Signals and Systems
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Lecture - 9
System Properties

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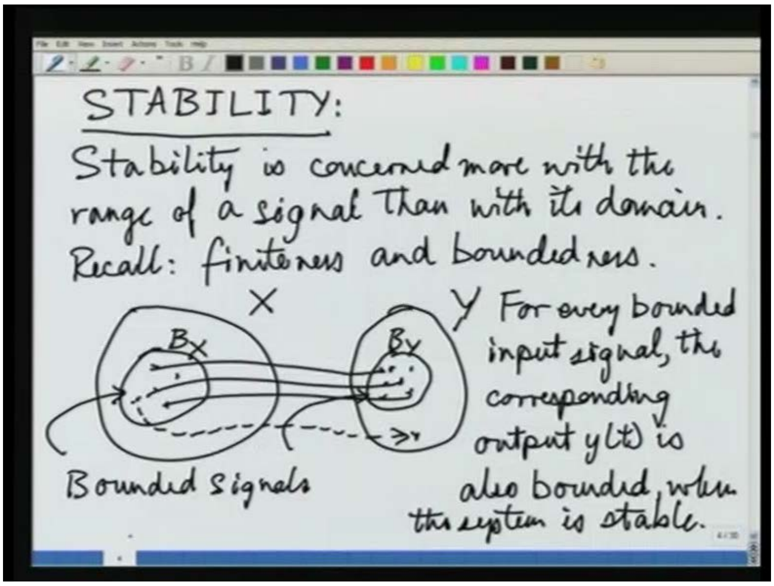

Let us move on now to a new property which we call stability. While causality and memory both referred to the regions of time over which the input signal provided information for computation of the output signal at a particular point. Stability has nothing to do really with the time access a such. Stability has more to do with the values of the signals with the range of the signal rather than the domain of the signal. So, let us put this down. So, what is stability? If we consider a particular system, as we said it can be understood as a look up table. For every kind of signal that we apply to the input there is always some output signal that exists; however, if we go back to our discussion about the preliminary properties of signals, we will recall the definition of finiteness and boundedness of signals.

So, recall boundedness was the property of signal. That was present if the absolute value of the signal never exceeded at any point of time a certain fixed value called the bound. So, if that is the definition of a boundedness signal, then if we take the word of all possible signals, we can extract from them the subset that consists only a bounded signals. So, the set of bounded signals is a subset of set of all signals. If this is the set of all signals X, then inside this is considered this subset of I will call this B of bounded signals, B is the set of all bounded input signals.

Now, let us go to the range set, which we use to call Y. It will be recalled that Y is the same set of signals as X except that; we draw it separately to indicate that it is the set of output signals. Though in every other way they are indistinguishable from input signals they are both signals. Now, even here there will be a same set subset B, I will not give it any other name, maybe I can call the first set B X and I can call this B capital y, B subscripted by capital y. So, these are also bounded signals in the output signal space. Now a stable system is one about which the following can be stated. For every signal that is bounded in the input the corresponding output is also a bounded signal. Let us put that down for every bounded input signal the corresponding output y t is also bounded, when the system is stable.

In short a stable system has this property about its look up table or about its map that it is implements on the signal space. That every member inside this set is mapped to some member of this set over here. what is not allowed is to map an input signal which is bounded to some output here, which is not bounded the dotted line that, I have just drawn is an instance of a signal of a map that cannot be allowed for a stable system. For a stable system only these solid lines represent associations between input and outputs, every bounded input signal must be associated with a bounded output signal. What restriction does the definition plays up on input signals, which are not bounded.

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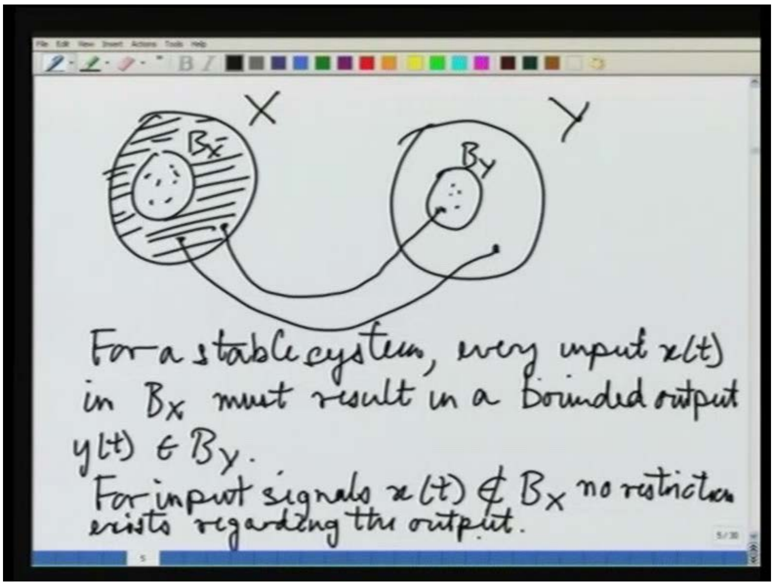

After all the input signal space had, this subset which we called B x and the entire space was this what about all the signals in this region. What should a stable system satisfy with regard to signals in this outer region? The answer is that there is no restriction on them. They can do what they like the response due to an unbounded signal that is a signal in the outer shaded region can still be a bounded signal. Thus for example, if this is the output space and this is the B y is set.

Then it is quite the legitimate for a stable system to take a signal in the unbounded subset and map it to a signal in the bounded subset. There is no problem here, it can take an unbounded signal and produce an output that is a bounded signal. Even if it does so it is still considered as stable system. If for all the bounded signals over here, the outputs corresponding outputs are bounded; however, even if an unbounded signal that is a signal in the outer region goes to some other signal in the outer region over here. Even so this system is stable if the signals in B X are mapped to B Y. In short therefore, we may say the following that, for a stable system every input x t in B X must result in a stable sorry, must result in a bounded output y t in B Y. Just to clarify further.

Let us also, talk about what it suppose to happen to signals not in B X? For input signals x t not in B X no restriction exists regarding the output. The output for a signal x t not in B X could be in B Y or may not be in B Y it does not matter for us. So, that is a rather dry if you like rather formal definition of stability. Let us try to understand more clearly, what it means in real life? If I place this pen at this point on the table and this place it is slightly with my hand it moves only slightly corresponding to the amount of force I applied it will move from place to place. But if I place it on a convex surface, there is no convex surface here to demonstrate. Except perhaps this slant of the desk, if I place it somewhere here and just even tilt its slightly it will slide all the way of, the response in short would be disproportionate to the input. Even if the input is bounded, even if the input disturbance applied to the system is bounded the output disturbance that is caused is unbounded.

Such a system would be called unstable. The system which maps the input disturbance to the output disturbance would be unstable, if the output is unbounded when the input is bounded. So, there are lots of instances in real life of stable systems. For example, an amplifier a normal amplifier properly designed amplifiers of sound signals is stable. If you apply an input signal which has some bound on it. It only gets scaled up in magnitude or amplitude by a certain factor, which is the gain of the amplifier and so the output will be a signal which has wider variations, but still is bounded as long as the input is bounded the output is bounded. And if the input is not bounded then as we just followed from our definition there is no restriction on the output. If the input is unbounded output is free to be unbounded, we will not blame the system and call it unstable. If it produces an unbounded output when the input is unbounded it is only important to produce a bounded output when the input is bounded.

So, there are many systems where things are unstable also. For example, if you take an object such as this and place it vertically I do not dare to do it, because it will fall down immediately. And if I have had let go of it and even applied the slightest force at the top. Irrespective of the force I apply being extremely small the response will be very large. The system will cause the pen to fall this is an unbounded system once again. So, stability is a very, very natural physical property, which we encounter in all kinds of mechanical situations, electrical situations all kinds of situations. In fact, the property of stability as we have defined it is considered to be the simplest form of stability.

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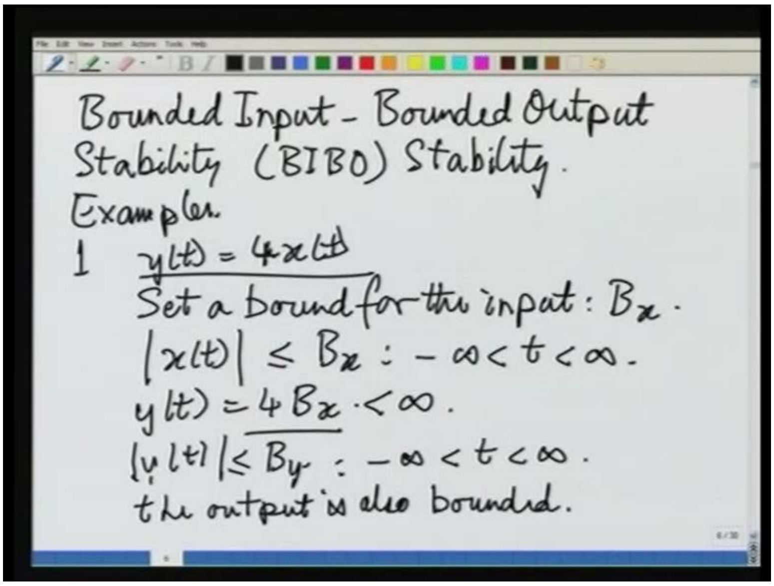

It is called bounded input, bounded output stability. Let us now start taking a few clothes from examples. And see if we can understand what we mean in practise by this definition of stability. Whether we can characterizes system property either a stable or non stable by looking at the closed form description of the system, how it maps the inputs to the outputs.

Let us take examples. One let y t be equal to say 4 x t is this system stable or unstable. The system is completely define, because for every input x t we can substitute the right side of this equation, there will be some y t that we will result automate naturally. And so this system is going to be stable anyway, this system is well defined anyway. The question is whether it is stable? In order to answer this question, the procedure is as follows; set a bound for the input, because we are concerned only with bounded input signals. That is the only part that is of concern when we consider the property or stability bounded input signals. So, set a bound for the input, call it is a B x. B x is a number, whose value is never exceeded by x t or by the absolute value of x t for any time instant t. In short we can say that x t, absolute value of x t is less than equal to B x for all time. Fine.

So, this is a bounded input signal. Now if the system is stable it art to produce as a output signal which is also stable. In some sense the bound is the worst case value of the input signal x t, because x t can be less than the bound or the absolute value of x t can be less than the bound, but never more than the bound. Hence suppose we apply a signal x t equal to the bound, then what would be the output? We can find the answer to that, by looking at the definition of the system namely this y t equals 4 x t. If you look at that then we find that if we applied B x the input, then we would get y t equals 4 B x. Now if B x is a finite number that bounds x t, bounds the absolute value of x t for all time. Then clearly 4 B x is also finite. If B x is a finite number, that is if B x is not infinity, if B x is less than infinity then 4 B x is also less than infinity.

So, long as it is less than infinity, this number 4 B x can be called a bound for y t. So, we can call this number B y and say that mod y t is never greater than B y for all time. This shows that the output is also bounded. Note the crucial point of this argument. The crucial point of the argument is that once we have set a bound B x for x t. And applied the system on the extreme case of the input signal namely B x that is the b t signal of value equal to B x for all time. Then calculate the output, the output is four times B x. The crux of the argument is to show that, when B x is finite, four times B x is also finite. And hence, y t is also a bounded signal, because 4 B x acts as for a bound for y t whenever, B x acts as a bound for x t. So, when the input is bounded the outputs also have the bound equal to 4 B x, that we have called B y.

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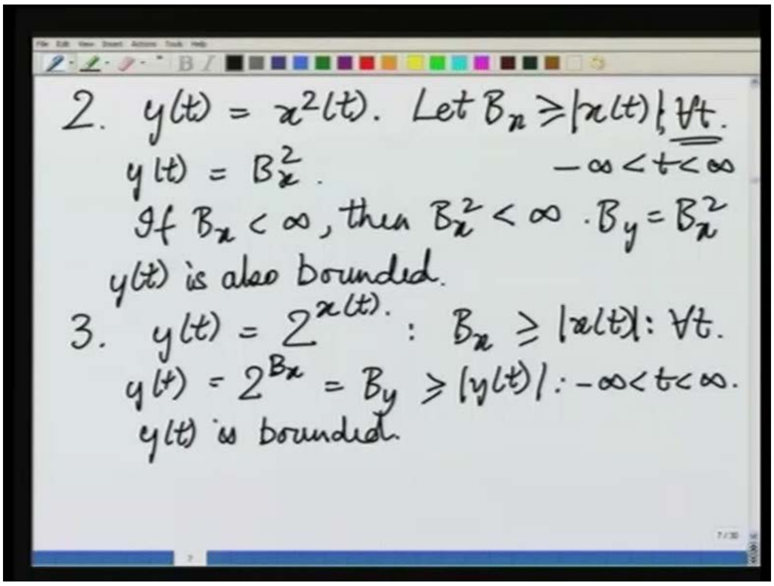

For example, this take y t equal to x squared t this seems a little more doubtful, because as we all know the square of a signal can be very large even when the signal is small. Hundred squared is ten thousand, thousand squared is one million. So, are we talking about an unstable system, when we make this definition? The answer is no, and the way we obtain the answer is same as before. We are concerned only with bounded input signals. So, let us say, let B x be a bound for x t that is B x is greater than or equal to mod x t for all time. This is just an abbreviation for writing minus infinity less than t less than infinity. So, B x is a bound for x t, we have chosen only a bounded input signal.

Now when we have a bounded input signal, what can we do with it? We consider the extreme case of the input signal, that is then x t achieves the bound, x t equals to B x or mod x t equal to B x. In which case y t would be equal to B x squared. Now, coming to the crux once again, what can we say about y t now. If B x is a finite number and number less than infinity then B x squared is liable to be very large. Particularly when B x is greater than one or B x squared will be much larger. If B x is between zero and one then of course, B x squared will be smaller than B x, but in general we can say without loss of credibility that B x squared is also certainly finite if B x is finite. If B x is finite then B x squared is also finite and that is the crux of the argument, which says that B x squared can be designated as a bound for y. And hence y t is also bounded.

So, if the output of a system is the square of the input signal. While the output can get very large, even though the input is not large that does not in any way make the system unstable. Because as long as the input is finite at every instant of time and has a bound B x the output will also have a bound which is the square of the bound of the input. So, this is also a bounded system. Three, let us going the same direction that we have gone, first we had a linearly scaled version of x t we had y t equal to 4 x t.

Now, we have y t equal to x t squared, let us try to see, if we can make a system unstable by making the y t and even faster growing function of x t. Suppose y t equals 2 to the power x t is this an unstable system. Going by past experience we should not be seemed is to thinking that the system is dangerously un stable, because it actually is again not unstable. If B x is a bound for a y therefore, for x suppose B x is greater than equal to x t for all t. That is B x is in bound for x, then can we found a bound for y as usual, we ask the same question can we find a bound for y? And the answer is again yes, because when x t achieves it is bound of B x. y t will be equal to 2 to the power B x.

Now, if B x is not infinite, if B x is a finite number. 2 to the B x is liable to the extremely large, but it will still be finite and that is the important point. Exponentiating B x is not making it infinite, exponentiation or raising it to a power or scaling it by a scale factor all these things can never make a finite number in to an infinite number. It will keep a finite number are finite number and therefore, 2 to the B x can be designated as a bound for y because 2 to the B x is also finite. Whenever B x is finite it is of course, quite possible that for a particular B x, 2 to the B x is much much larger than B x, but larger is not our problem, infinity is the only problem.

And therefore, B y is certainly greater than equal to mod y t for all t. So, again y t is bounded. So, again and again we have found that for three successive examples x t was bounded and for an every instance of a bounded x t the output was also bounded. Let at last turn to some example of a system where the output need not be bounded, even though the input is bounded. What about y t equals tan x t the tangent trigonometric function. What if y t equals tan x t, tan x t is not a safe function.

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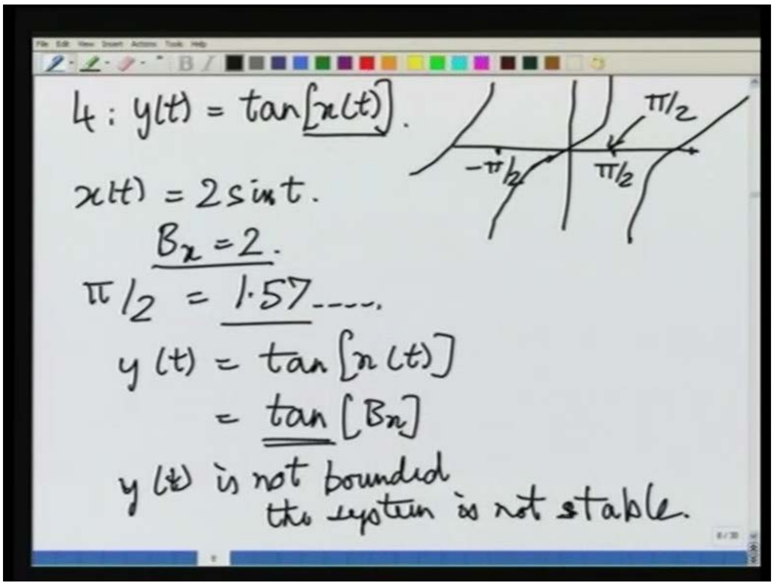

There can be an instance where x t is a finite bounded signal not just a finite bounded signal, but y t can become unbounded. It is very easy to see when this happens? Let us now, take a fresh example, this time for the first time an unstable system. Four, let a y t be define as tan x t the trigonometric tangent function. If you recall the graph of this function is one where, for even finite values of the argument the value of the function goes to infinity. Let me just plot on the side here some instances of the tan functions. So, if this is pi by 2 and this is minus pi by 2 then in this region.

The tan function stood towards infinity starting from very innocent values and goes this way and towards negative angles. It goes this way; it goes towards minus infinity near minus pi by 2 and towards plus infinity at plus pi by 2. Now, beyond this it repeats itself it has a periodicity of pi. So, this is how the tangent function looks. Now you clearly see here, that for x the value of the input equal to pi by 2 the output is infinite. In short then if we consider an input signal x t over here which achieves a value close to pi by 2 the output will start increasing without limit the output will be unbounded even when the input is bounded. For example, if I take a signal such as x t equals 2 sin t the sin function is bounded from minus 1 to 1. The 2 sin t function will be bounded from minus 2 to 2.

So, about this signal we can say that, B x equals 2, because mod x t never exceeds 2. So, input is clearly a bounded function, but pi by 2 is approximately equal to 1.57 dot dot keep goes on forever. So, this is pi by 2 and it is clearly less than the maximum value achieved by x t, whenever x t goes through this value pi by 2 which it will repeatedly do because it is a periodic signal. At every one of those instances y t will tend to infinity. This is an instance of a system where a bounded input will yield a signal at the output which is not bounded. What happens if, we proceed in to examine the stability of the system along the same line that we have applied so for. Let B x be a bounds for x t we already have a B x, B x equal to 2 is a possible bound even B x equal to 1.8. If you had used different x t would have been fine for this particular x t of course, we would can any number greater than 2 is also a bound on x t.

Now, if we let x t cross its bound in this case, the bound is 2. The extreme value of x t is not of danger to us; because at x t equal to 2, y t will be finite that is not the problem. But for values even within the bounded within the bound namely at a value will at 1.57 this output signal will tend to infinity. So, if we say that y t equals tan x t and substitute the bound for x t that is we write tan B x thus this give us an indication of the output being unbounded. It does because the tangent function is not a bounded function as can be seen from the graph drawn on the side. The tangent function is not a bounded function and for a function which is of this sort even though the argument of the tangent function is bounded the value of the tangent function need not be bounded. So, y t is not bounded, that is to say the system is not stable. One more example also an unbounded system, unbounded systems or rather an unstable system can arise.

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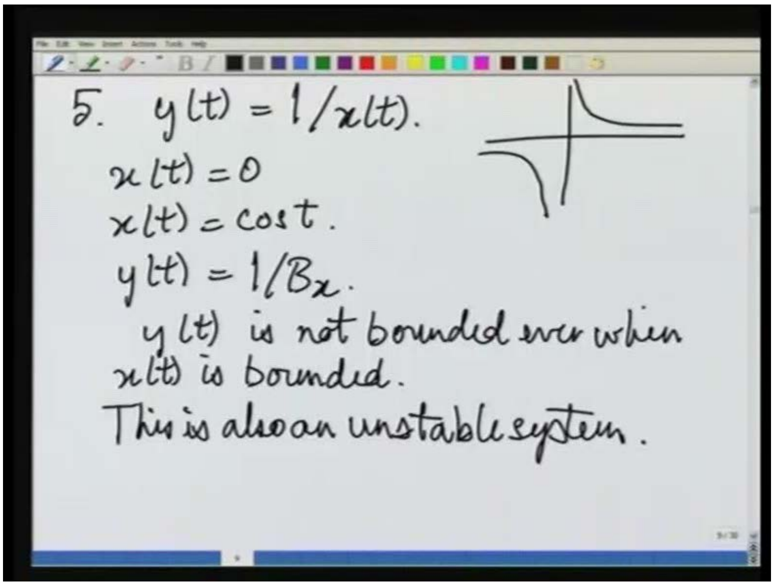

If y t is given as 1 by x t, this is also an unstable system. Why so because 1 by x which is the function which relates y t to x t is again not a bounded function, recall what 1 by x looks like we will let us make a graph in same place on this paper. This is what 1 by x looks like, even when especially when x takes a very innocent value like x equal to zero, 1 by x will tend to infinity. Thus even if we set x t equal to zero or x t equals to cos t any of these input functions they are all bounded, but the output would not be bounded if y t equals 1 by x t. And clearly that can be seen if you set x t equals to the bound and say y t equals 1 by B x.

Now, even when B x is a finite number, 1 by B x need not be a finite number, because for B x equal to zero as seen from the graph. The value of y t that will arise is infinity. So, here again y t will not be bounded even when x t is bounded. So, this is also an unstable system. Thus the closed form expression of the system gives us information indirectly about the stability of a system. If the system is described some by something like y t equals 1 by x t or y t equals tan x t, then we have on our hands an unstable system. Now, the funny thing is that 1 by x t and tan x t probably look more innocent than functions. That we discussed in the earlier examples, such as y t equals 4 x t that sounded very lubious because y t was an amplified version of x t.

This would give at first class and impression that the y t is going to be very large even though x t is not large, but that kind of largeness is not going to scare us with regard to the stability of the system. Nor the subsequent examples such as y t equal to x squared t though we were squaring the input signal to obtained output signal and therefore, often likely to get a larger output signal. A signal with greater amplitude than the input signals still this system is perfectly stable.

So, also the third example, exponentiation the output could be much much larger than the input, but still this system is bounded what. In fact, is not bounded is a system such as this or this all these are unbounded systems, are unstable systems sorry. So, stability is this property of maintaining boundedness of signals that is to say if the input signals are bounded and that boundedness property is maintained in the output signal then we will say that the system is stable. What can we say about real life systems? We could say for example, about real life systems that they were all causal.

So, with regard to the property of causality, it was clear that real life system for always causal. What can we say about real life systems with regard to stability? There is no clear statement we can make, because in the real world we have instances of systems which as stable instance of systems, which are not stable all kinds are available. We move on to another important definition, another property of systems, which we call linearity. Linearity has again got something to do with the range of a signal is concerned with a range of a signal.

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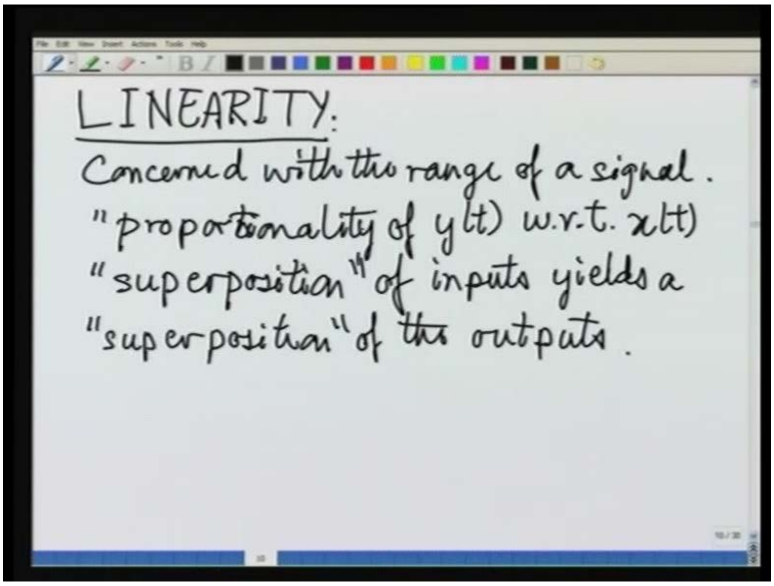

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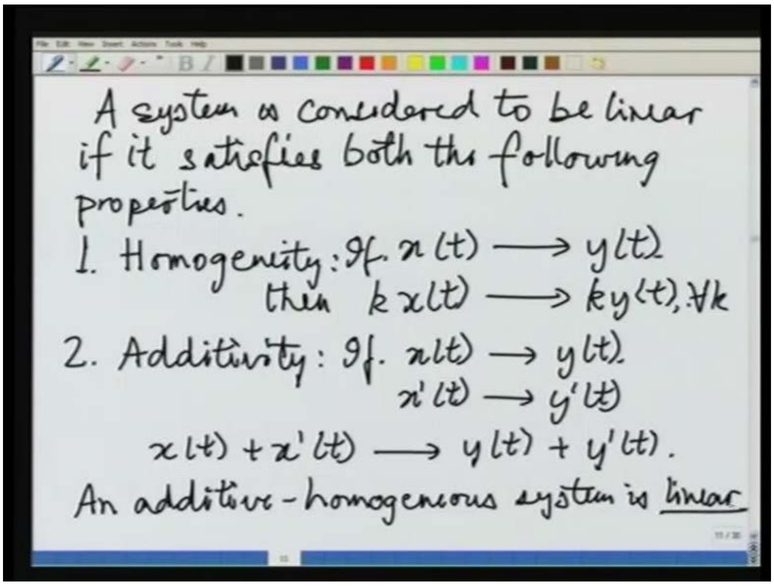

What is linearity? Linearity has something to do with proportionality of y t with respect to x t. It can also be said to support super position or to related to super position of inputs yields a super position of the outputs. Let us discuss this in terms of few examples, let us make a definition first. We will say that, a system is linear.

If it satisfies simultaneously two distinct properties. The first is called homogeneity; this is the property that I had earlier called the proportional to y t to x t. If the system takes the certain signal x t to an output y t, then the system must necessarily take k times x t which is a different signal it is the signal x t multiplied by the constant number k to the signal which is k times y t. So, if x t goes to y t then k time x t has to go to k times y t, this makes a system homogenous. The second property is called additivity; the property of additivity refers to super position of inputs and outputs. What we say over here is this, if as we said here, if x t goes to y t then k x t has to go to k y t for all k. Thats what we had for homogeneity, what we will say over here, is this. If x t goes to y t and say x prime t goes to y prime t where x t and x prime t are two different signals and their corresponding outputs produced by the system in hand or y t and y prime t.

Then a signal, a system which is linear, which is additive to be more particular must take x t plus x prime t. Remember that when we add two signals the result is still a signal. In fact, when you add two signals, then the some of their respective bounds is a bound for the overall signal this is a property. Therefore, when you add two bound signals you can be sure that the sum is also a bounded signal this is by the side.

Going to our definition of additivity, if x t goes to y t and x prime t goes to y prime t, then if we applied the input x t plus x prime t which is a new signal. The output must necessarily be y t plus y prime t, whenever this is the case for all x t and x prime t. And homogeneity holds for all values of k and for all x t, then we will say the system is both additive and homogenous. That is we will in short say that an additive and homogenous system is simply set to be linear. We will say that it is linear. The best way to understand linearity also is by invoking a lot of examples.

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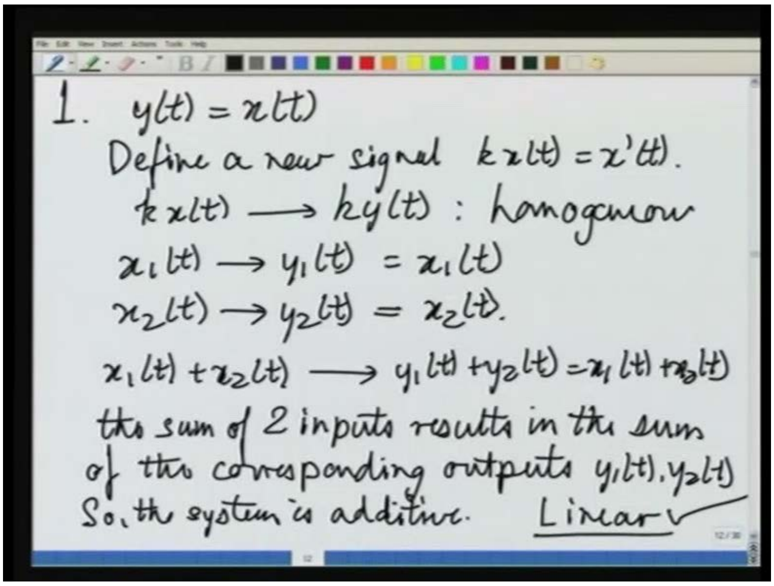

Let us do that starting with a very simple example. What if y t equals x t is this a linear system. To answer that, we will have to test whether the system given by this close form definition satisfies homogeneity and satisfies additivity. If the answer is yes to both these questions then and only then can we say that the system is linear.

So, let us test it for homogeneity, if y t equals x t then let me define a new signal k x t. The definition of the system simply says that whatever the input is, the output is the same. Neither larger nor lesser, neither delayed nor advanced in time, it is just the same signal. So, if I apply a x prime t to the input, then y t must be equal to x prime the output must be equal to k x t are coming to this definition it should be equal to k y t. So, if we apply k x t to the system, then you should get k y t that the output.

So, the system is homogenous, very very straight forward. What if whether the system is additive is what will have to look at next whether the system is additive. So, let x 1 t go to y 1 t which according to our definition equals x 1 t, then x 2 t let it go to y 2 t which by the definition of the system will be equal to x 2 t ,then when x 1 t plus x 2 t is applied as a new input the output will simply be the same as the input as we have been told, it will be equal yield y 1 t plus y 2 t equal to x 1 t plus x 2 t. Since y 1 t plus y 2 t is the output corresponding to the sum of the inputs, we again see that the sum of the outputs. The sum of the inputs results in the sum of the outputs.

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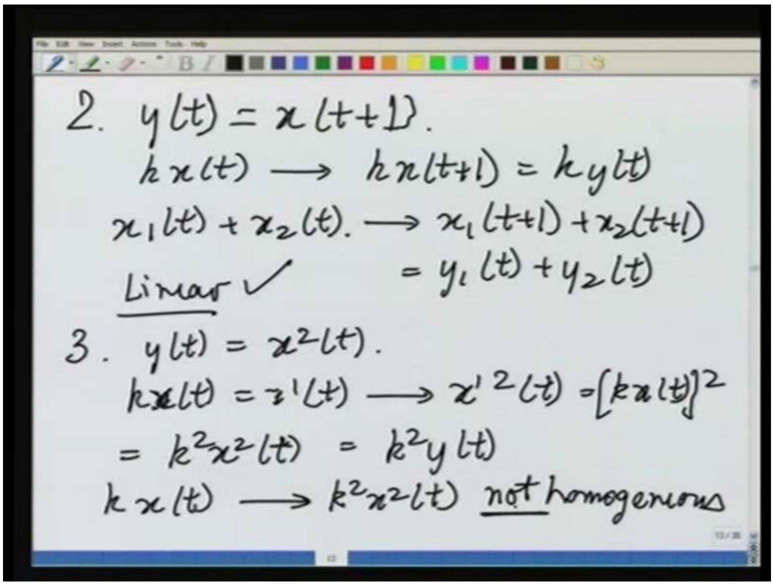

So, the system is additive. So, the system is already been shown to be homogenous also and jointly therefore, we can declare the system to be linear. Second example, here the system which gives y t to be equal to x of t plus 1 that is it. You can clearly see that this is a non causal system, because to determine y at t equal to zero, it requires x at t at the input at one. So, clearly the system is not a causal system, but that not our concern here be the in this case with studying linearity is the system homogenous. If y t equals x t minus 1 and if I apply k times x t as the input, the output will correspondingly be k times x t plus 1 which is equal to k times y t. So, k x t as yield at k y t; that means, the system is stable that the system is homogenous. Now let see what happens, if we test for super position that is, if we test for additivity; suppose x 1 t yields y 1 t, x 2 t yields y 2 t and we apply the inputs x 1 t plus x 2 t.

So, let us see what happens when we apply x 1 t plus x 2 t, for this new signal what will be the output. The output will be x 1 t plus 1 plus x 2 t plus 1 which is equal to y 1 t plus y 2 t by definition. Hence again the systems shows up to be additive as well, additive and homogenous both. So, this system is also linear. Let us now try to find a nonlinear system, a system which defines this definition. Suppose we take y t equals x squared t is this system homogenous. Suppose I apply k x t to the input k x t we will call this say x prime t. We apply this at the input what should I get at the output I will get x prime squared t that will be the output. What is x prime squared t? Substituting the value of x prime t will get that this equals k x t whole squared, which is equal to k squared x squared t, which is equal to k squared y t.

So, you see putting all these multiple steps and together and summarizing our calculation. We find that if you applied k x t to the input, the output you get should have been k y t which is k x squared t if the system are homogeneous, but what you have got is k squared x squared t. So, the system is not homogeneous. Once homogeneity has failed it is reasonable to expect in most cases that additivity will fail also, but let us go ahead and prove it. If y 1 t is the result of applying x 1 t to the input, that is if we apply x 1 t you should get x 1 squared t as the output ,we call that y 1 t. Right.

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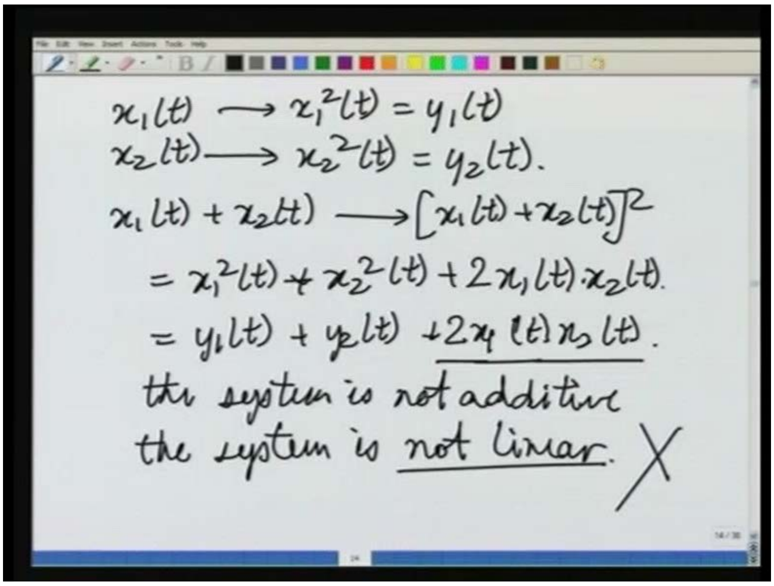

That is just the definition of our system call this y 1 t, x 2 t would yield correspondingly x 2 squared t called this y 2 t. Now let see what happens when you apply both the signals simultaneously. You apply both these together, what happens the output signal we have told is the square of the input signal. So, if this is the input signal the output you get will be the whole thing squared x 1 t plus x 2 t whole squared.

Which as we know expands to x 1 squared t plus x 2 squared t plus twice of x 1 t times x 2 t this on simplification comes to y 1 t, x 1 squared t is after all y 1 t plus y 2 t plus this unfortunate term called 2 x 1 t x 2 t. So, when you applied x 1 t plus x 2 t at the input you add to have got only y 1 t plus y 2 t at the output, if this had happened then we would set that the system is additive. But apart from these two terms there is this additional term 2 x 1 t x 2 t this term spoils the additivity of the system. The system is not additive. It is not homogenous as we have already shown; it is also not additive as we just shown. So, together the system is not linear, an example of a nonlinear system.

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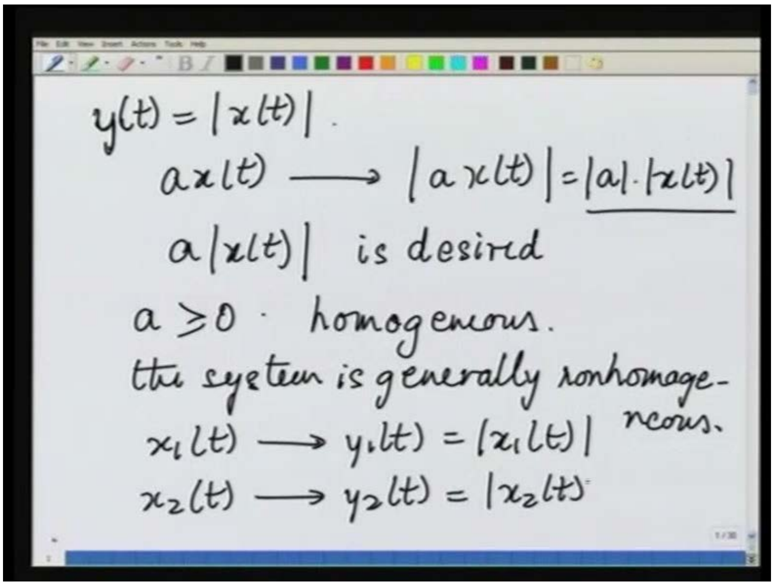

Let’s take another example, and see what happens? If x t is taken stopped here. Let us pick a second example, let the system be given by this equation y t equals mod x t the absolute value. Let see if this system is linear, the right way to do it of course, it is a first test for homogeneity and then subsequently test for additivity. So, let us start with testing for homogeneity, if y t equals mod x t then what is the output corresponding to a scaled version of the input.

So, instead of x t lets apply an input like a times x t suppose this is the input, then the output we should get would be mod a x t which as we all know is equal to mod a times mod x t. Now, clearly under all circumstances this will not be equal to a times mod x t which is what it should be if the system is homogenous. It should be equal to mod a x t. Instead we get mod a times mod x t and therefore, this system can be considered homogenous under certain conditions.

And the conditions being that, a must be greater than equal to zero. So, if a is greater than equal to zero, it is homogenous only, if otherwise it is not and therefore, all we can say finally, is that the system is generally non homogenous. Moving on to testing for linearity. Let us consider two inputs x 1 t yielding an output y 1t equal to mod x 1 t, x 2 t yielding an output y 2 t equal to mod x 2 of t.

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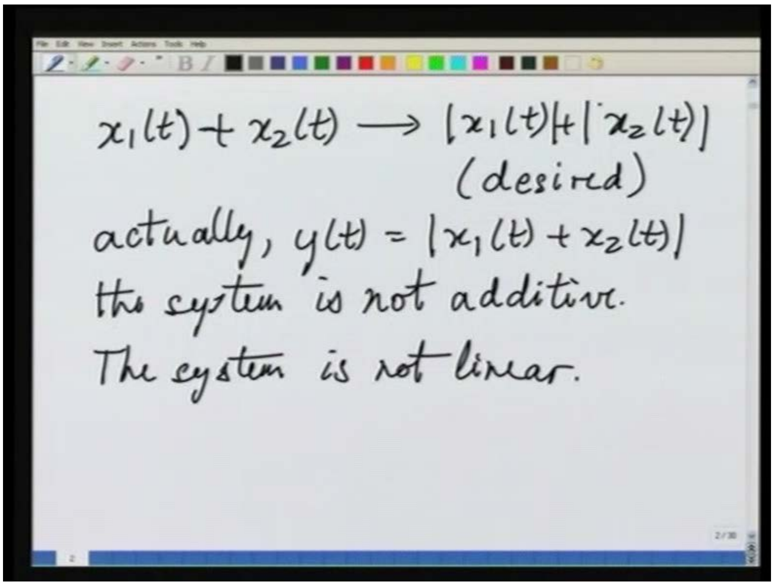

Now, let see what happens? If we apply both this inputs simultaneously, then the input becomes x 1 t plus x 2 t and this add to yield not it actually yields, this add to yield mod x 1 t plus mod x 2 t, this is what is desired. But what happens actually is that we get instant the absolute value of the entire input. So, that actually the combination which we call y t the overall output for this overall input, will be equal to mod x 1 t plus x 2 t. Certainly it is not equal to mod x 1 t plus mod x 2 t except when both x 1 t and x 2 t happened to be greater than or equal to zero. So, this does not hold in general. So, the system is not additive. We thus have a nonlinear system. The system is not linear. So, this is a second example in a study of linearity. We will now want to develop more rigorous tests for this discovering, whether a given system is linear or not linear and this is what we will next proceed to do.