Signals and Systems

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Lecture - 44
Inverse Z Transform

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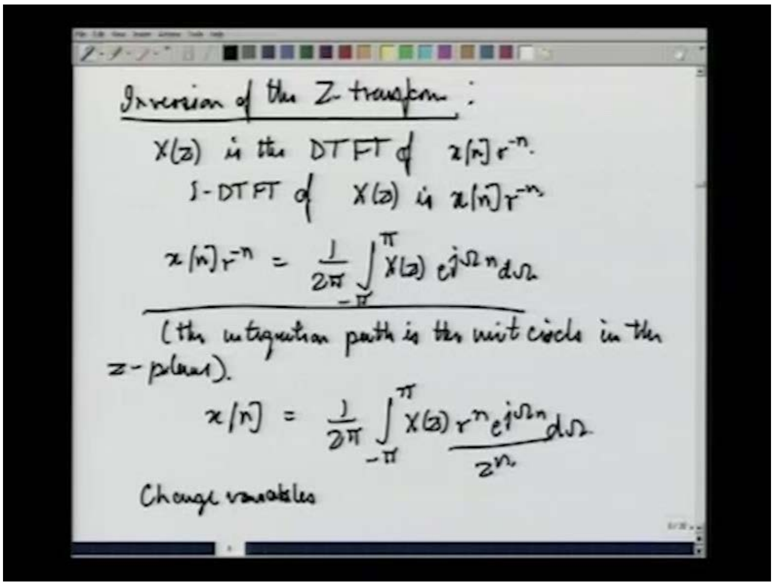

Now, let us find a formal expression for the inversion of the Z transform, as with the Laplace transform, we start the process of deriving an expression for the inversion of the Z transform using the fact that the Laplace, the Z transform is the DTFT of the modified sequence, that is to say that x (z) is the DTFT of x and r to the minus n, keeping this in mind, we can write that the inverse DTFT of x(z) is x(n) r to the minus n. So, you can write we have an expression for the inverse DTFT of any sequence, we can write the inverse DTFT of x(n) of x(z) is r to the x(n) r to the minus n equals 1 by 2 pi integral minus pi 2 pi x(z) e to the j omega d omega.

The integration is done here over values of omega from minus pi to pi, which is to say the part of integration is the units circle in the z plane, fine; because on the unit circle we do know that this converges for whatever value of r n, this does convert; it is only when that is the case that this expression this entire expression is meaning full alright so… Now, what we will say next is this, if the integration path is the unit circle on the z plane for this expression, let us continue to see what happens? If we multiply both sides by r to the n. So, that we just get x(n), x(n) equals 1 by 2 pi integral minus pi 2 pi x(z) r to the n e to the j omega n d omega, this is just equal to z to the m as you can see, finally we need to change variables.

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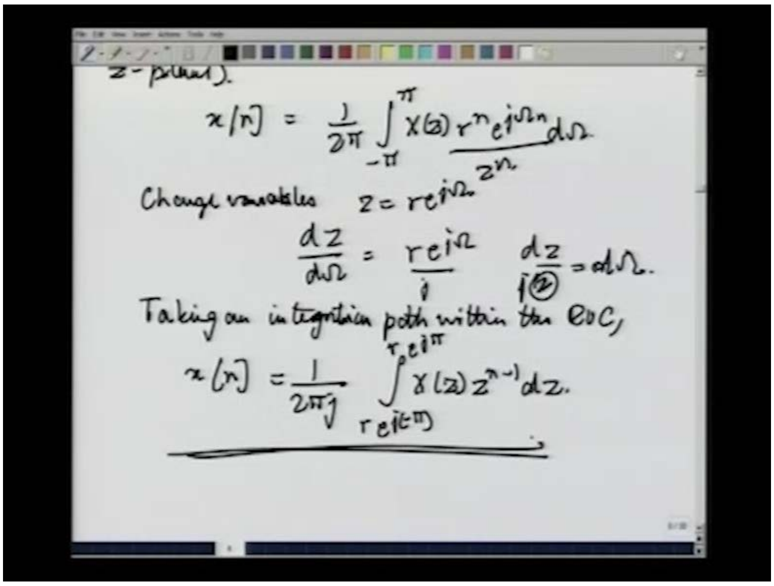

We need to change variables on the k, we have to use z equal to r e to the j omega. So, d z by d omega equals r e to the j omega by j that is to say that d z by j z equals d omega, if you make the substitution then the variable of integration is z, and therefore the path of integration has to be through the ROC. So, taking a path through the ROC integration path within the ROC x(n) becomes equal to 1 by 2 pi j integral r e to the j minus pi to r e to the j pi x(z) z to the power n minus 1, because we have 1 by z over here d z, this is the formal expression for the z transform inversion.

Though we have this formal expression for the inversion, we shall not just as with the Laplace term from use it very often, since our concern will largely be with exponential functions and a few other standard forms for which the Laplace transform, the z transform inversion is given by a familiar formula we do not really have to do this integration. So, let us see what we really need to do if we have a polynomial rational form which is the usual thing that we get when we start with the difference equation describing a system that we apply the z transform to on both sides, and then get an expression for h z.

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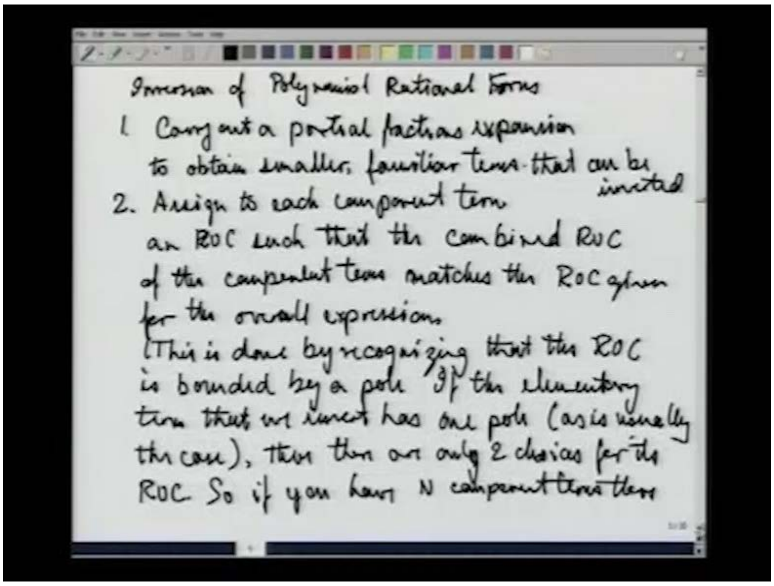

So, what we will have is a polynomial rational form, infer inversion of polynomial rational forms, inversion of polynomial rational forms. What we have to do is to do a partial faction expansion to obtain smaller terms, smaller and more familiar. Now, second assign well former a smaller familiar terms that can be inverted, assign to each component term and ROC such that the combined ROC of the component terms matches, the ROC given for the overall expression. In order to do this, you will have to recognize a simple fact that was reasonably apparent in the examples that we have work out so for, that is that the ROC is bounded by the existence of poles by recognizing that the ROC is bounded by a pole. If the elementary term that we invert has one pole as usually the case, then there are only two choices for its ROC, there only two choices for its ROC. So, if you have N terms N component terms.

There are 2 to the N ROC combinations, and the right one is chosen by checking for a match with the given ROC for the overall expression. So, that is what we have to do? Now, let us go through a list of standard z transform pairs common z transform pairs which are likely to occur as the component terms.

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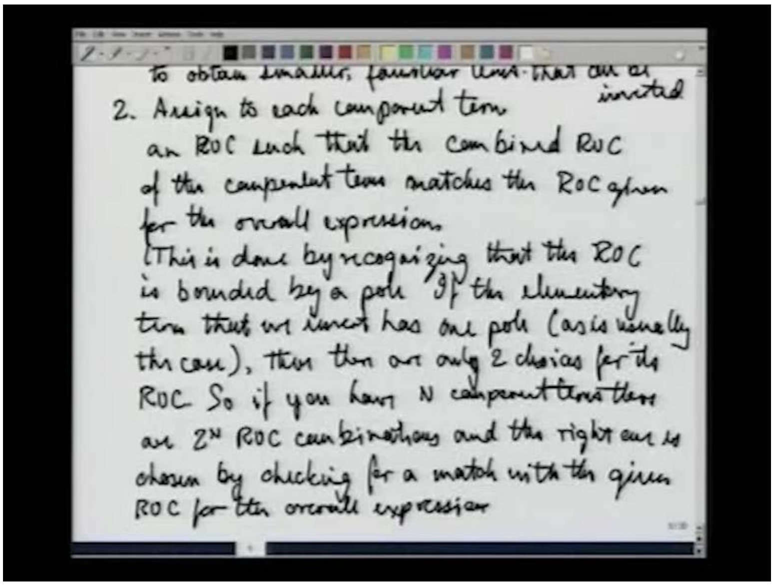

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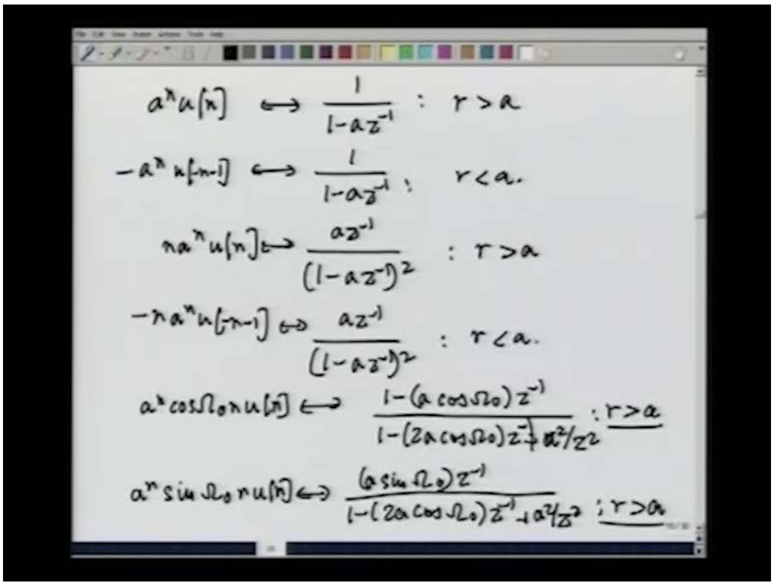

We are already familiar with two of them, a to the n u n transforms to 1 by 1 minus a z inverse with an ROC of r greater than a minus a to the n u minus n minus 1 transforms through the same thing 1 minus a z inverse r or less than a, then there are some more expressions which we will just note down. n times a to the n u n transforms to a z inverse divided by 1 minus a z inverse whole squared with an ROC of r greater than a.

This comes out from the differentiation in frequency property which we will discuss later for the z transform, then you have minus n a to the n u minus n minus 1, which also transforms to the same expression a z inverse by 1 minus a z inverse whole squared, but this for r less than a fine. So, this is a typical expression you get when you differentiate in frequency differentiate in respective z, next you have a to the n sin cos omega naught n u n which transforms to 1 minus a cos omega naught z inverse 1 minus a cos omega naught z inverse over 1 minus 2 a cos omega naught z inverse 1 minus 2 a cos omega naught z inverse plus r squared r squared by plus a squared by z squared for r greater than a does the ROC.

Finally you have a to the n sin omega naught n u n which transforms to 1 minus, there is no 1 minus, just I have a sin omega naught z inverse over 1 minus 2 a cos omega naught z inverse plus a square by z squared with r greater than a is the ROC . So, these are some of the standard forms. Now, the last 2 forms that we have over here namely that corresponding to cos omega naught n and sin omega naught n with an exponential term like a n attached multiplied will simplify appropriately to the case when a is taken as 1, for example the first expression a n cos omega naught n u n will just become cos omega naught n u n, and oscillatory function and that will have z transform as you can evaluate from this expression as 1 minus cos omega naught z inverse by 1 minus 2 cos omega naught z inverse plus 1 by z squared alright.

So, these are the standard forms, one unit worry beyond the standard forms. Now, I will work out an example in detail to demonstrate how? If you have an certain h z given to you or x(z) whatever you want to call it, and you are given the task of inverting by using partial fractions expansion and identification of individual ROC's and then inversion of the individual terms. So, let us try to do it for a concrete example to make us more confident about it. Before we do this it is convenient and in fact essential to learn some of the standard properties of the ROC of this a transform, these properties will very closely resemble the properties of the ROC of the Laplace transform and so we will go through them in more or less the same order and point out the analysis where as in where we meet them.

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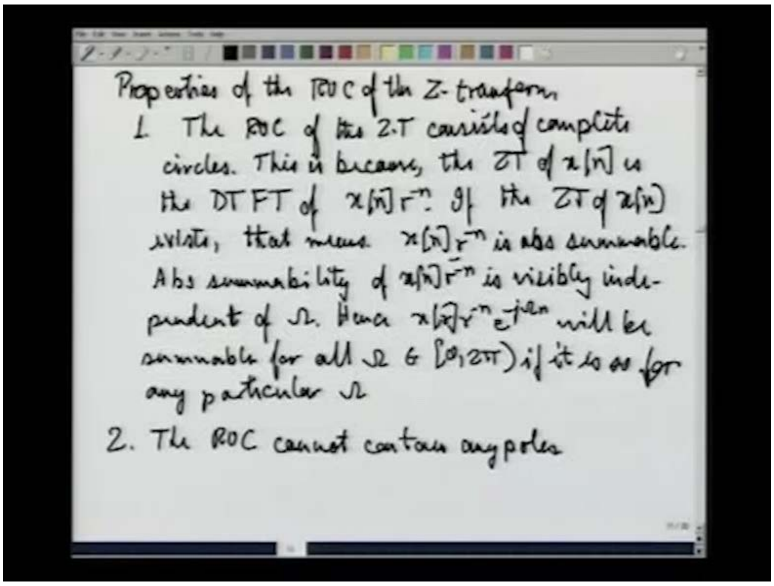

So, properties of the ROC of the z transform, the first property analogs to the first property we discuss there, is that the z transform the ROC of the z transform, the ROC of the z transform consists of complete circles. In the case of the Laplace transform we found that it consisted of complete vertical lines well, the reason here is similar to the reason over there, this is because the z transform of x(n) is the DTFT of x(n) r to the minus n. So, if the z t of x(n) exists it means, that means that x(n) r to the minus n is absolute absolutely summable alright, if it is absolute summable then you see that absolute summability has nothing to do with omega. It does only got to do with r.

So, absolute summability of x(n) r to the minus n is visibly independent of omega, hence x(n) r to the minus n will be will be summable sorry, x(n) r to the minus n e to the minus j omega n will be summable, if it is will be summable for all omega in 0 to 2 pi, if it is so for any particular omega that completes the proof, so to speak. The next thing of course, the next familiar statement we will make the next property of the ROC of the transform is that the ROC by its very definition is a region of convergence and so it cannot contain any poles, the ROC cannot contain any poles.

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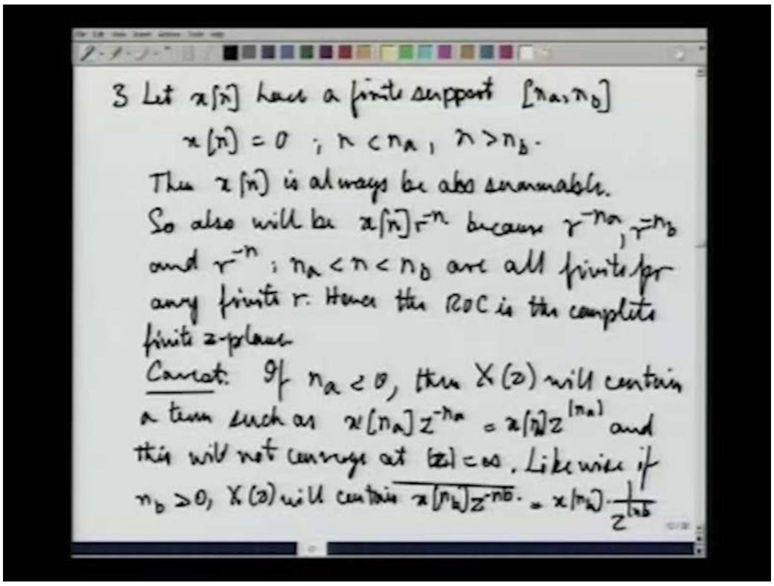

Now, the third property, suppose we have x(n) as a finite support sequence, a sequence we just runs from say some n a to n b. Let x(n) have a finite support, then we have let the support we say n a comma n b, that is to say that x(n) equals 0 for n less than n a, and n greater than n b, this is what we have. Then clearly x(n) will always be absolutely summable. So, also will be x(n) r to the minus n correct, x(n) r to the minus n will also be absolutely summable, because r to the minus n a, r to the minus n b, and r to the minus n for n a less than n less than n b are all finite for any r for any finite r. Hence the ROC is the complete finite z plane, however there is a probable zone, there is a caveat; if n a is less than 0 ,then x(z) will contain a term such as x n a z to the power minus n a, fine.

Now, since n a since n a is negative z to the power minus n a is just equal to z to the power mod n a, and this will not converge at mod z equal to infinity, remember that unlike the Laplace transforms context, here we include infinity as part of the z plane as well as the origin z equal to 0 as part of the z plane. So, this means that this has to be taken out of the finite z plane is completely covered is completely part of the ROC, but z equal to infinity is not part of the ROC. Likewise if n b is greater than 0, x(z) will contain x n b z to the power minus n b where n b is greater than 0.

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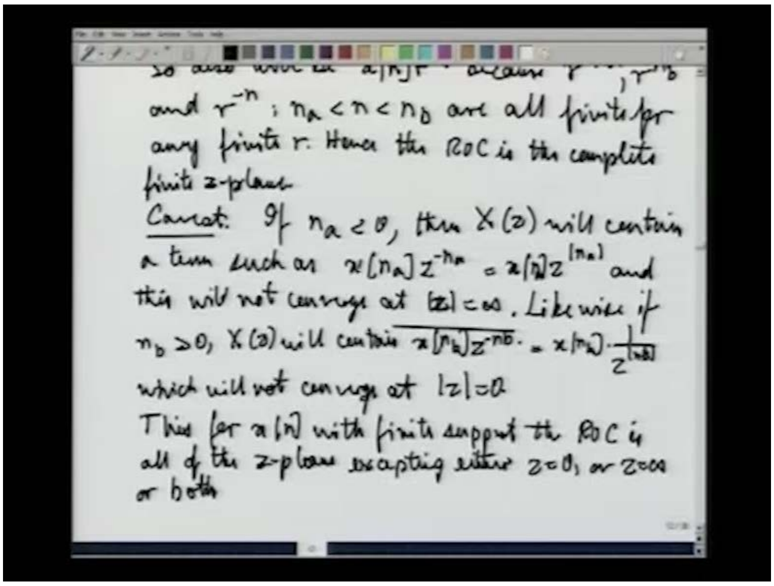

So, you have 1 by z to the power mod n b x n b times 1 by z to the power mod n b, which will not converge at mod z equal to 0, thus summarizing thus for a finite sequence thus for x(n) with finite support, the ROC is all of the z plane. The z plane excepting either z equal to 0 or z equal to infinity or both. So, that is for the finite length sequence, finite support sequence.

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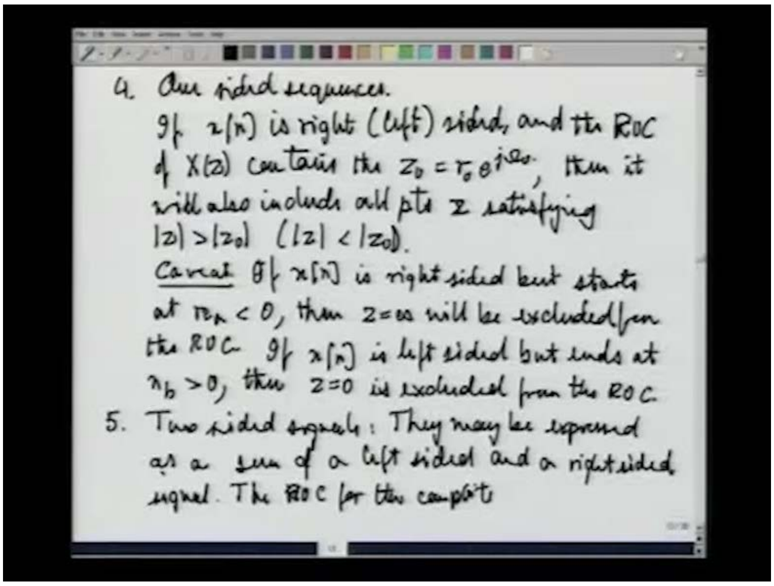

Now, let us look at the next case next property of one sided sequences, we will handle both the left and right sided sequences at 1 go by writing that if x(n) is right parenthesis left sided, and the ROC of x(z) contains the point z naught equals r naught e to the j omega naught, then it will also include all points z satisfying mod z greater than mod z naught in parenthesis mod z less than mod z naught. Now, this is proved by a process that is extremely analogous to similar to for the Laplace transform case.

So, we will not bother to go into this further, all we need to point out is by making this combined statement for the left and right sided sequences, we are saying that this is actually two statements; if x(n) is right sided and the ROC of x(z) contains the z naught, then it will also contain all points of z satisfying mod z greater than mod z naught. The second statement would be if x(n) is left sided, and the ROC contains the point z naught, then it will also include all points z satisfying mod z less than mod z naught. So, these two statements are being made together, in addition there is the caveat that came up with the previous case if x(n) is right sided, but starts at n a less than 0 then z equal to infinity will be excluded from the ROC, likewise if x(n) is left sided, but ends at n b greater than 0 then z equal to 0 is excluded from the ROC, right.

So, this is just the application of the same arguments that we found for the finite support sequence case. Finally, we have the case of two sided signal, fine. Now, two sided signals they are split expressed as a sum of a left sided, and a right sided signal, we expressed as a sum of a left sided and a right sided signal alright. Then we applied the two criteria that came out of the previous condition for one sided sequences simultaneously, and we have an ROC for one sequence for one part of the sequence an ROC for the second part of the sequence, and we know then that the ROC for the combination will be the intersection of the component ROC's.

The ROC for the complete signal will be the intersection of the, ROC of the complete signal will be the ROC of the will be the intersection of the component ROC's, if they both exist and intersect, fine. That is for the tw0 sided thing, so let us summarize what we have found?

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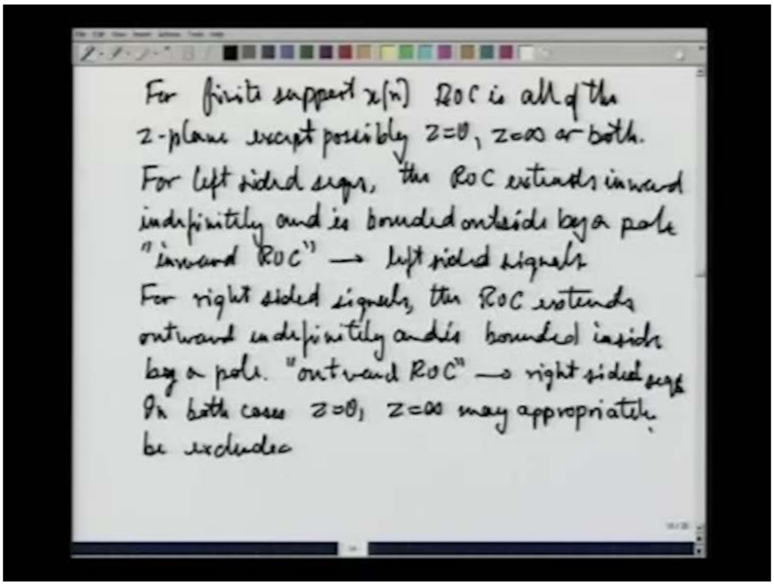

For finite support x(n) ROC is all of the z plane except possibly z equal to 0 z equal to infinity or both for one sided sequences for left sided sequences, the ROC extends inward indefinitely, and is bounded outside by a pole, we call such a configuration an inward ROC, inward ROC, and it happens for left sided signals. Next for right sided signals, signals, the ROC extends rightward or rather outward indefinitely, and is bounded inside by a pole, thus the outward ROC is what you call this, and it corresponds to a right sided sequences right sided sequences, fine.

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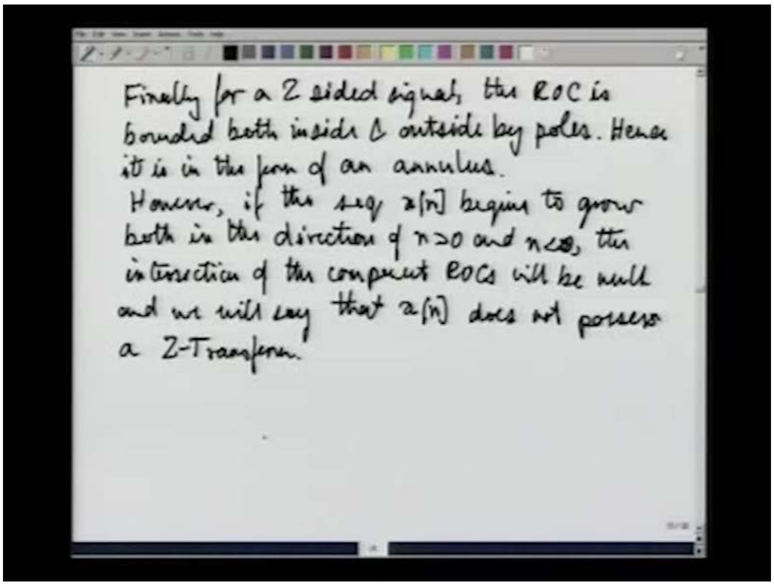

So, it is very nice inward ROC for left sided signals, outward ROC for right sided signals; of course, with in both cases we will have to say that z equal to 0, z equal to infinity may appropriately be excluded in both cases, z equal to 0, z equal to infinity may appropriately be excluded.

Finally, for a two sided signal the ROC is bounded both inside and outside by poles, hence it is in the form of an annulus. However, if the sequence x(n) begins to grow both in the direction of n greater than 0, and n less than 0, the intersection of the component ROC's, ROC's will be null. And we will say that x(n) does not possess a Z transform.

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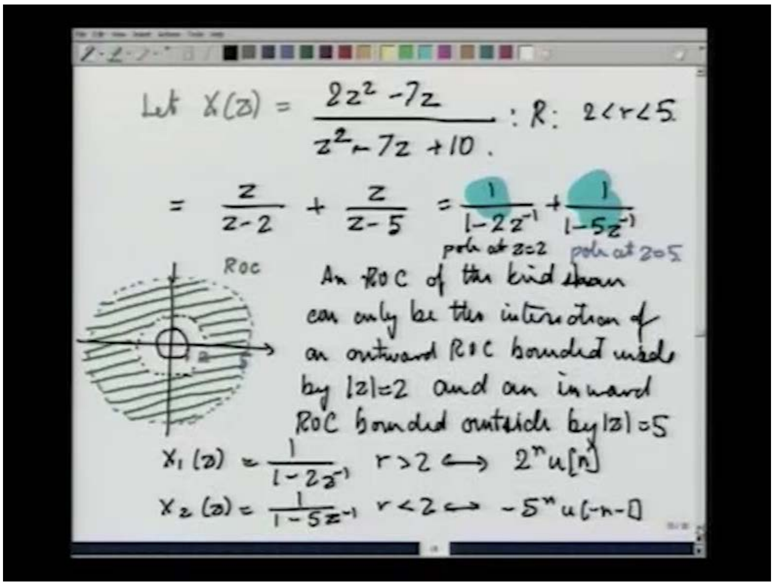

So, let us stop at this point and continue the discussion in the next session. So, let us work out an example that brings out many of the features of the manner in which we solve the Z transform the way be invert the z transform. So, here is the example, let x(z) be given by 2 z squared minus 7 z over z squared minus 7 z plus 10. This is a sufficiently cryptic looking x(z), and let us say that the ROC R is given as follows, 2 less than r less than 5, so this is an annular ROC. Now from our knowledge of the properties of the z transform that we of the z transform ROC that we have just required, this seems to suggest right on the phase of it that we have a two sided signal, because the ROC is annular. The first task is to do a partial fractions expansion, now if we carry out a partial fractions expansion it turns out that this is equal to z by z minus 2 plus z by z minus 5, this is what we have which can be written in the more familiar form as 1 by 1 minus 2 z inverse plus 1 by 1 minus 5 z inverse, so we have this.

Now this is the real task of choosing appropriate ROC's for each of these two component terms, components of the partial fractions expansion. Now, consider 1 by 1 minus 2 z inverse, this can either represent a leftward ROC with r less than 2 or a rightward ROC with r greater than 2, it cannot be any third possibility, because we have two as a pole in the function. So, let us look at the ROC of the complete function, let us make a plot of the ROC of the complete function, that is might be unit circle, then you have 2, this is at 2, this was at 1, this is at 2, and then there is the third boundary at 5, which have the third boundary at 5. We have told that the ROC is annular; that means, to say that the ROC fills up the space between these two dotted boundaries that we have drawn like this.

This is the angular ROC, we have, that is the ROC. Now the ROC is bounded inwarded at 2 inside at 2 and outside at 5, such an ROC can only be found as the intersection of an outward ROC's starting from 2, and an inward ROC ending at 5. So, that is the common we have to make. Now an ROC of the kind shown can only be the intersection of an outward ROC bounded inside by mod z equal to 2, and an inward ROC bounded outside by mod z equal to 5, that is the only way you can do it, right.

So, when this is clear to us we have to only assign the appropriate ROC to the appropriate term, the first term over here, the first term over here has a pole at z equal to 2, this second term over here, the second term over here has a pole at z equal to 5. This means simply that, it is the first term that should have an ROC which is outward, and it is the second term that should have an ROC which is inward, now we are in a position to completely separate the 2 terms, and assign them their respective ROC's, we will say therefore that x 1 sorry, x 1 (z) equals 1 by 1 minus 2 z inverse with r greater than 2, x 2 z equals 1 by minus 1 minus 5 z inverse z inverse with r less than 2.

Now, what does this transform to the inverse transform of this will be the even outward ROC, an outward ROC will have a signal which is right sided. So, the right sided signal for a form which is a standard form such as this will simply be 2 to the n u (n), and similarly for this expression which is 1 minus 5 z 1 by 1 minus 5 z inverse which has an inward ROC, it will be of the form minus 5 to the n u minus n minus 1.

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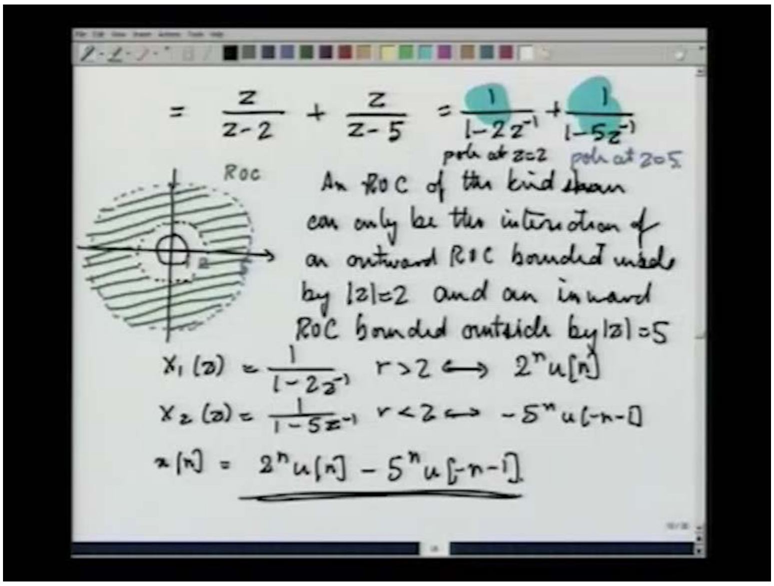

So, adding the 2 terms together, we can write that x(n) which is the inverse transform of the complete x(z) that was given to us is 2 to the n u n plus rather minus 5 to the n u minus n minus 1, this is the complete exercise of inversion, but let us just explore other possibilities that could have been given could have been supplied as the ROC for x(z) briefly.

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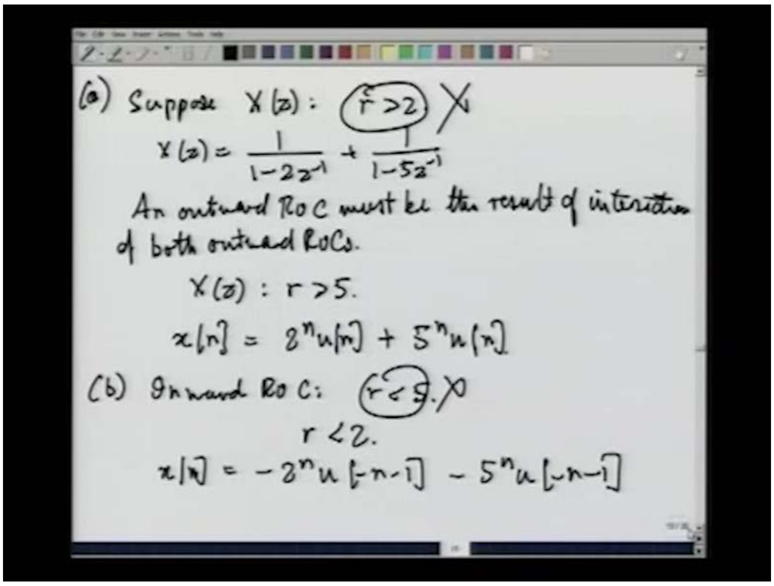

Suppose x(z) had been the same, but we had been told that the ROC was r greater than 2, suppose we have told that the ROC was r greater than 2. Now, is it possible to have an

ROC which is r greater than 2 for an expression like this, it is not possible to have an ROC for r greater than 2, because you remember that x(z) equals 1 by 1 minus 2 z inverse plus 1 by 1 minus 5 z inverse.

Now, if you have an outward ROC, that outward ROC would be the intersection of 2 outward ROC's, an outward ROC, ROC must be the result of intersection of both outward ROC's, but if you produce an outward ROC from the first term x 1 of s, x 1 of z that will be outward from 2. The second term we produce an ROC which is outward from 5, and the intersection of these 2 will only be outward from 5, it will not be outward from 2. Hence this specification is incorrect, there can be no signal x(n) which has this given x(z) as the algebraic expression and an outward ROC is starting from 2.

However, if you had been told that the outward ROC was starting from 5 that is to say x(z) with r greater than 5, then clearly we would assign outward ROC's to both these terms, and we would get x(n) if I may be allowed to write it right away as 2 to the power n u n plus 5 to the power n u n. Both right sided signals, both outward ROC's, and hence the overall ROC is the intersection of 2 outward ROC's, hence it is outward from 5 from mod z equal to 5 right.

Now, suppose we had both inward ROC's that is the next possibility. So, we will call this possibility a and we will look at possibility b, suppose we had been told inward ROC is starting from 5, this is also an impossible case, because an inward ROC can be the result only of intersecting 2 inward ROC's, and when we have 2 inward ROC's the inward ROC that x 1 can give is inward from 2, x 2 can give an inward ROC's starting from r equal to 5, that is r less than 5. The intersection of these 2 will only consists of an ROC starting an ROC for r less than 2, it will not consist of r less than 5. Hence this specification is incorrect; however, if we are said inward ROC r less than 2, then that is fine, because we would had said that we have 2 sequences both left sided sequences.

And you would write x(n) equals minus 2 to the n u minus n minus 1 minus 5 to the n u minus n minus 1, this is fine, it is a perfectly acceptable. There is only one last possibility which we will called as possibility c, this possibility c is not actually a possibility it should be called an impossibility c.

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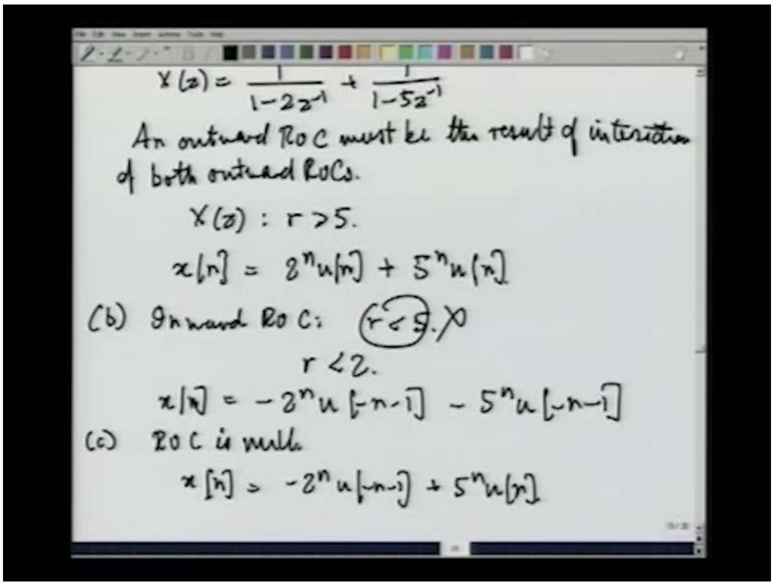

What if x 1 of s had an inward ROC and x 2 of s as an outward ROC, then you would have been told that ROC is null interestingly, and almost humorously even when we are told that the z transform does not exist effectively. So, long as we are given the expression which was given to us namely the expression of the z transform is 2 z square minus 7 z by z square minus 7 z plus 10, if you are just given this expression.

Then given this expression as well as told that the ROC is null, we can still invert it, we need an inward ROC for x 1 s, and an outward ROC for x 2 s; that means, a left sided sequence for x 1 s, a right sided sequence for x 2 s. So, you would get x(n) equals minus 2 to the n u minus n minus 1 minus plus 5 to the n u n, this sequence will not have a Laplace z transform for any value of r; that means, it will converge at no point on the z plane. So, that was a thorough work out of one example we have beaten it from all directions and seen what comes and what.. .