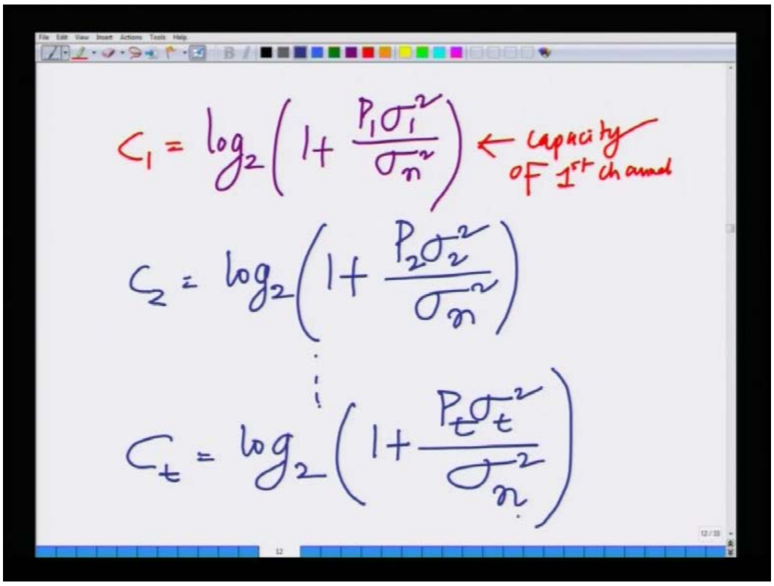

In fact, the capacity of all the channels can be written as capacity of the first channel is log 2 1 plus P 1 sigma 1 squared divided by sigma n squared. This is the capacity of first channel associated with singular value sigma 1. Capacity of… Let me write this as c 1. Capacity of second channel c 2 equals log 2 1 plus P 2 sigma 2 square divided by sigma n squared, so on. Capacity of the t-th channel is log 2 1 plus P t sigma t square divided by sigma n squared. That is the capacity of the t-th channel, that is, c 1, c 2, so on up to c t; that is, these are the capacities of the individual t of the channels. Hence, the net MIMO capacity is nothing but the sum of all these capacities. Remember, the MIMO capacity – you are transmitting t symbols of information. So, the capacity of each – you can think of it as t individual pipes; the capacity of some total capacity is nothing but the sum of the individual pipes.

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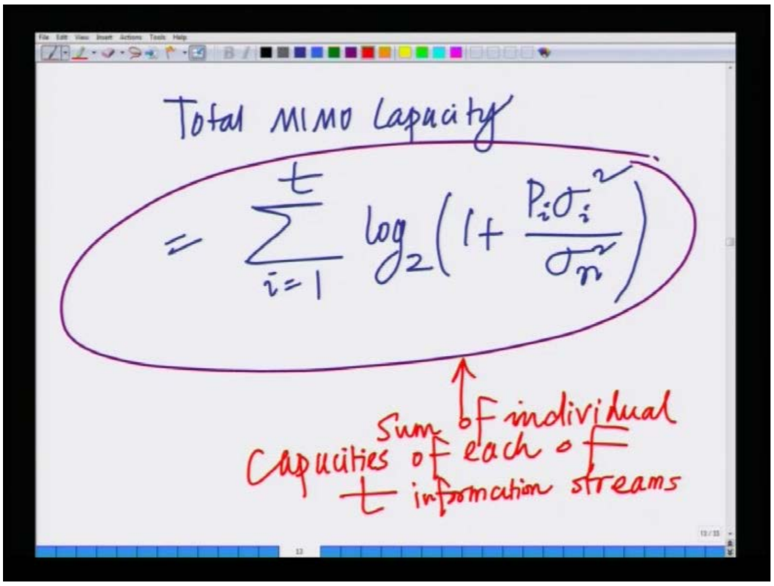

Hence, the total MIMO capacity is sum of the individual capacities, which is sum log 2 1 plus P i sigma i squared divided by sigma n squared. That is the sum. If you have to write this, this is nothing but sum of the individual capacities of each of the t information streams – each of the t parallel information streams. That is in fact, the total capacity of the MIMO systems. Now, we can clearly see the effect of spatial multiplexing. And spatial multiplexing is nothing but parallel transmission of information.

And you can see here that, now, you have capacity, which is the sum of the capacities of each of those terms. Hence, you are in fact transmitting information in parallel; that is, each… It is as if the channel capacity of each channel is log 1 plus SNR; you have t such log 1 plus SNRs. So, you have log 2 1 plus SNR 1, log 2 1 plus S N R 2, so on log 2 1 plus SNR t. So, you have t such terms, t such capacities – the total M I M O capacities – the sum of the capacities of this t information streams. That is nothing but spatial multiplexing.

Now, we have an interesting problem here. So far, we have said that, the capacity is proportional to… depends on the power. However, we have not said about how to allocate the power to these different information streams. Remember, you have one transmitter, which has the power P, which has to be divided amongst these information streams. How do you allocate power optimally to these different information streams?

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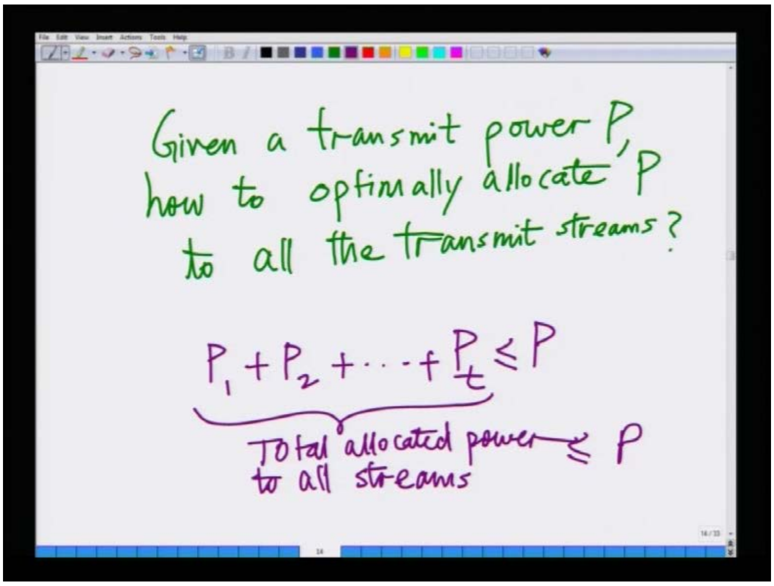

That is what we want to look at next; that is, given a transmit power P, how to optimally allocate P to all the transmit streams? Optimally means how to efficiently allocate it. So, we have power allocated to stream 1 as P 1; power allocated to stream 2 as P 2; power allocated to stream t as P t. Now, we know that, the total power allocated has to be P, because you cannot allocate more than P power, because that is the available maximum power. So, we know, P 1 plus P 2 plus P t equals P; that is, power allocated or… In fact, strictly speaking, we have to say, less than or equal to P.

So, sum allocated power or this is the total allocated power to all streams is less than or equal to P. We said that, the total allocated power to all the information streams has to be less than or equal to t. But, how do you optimally allocate the power? The reason… The way we want to optimally allocate the power is to maximize capacity; that is, we want to allocate power in such a fashion, so that we want to achieve the maximum information rate. And that is the problem of optimal power allocation.

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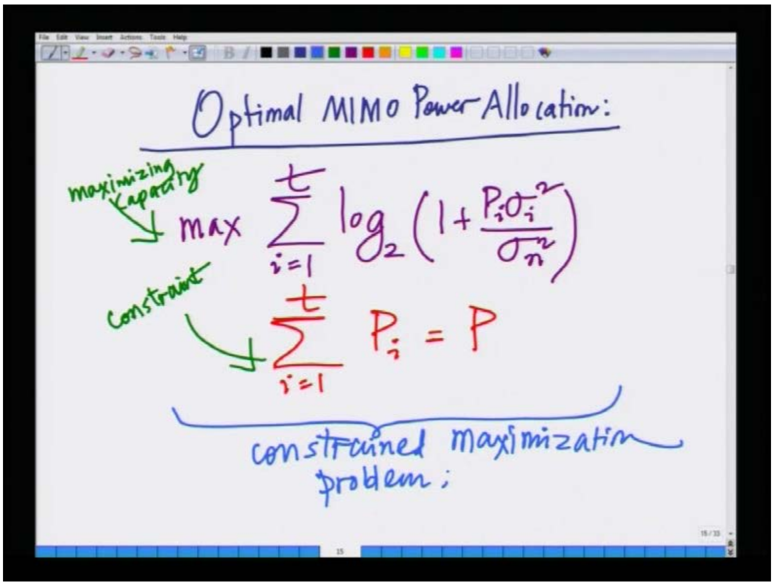

So, this is the problem of optimal MIMO power allocation. What is the problem of optimal MIMO power allocation? It is nothing but I want to maximize the capacity; that is, maximize log 2 1 plus – in fact, there has to be a summation here – summation i equals 1 to t log 2 of 1 plus P i sigma i square divided by sigma n square. I want to maximize this. So, this is nothing but maximizing the capacity subject to the constraint that, summation i equals 1 to t P i equals P t. This I will call as constraint; that is, maximize this capacity subject to this constraint that, summation i equals 1 to t P i equals P t. That is the constraint; that is, P 1, P 2 up to P t; that is, the sum of all transmit powers allocated to all the transmit streams should be limited by the maximum powers.

This is not P t, but this is simply P; that is, sum of all powers should be limited by the maximum transmit power, which is P. So, this is maximization with a constraint. So, this is nothing but a constrained maximization problem. This is nothing but a constrained maximization. In your high school level calculus, you might have known how to handle such problems. This is known as the technique of Lagrange multipliers; that is, for a maximization, normally, I have to take the different derivative and set it equal to 0. However, this is constrained maximization. Hence, I have to consider the Lagrange multiplier.

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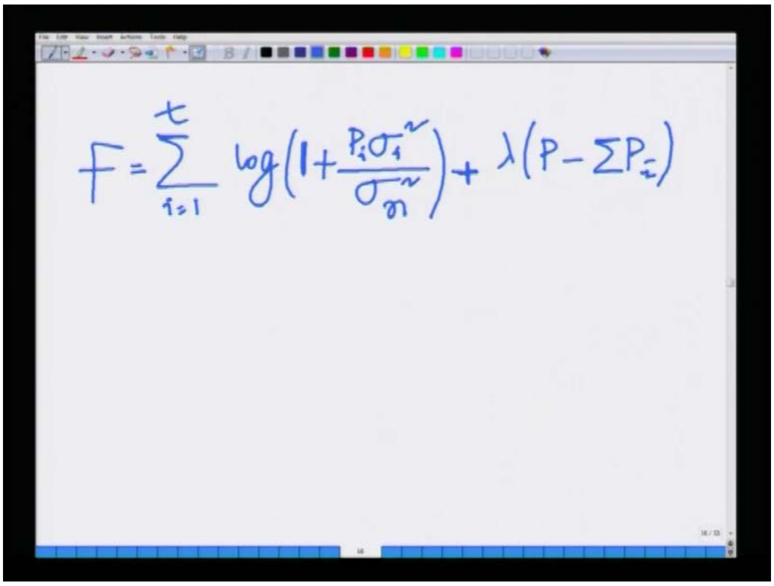

Hence, I will follow that procedure, which is, I want to compute the capacity… maximize the capacity, which is the function F, which is a vector function of the allocated powers; which is i equals 1 to t log 1 plus P i sigma i square divided by sigma n squared plus lambda P minus summation… – minus summation 1 over… This is lambda summation 1 over P i. So, this is nothing but the Lagrange multiplier.

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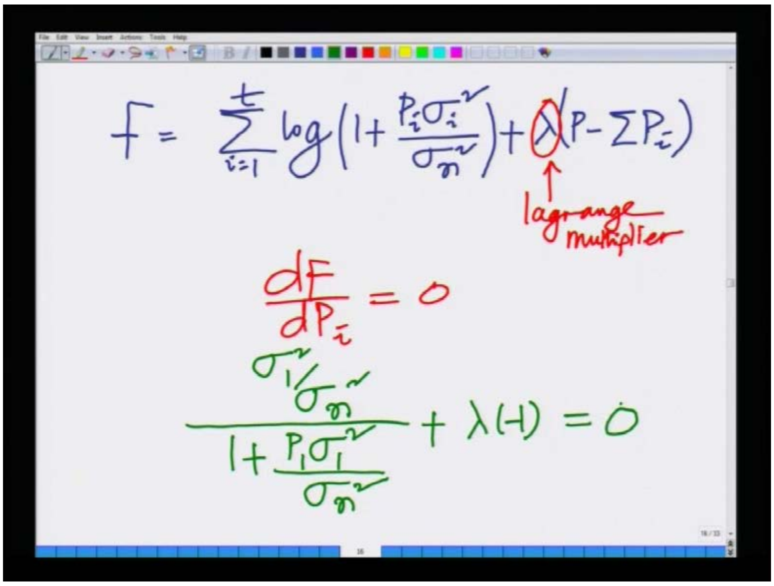

Hence, the capacity of the MIMO channel… Hence, the constrained maximization problem can be represented as F equals summation i equals 1 to t log of 1 plus P i sigma i squared divided by sigma n square plus lambda P minus summation of P i. This lambda is nothing but… This is nothing but the Lagrange multiplier. This lambda is nothing but this is the Lagrange multiplier associated with this optimization problem. Now, I know how to maximize this. I want to consider dF by dP i for each power and set this equal to 0.

Now, if I look at dF with respect to dP 1; let say I consider dF with respect to dP 1; then the derivative of log 1 plus P i sigma i square sigma n squared is nothing but 1 plus P 1 sigma 1 square divided by sigma n squared. And derivative of 1 plus P 1 sigma 1 square by sigma n squared; that is nothing but sigma 1 square divided by sigma n squared plus there is also another term lambda; derivative of P is 0 minus P 1 P 2 plus P 1 P 2 P 1 plus P 2 plus P n. This is minus lambda; that is, lambda into minus 1. That I will set it equal to 0.

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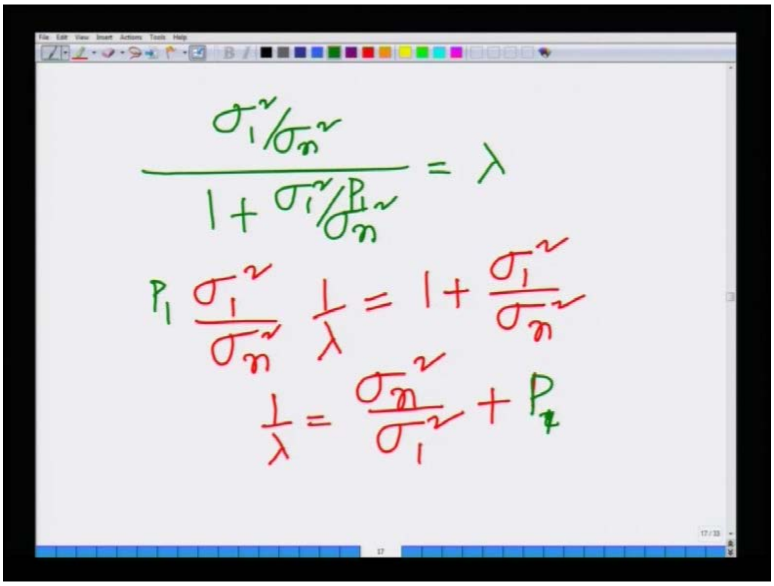

Hence, I have the net expression sigma 1 squared divided by sigma n squared divided by 1 by sigma 1 squared plus sigma n squared equals lambda. I will manipulate this further. In fact, I have sigma 1 squared divided by sigma n squared into 1 over lambda equals 1 plus sigma 1 squared divided by sigma n squared. In fact, 1 over lambda equals sigma n squared divided sigma 1 squared plus 1. This is the result of the manipulations. And hence, I have… This has to be P 1 over here; I apologize for that; there has to be a P 1 over here. So, this is P 1. So, this is plus P 1.

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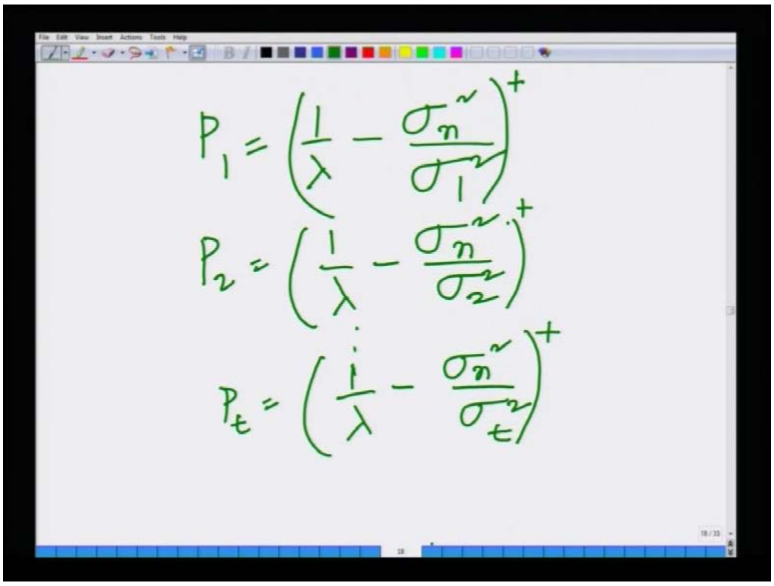

So, we have nothing but P 1 equals 1 by lambda minus sigma n squared divided by sigma 1 squared. So, we have P 1 equals 1 by lambda minus sigma n squared divided by sigma 1 squared. P 2 is also similarly 1 by lambda minus sigma n squared divided by sigma 2 squared, so on and so forth. P t is 1 over lambda minus sigma n squared divided by sigma t squared. Hence, we have P 1 equals 1 over lambda minus sigma n squared divided by sigma 1 squared; P 2 equals 1 over lambda minus sigma n squared divided by sigma 2 squared, so on. Except there is one caveat here; the power can only be positive. Hence, I will place a plus sign over here.

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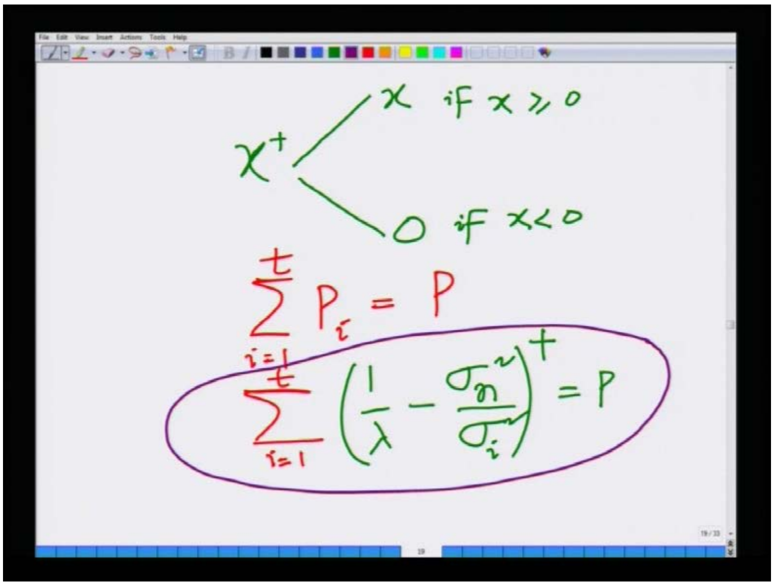

What that means is what is this function x plus? x plus means it is equal to x if x is greater than or equal to 0; it is 0, if x less than 0. Hence, the power equals 1 over lambda minus sigma n squared divided by sigma 1 squared, if this is greater than 0; if it is less than 0, it is simply set to 0. Now, looks like all the powers depend on this parameter 1 over lambda. How do we find this parameter – 1 over lambda or lambda? And the answer to that is straightforward. Remember, we still have to use our power constrained; we still have to use the condition that, summation P i i equals 1 to t equals P.

Substituting the expression for P i, I have summation i; i equals 1 to t 1 over lambda minus sigma n square over sigma i square plus – notice the plus sign here; that is P. That is how I compute the powers; that is, now, what is the optimal capacity maximizing power allocation? That capacity maximizing power allocation is given as P i equals 1 over lambda minus sigma n square divided by sigma n squared plus; that is, this quantity, if it is greater than 0; it is 0, if this quantity is less than 0.

And how do I find this parameter 1 over lambda? I find this using… This is nothing but the transmit power constrained; that is, the sum of all transmit powers should add up to P. Let us look a little bit more at this power constrained qualification or at this optimal power allocation.

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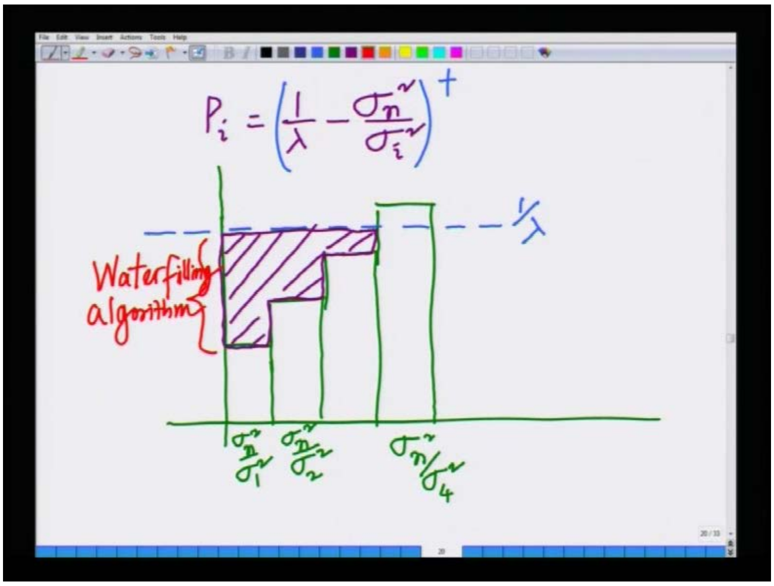

We have the optimal power allocation P i equals 1 over lambda minus sigma n square divided by sigma i square. Let us look at this quantity sigma n squared divided by sigma i squared.

Sigma 1 square is largest; which means 1 over sigma 1 square is smallest. Hence, this quantity sigma 1 squared over sigma i square is smallest for sigma 1. So, what I will do is here I will draw a picture of this MIMO communication channel. Here I will draw this different bars corresponding to sigma n squared divided by… This is the smallest bar. This is sigma n squared divided by sigma 1 squared. This is slightly larger, because sigma 2 square is slightly smaller – sigma 2 square. Finally, you have another bar here and you have another bar corresponding to sigma n squared divided by sigma 4 square. Assuming there are four channels, sigma 4 is smallest; so, sigma n squared divided by sigma 4 square is the largest.

Now, let us consider the level 1 over lambda. I will draw that level over here, that is, the level 1 over lambda. This is the level 1 over lambda. Now, this means 1 over lambda is greater than sigma n squared by sigma 1 squared. Hence, this power allocation is positive. 1 over lambda is greater than sigma n squared by 2 square. Hence, this power allocation is positive. Similarly, it is greater than sigma n squared by sigma 3 squared. Hence, this power allocation is positive.

However, it is less than sigma n squared by sigma 4 square. Hence, using that plus function that we have here, this power allocated to the fourth channel is 0. Now, if you can look at it; I will erase this for a moment. If you can look at this; it seems as if we are trying to fill this odd shaped bowl with some water and the level of water is exceeding some bars, but it is not exceeding some bar. So, all water… The water fills all these levels. So, you can think of it as if I have different bars corresponding to this thing and the water is filling.

Now, when I pour water into this, the water fills this odd shaped container. So, if you can think of it as different shaped bars, I am pouring some water such that the level of water is 1 over lambda; and the water fills this region; that is, it fills allocated power to three channels, but does not allocate power to the fourth channel. This is known as the water-filling algorithm. This algorithm has a name; this is known as the water-filling algorithm for power allocation. This can be thought of as trying to fill these different MIMO channels with this troughed, this shaped container – this weird shaped container with water. That is at the amount – the level – water level on each of these channels is nothing but the optimal power that has to be allocated to the channel.

Now, that is a slight issue, because this is a non-linear equation. Remember, this plus sign makes it non-linear. How do you solve this thing? The way to solve this thing is as follows.

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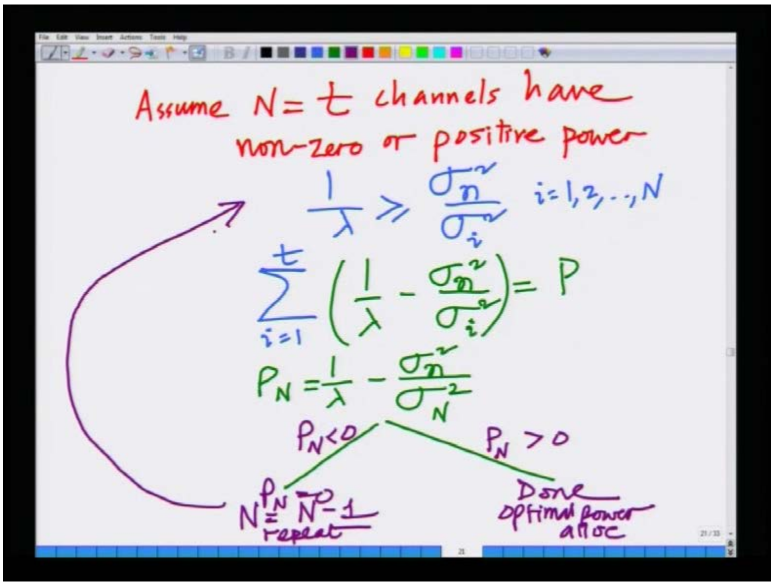

First, we start with – assume that all n… Assume all N equals t channels have positive; or, have nonzero power or positive power. We start with the assumption that, 1 minus lambda minus… 1 over lambda minus sigma n squared by i square is greater than 0 for all the channels. We use this to compute the equation; that is, we start with the assumption that, 1 over lambda greater than equal to sigma n squared by sigma i square for all or for i equals 1 comma 2 comma n. Now, what we do is we set summation i equals 1 to i 1 over lambda minus sigma n squared divided by sigma i squared equals P. We solve this equation; we compute 1 over lambda or we compute lambda essentially. And then we compute the power.

Now, if this fails, observe sigma small n squared by sigma capital N squared, because sigma capital N squared is the smallest; sigma 1 over sigma capital N squared is the largest. Hence, if it fails, it fails at the largest constraint. Hence, all we need to do is we need to check 1 over lambda minus sigma small n squared, which is the noise variance by sigma capital n squared, which is the singular value now, which is P N. This is nothing but P N, which is the power allocated to the N-th channel.

If P N is greater than 0… Now, if P N is greater than 0, then this procedure terminates; then this is done. This is optimal power allocation. However, if P N is less than 0, then that means 0 power is allocated to N-th channel, because remember, you have to take the plus above here; which means P N equals 0. Substitute N equals t minus 1; that is, no power is allocated to the t-th channel; and repeat. Or, in other words, N equals… set N equals N minus 1 and repeat from here – this position onwards; that is, this has to be solved in an iterative fashion; that is, first you assume that, all channels are allocated power. Solve for the Lagrange multiplier 1 by lambda. Now, compute that if the…

Now, see, check if the computer solution is in fact, consistent; that is, if lambda… is this computed lambda is in fact satisfying the criterion that all channels have been given positive power. If the constraint is violated, then you set that channel to 0; go back, repeat the procedure, solve it and so on and so forth. And then you will get the optimal power allocation corresponding to this MIMO system. To reinforce this idea, let us just do a simple example.

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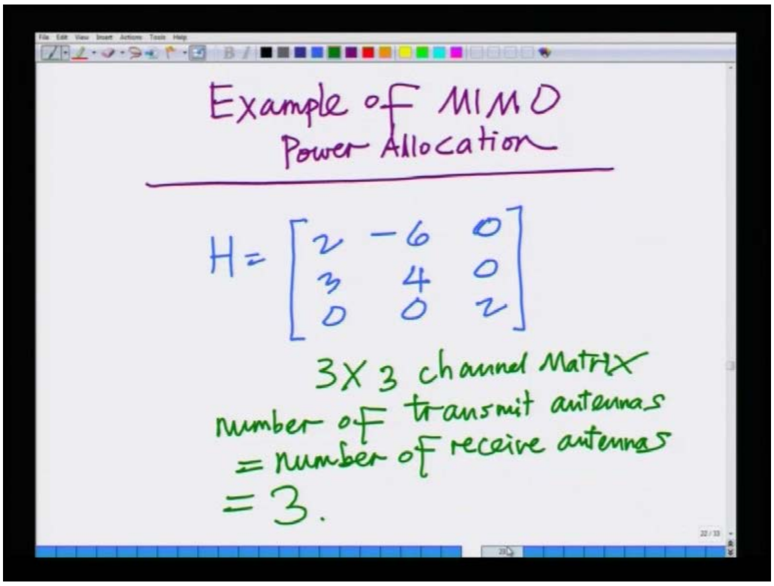

Let us do an example of MIMO power allocation. So, let us do a simple example of MIMO power allocation. So, let me start with a channel matrix. I start with this channel matrix – 2 comma 3 comma 0; minus 6 comma 4 comma 0; 0 comma 0 comma 2. This is the channel matrix that I am considering. Observe that, this is a 3 cross 3 channel matrix. This is a 3 cross 3 channel matrix. And number of singular values – the number of… This is a 3 cross 3 channel matrix. Number of transmit antennas equals number of receive antennas equals 3. The number of transmit antennas equals number of receive antennas equals 3. Hence, we said let us compute the optimal capacity of a system at transmit power.

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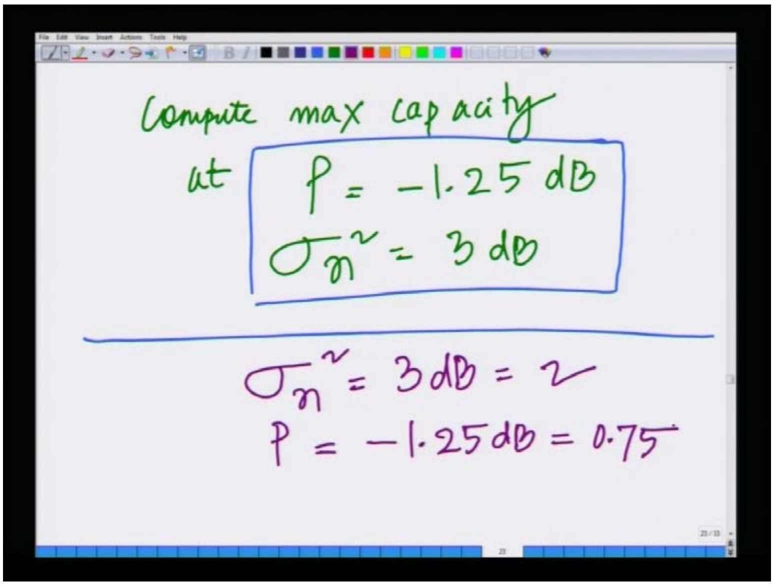

Compute maximum capacity or optimal power allocation at P equals minus 1.25 dB; that is, the total transmit power is minus 1.25 dB. And the noise power sigma n squared equals 3 dB. So, these are the specs; that is, the total transmit power if the transmitter is minus 1.25 dB and sigma n squared, that is, the noise power equals 3 dB. Remember powers are always specified in dB; that is, its tradition to always specify powers in dB. So, first step – let us convert these into linear – sigma n squared equals 3 dB. That is straightforward – sigma n squared equals 3 dB; that is, 2. And also, P equals minus 1.25 dB in linear terms; that is, equal to 0.75. So, we are trying to compute the maximum capacity of a MIMO channel given by this expression H equals 2, 3, 0; minus 6 4 0; 0 0 2. That is the MIMO channel under consideration.

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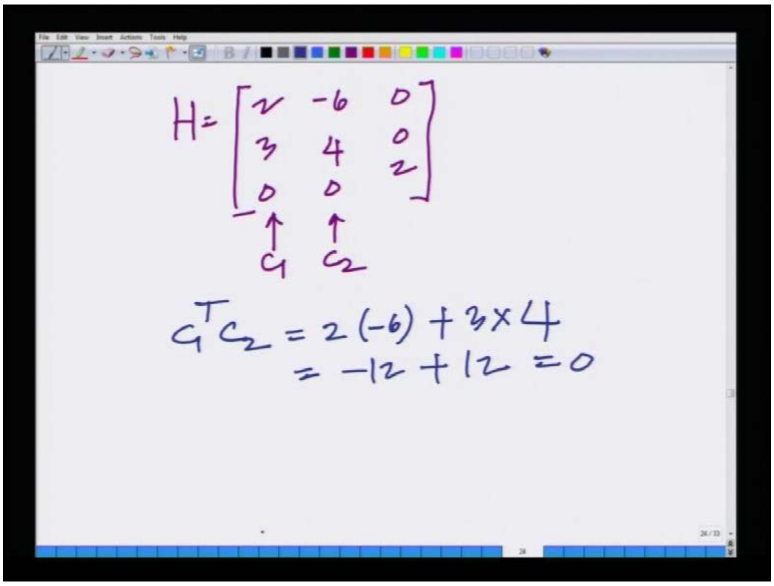

Now, the next thing we need to compute the capacity is to compute the singular value decomposition of this channel. So, let me rewrite this channel here again. So, this channel is 2, 3, 0; minus 6, 4, 0; 0, 0, 2. Now, if you take a look at this columns; look at c 1, c 2; look at c 1 transpose c 2. That is nothing but 2 into minus 6 plus 3 into 4 equals minus 12 plus 12 equals 0. Hence, these two columns are in fact… They are in fact orthogonal to each other; these two columns – they are orthogonal to each other.

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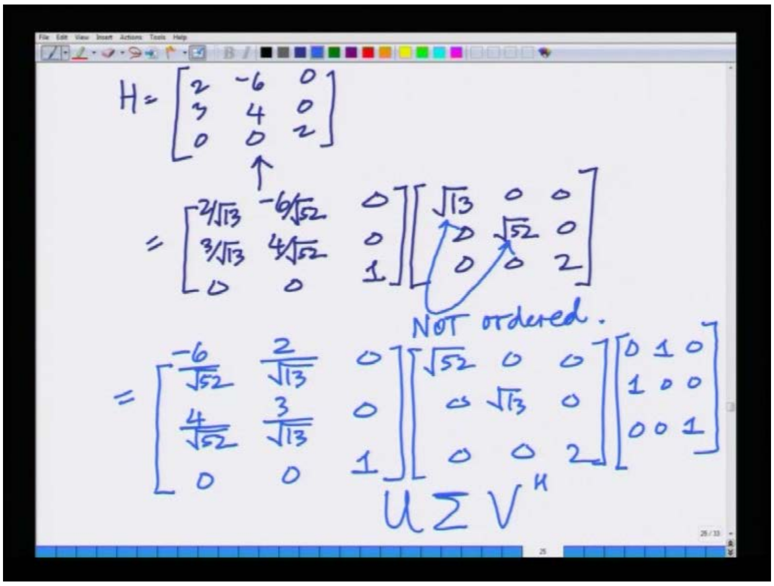

Hence, I can simply write this singular value decomposition as… I can first normalize this column. So, let me start to write the singular value decomposition of this channel – 2, 3, 0; minus 6, 4, 0; 0, 0, 2. First, I will normalize these channels; that is, 2 divided by square root of 13; 2 square plus 3 square, that is, 4 plus 9, which is the norm square is 13. Since the norm is square root of 13, 3 divided by square root of 13, 0; minus 6 divided by square root of 52, 4 divided by square root of 52, 0; 0, 0, 1. Normalizing this column is simply dividing by 2. Hence, I have here square root of 13, square root of 52 and 2. This is the diagonal matrix.

Now, remember we said this is still not a singular value decomposition, because these singular values are not ordered. These are not ordered singular values. Hence, I will do the same trick that I did earlier for the singular value decomposition; which is, I will multiply with the matrix – 0, 1, 1, 0. Except here it is slightly different; only this part has to be multiplied with that matrix 0, 1, 1, 0.

Hence, it can be written. This singular values decomposition – you can verify. This can be written as minus 6 divided root 52, 4 divided by square root of 52, 0; 2 divided by square root of 13, 3 divided by square root of 13, 0; 0, 0, 1 into square root of 52, square root of 13, 2; 0, 0, 0… And this matrix is nothing but 0, 1, 0; 1, 0, 0; 0, 0, 1. This is the matrix V hermitian. And you can verify that, this is the singular value; this is the matrix U; this is sigma; this is V hermitian. This is the singular value decomposition of the H matrix that we had illustrated before; which means what are the singular values?

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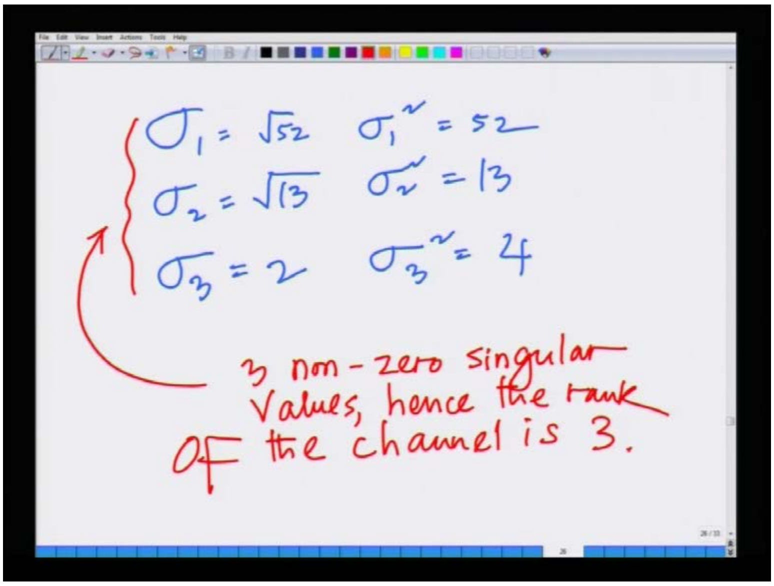

Singular values sigma 1 equals square root of 52; sigma 1 square equals 52; sigma 2 equals square root of 13; sigma 2 square equals 13; sigma 3 equals 2; sigma 3 square equals 4. You can observe that, this has 3 nonzero singular values. Hence, the rank of the channel is 3. So, what we are saying is this channel matrix has 3 nonzero singular values: square root of 52, square root of 13, and 2. Hence, the rank of this channel matrix is 3. And…

So, now we have the singular values using the frame work that was illustrated previously. We have to compute the power that has to be allocated to this different (( )) that is, the optimal power allocation corresponding to… Remember, we still have to compute the optimal power allocation corresponding to this transmit power of minus 1.25 dB, that is, 0.75. Remember, that is the problem we started out with. Due to lack of time, I have to stop this lecture here. In the next lecture, we will start with the procedure to compute the optimal power that is to be allocated to this different MIMO parallel channels so has to achieve the maximum capacity.

Thank you very much.