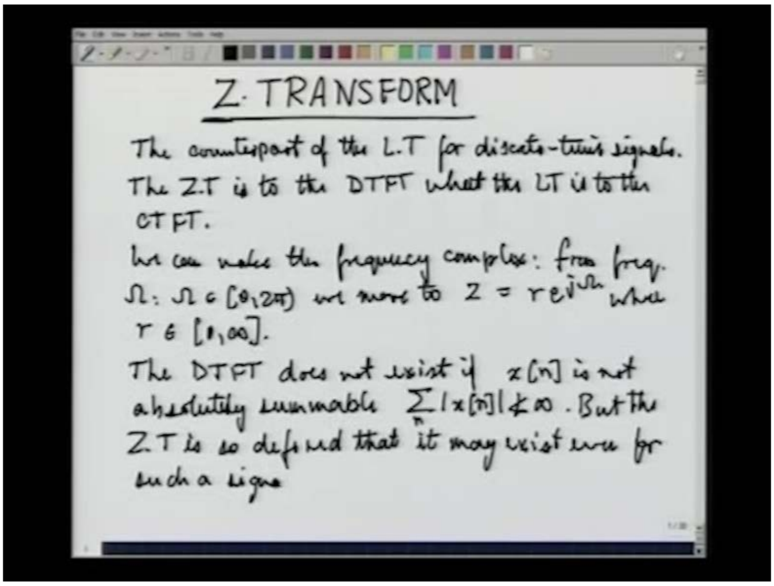
Signals and Systems
Prof. K. S. Venkatesh
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 43
Z – Transform

(Refer Slide Time: 00:29)

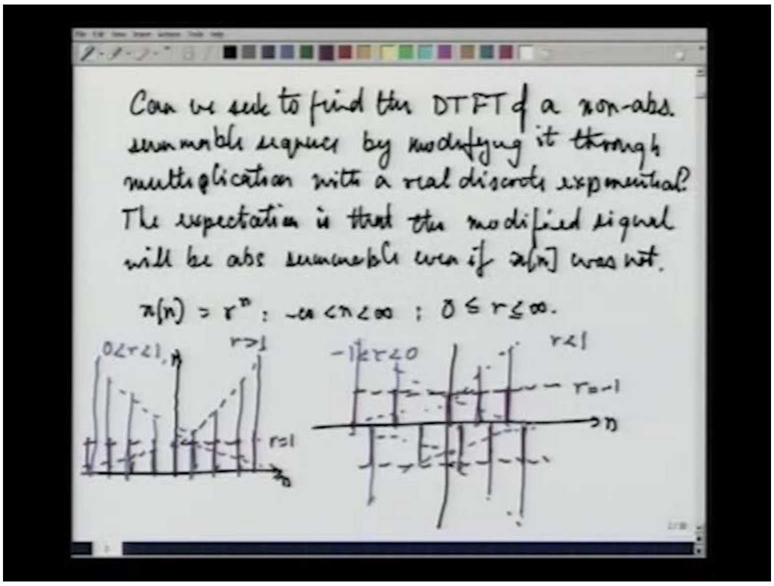


The final item on the agenda of this course is the transform that we call the Z transform. The Z transform has many, many analyses with the Laplace transform. In fact, the process by which I will treat the subject will go more or less through the same steps, and often include the same remarks except for the addition of drawing analogies with the Laplace transformation that I did when I handled the Laplace transform. So, there are too many things in common, and yet there are significant differences in appearance.

However if one is able to get beyond the superficial differences, one does see that there is a lot of commonality between the Z transform and the Laplace transform. So, let us start by telling you what it is? The Z transform is the counter part of the Laplace transform for discrete time signals. For discrete time signals that is to say that the Z T as I will abbreviate it to is to the DTFT, what the LT is to the CTFT.

The reasons we have invented a Z transform are also similar to the corresponding reasons we gave for the case of the Laplace transform. One way of looking at it is that we can make the frequency complex, so instead of frequency from the from frequency omega, where omega was from 0 to 2 pi. We move to set given by r e to the j omega, where r is a real number in the range 0 to infinity. Clearly you can see for what I have just written that even r equal to infinity and r equal to 0, I have considered which is unusual for normal mathematics. So, this is one way of looking at it the other ways to recall that the DTFT does not exist the DTFT does if x n is not absolutely summable write that is if summation over all n of mod x n is not finite, then the DTFT may not exist, but the z transform is so defined that it may exist even for such a signal.

(Refer Slide Time: 05:39)



Basically we ask the question can we seek to find the DTFT of a non absolutely summable sequence by modifying it through multiplication with a real discreet exponential. The expectation is that the modified signal will be absolutely summable even if x n was not. So, that is the idea whether we can somehow arrange this. Now, let us quickly go through the appearance of different kinds of discreet real exponentials or different values of the parameter. Now the general discreet exponential is written as x n discreet real exponential is x n equal to r to the power n fine for minus infinity less than n less than infinity.

So, this is where and r is 0 less than equal to r less than equal to infinity right. Now, let us make some fluid plots first for r greater than 1 r greater than 1, you have r to the power n. So, at n equal to 0 this will be equal to 1, and for positive n it will increase rapidly for negative n it will decrease exponentially. So, you have and so on, and on this side you have this is r greater than 1 next case which we will plot on the same graph for comparison is r lying between 0 and one.

When r lies between 0 and 1, 1 by r is greater than 1, and thus what you find is 1 by r to the power n is a sequence of the sort we have just drawn and r to the power n is the reciprocal of these values that we have plotted over here. In short it will be tall for values n less than 0 and short for values of n less than greater than 0 as n goes towards minus infinity it will blow up exponentially and get exponentially infinitely large as n goes towards plus infinity it will get smaller.

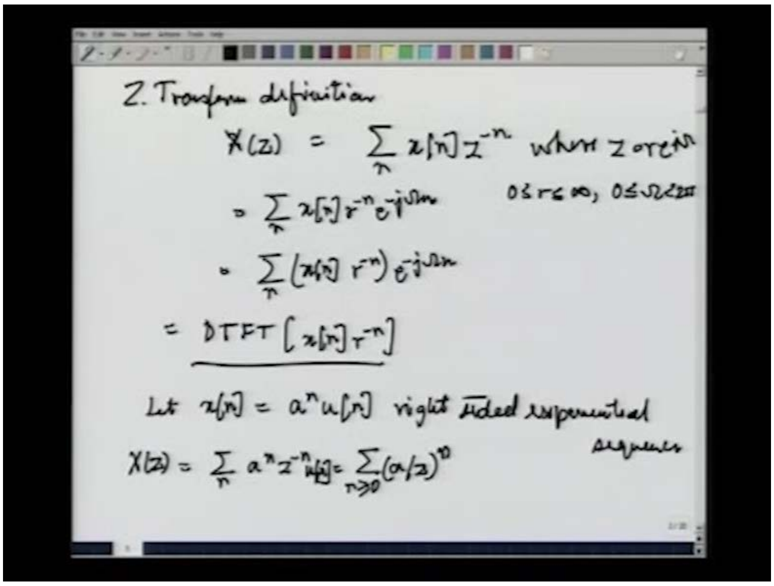
So, you have this. So, this is 0 less than sorry, 0 less than r less than equal to 1. what about r equal to 1 when r is equal to 1 you will get that curve which lies exactly in between these curves which is the fixed sequence with let us say the constant sequence . So, you see how the tops of these 3 varieties for r lying between 0 and 1 and beyond 1 are shown.

So, this is as far as positive values or non negative values of r are concerned what happens when r is 0, when r is 0, 0 to the power is identically there is nothing to discuss. Next we talk about what happens when r is negative, when r is negative you get sequences that look like what we have already drawn, but they are alternating in magnitude. So, thus for r less than 1, what you get is a number like say minus 2.

So, minus 2 to the n increases in magnitude as n increases and decreases as n goes towards negative infinity, but it alternates from 2 to the n to minus of 2 to the n to plus 2 to the n minus 1 and it keeps changing in sign. So, you have this for a 0 it is of course, this, then it is this, then it is this, this is for r less than 1. What happens if r lies between 0 and minus 1, there is a small correction I am making over in this place 0 less than r less than 1. Here I am now going to plot well if I continuing this plot towards the left side of the axis I would get this for I am drawing for r less than 1 sorry.

So, that it has a pair of envelopes like this. This is for r less than 1, now we move to r lying between 0 and minus 1. So, minus 1 less than r less than 0 you would now have 1 by r mod 1 by r greater than one. So, you get a curve similar to this expect that it gets flipped along the time axis about the time axis. So, you get an upper and a lower envelope like this that is what you get, but both these as you see are alternatives now the last case of what happens if you take r equal to minus 1. Here also we have plotted sorry here we will give the different color when r equals minus 1 you get an alternate pattern of pulses all of magnitude unity, but continuously reversing sign. This is r equals to minus 1. So, we have all these information, now about how real exponentials discreet exponentials look when they are subjected to various values of r.

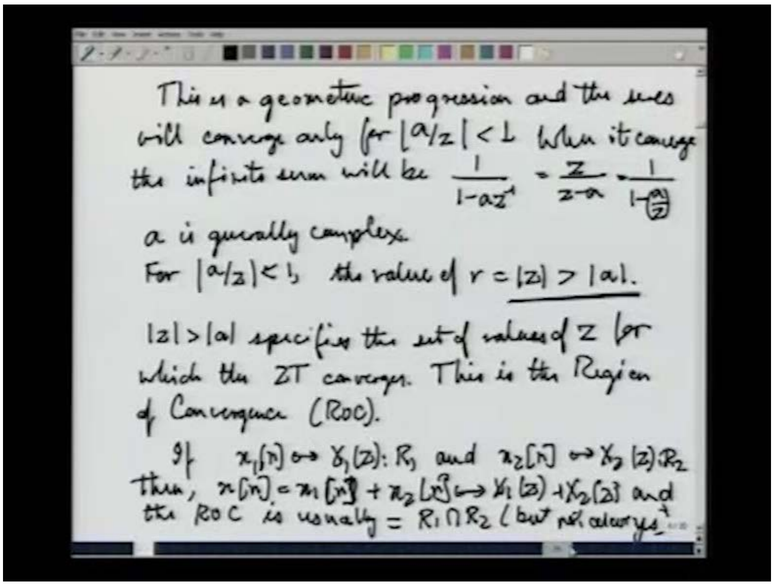
(Refer Slide Time: 17:15)



Now, let us make a formal definition of the z transform. x z as it is called equals the summation over all n of x n z to the minus n where z equals r e to the j omega, 0 less than equal to r less than equal to infinity, 0 less than equal to omega less than 2 pi that is z transform . Now, we can therefore, rewrite it as equal to summation over all n x n r to the power minus n e to the minus j omega n which equals summation to the over n of x n r to the minus n, e to minus j omega n which is simply the DTFT of x n r to the minus n this is what we said when we claimed that even if x n does not have a DTFT x n r to the minus n the modified function might have a DTFT.

So, the modification is being carried out by means of a multiplication with a real exponential exactly what we did in the case of the Laplace transform where the real explanation we multiplied with e to the minus sigma T here we are multiplying by r to the minus n fine. So, at all times let us not forget that the z transform is the DTFT of a modified version of the original x n. So, now, having seen this much of it we will see if we can compute the z transform for some simple functions for example, let x n be equal to a to the n u n which is a right sided exponential sequence. We have this right sided exponential sequence. So, let us just substitute it in the expression and see that x z equals summation over all n a n z to the minus n, which is equal to summation over all n greater than equal to 0, because there is u n over there u n should have been written over here which is now no longer required.

(Refer Slide Time: 20:52)

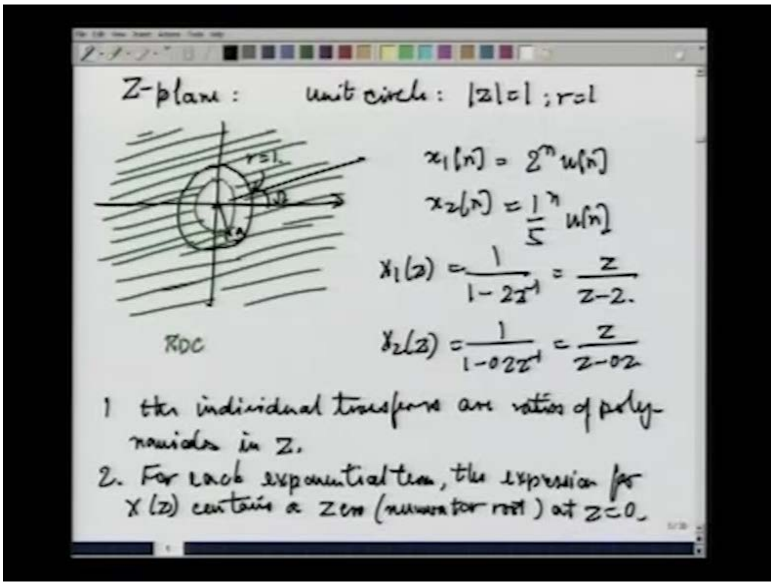


So, you write a by z to the power minus a by z to the power n this is a geometric progression and the series will converge only for mod a by z less than 1 when it converges the infinite sum will be 1 by 1 minus a z inverse which is nothing but z by z minus a or 1 by 1 minus a by z this is what we have. So, now let us see we have an expression for the infinite sum if it converges now when does it converge it converges when a by z is greater than is less than 1.

Now when this mod a by z less than 1 a is generally complex right and a by z for mod a by z to be less than 1 we need to ensure the value of r equal to mod z must be greater than mod a. So, this defines what is called the region of convergence the set of points the set of values of z for which the z transform exists right, now this is called the region of convergence this is the ROC the concept of the ROC is very much analogues to the corresponding concept in the case of the Laplace transform just as was the case with the Laplace transform we will largely confine ourselves to time functions which are exponentials and relatives of the exponentials, but before we study more examples let us just examine a fundamental property of this z transform that also correspond to a similar property with the Laplace transform.

Namely how it handles the sum of multiple component functions that is to say if x n had a z transform of x z sorry if x 1 n had a z transform of x 1 z and an ROC of r 1 and x 2 n had a z transform of x 2 z with an ROC of r 2 then x n equal to x 1 n plus x 2 n will have a transform of x 1 z plus x 2 z, and the ROC is usually equal to r 1 intersection r 2, but not always more on this later more about the expectations at a later point. So, all we have to note write now is that there is an ROC in addition to the algebraic expression of the function. Now, let us look at a plot of the ROC of the z transform just like we had made plots of the ROC of the Laplace transform of a function in the case of the Laplace transformation rectangular coordinates sigma and omega sigma along the horizontal axis omega along the vertical axis

(Refer Slide Time: 27:56)



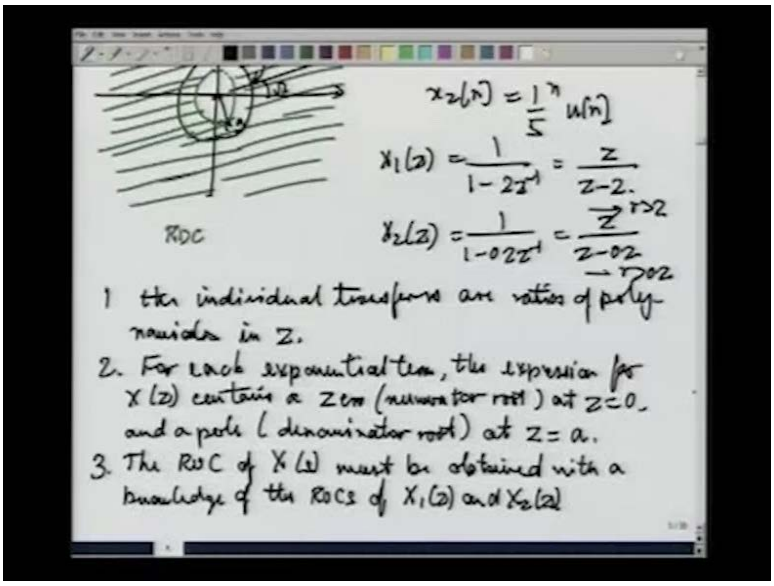
Now, for the z transform it turns out to be more convenient to use a polar format a polar coordinate system where we have omega along the rotational axis and r along the radial axis. So, we have the z plane and in the z plane a very important entity is the so called unit circle is the set of points satisfying mod z equal to 1 or simply r equal to 1.

It is a circle of radius 1 about the origin that is mod z equal to 1 that is the unit circle. Now this is the point at which r equal to 0 and r increases radially outward like this and omega increases like this is the value of omega at for this point and omega changes is the angle in the polar coordinate system that is omega. So, now, given 1 ROC and another ROC we can try to see how to plot them if we take the first example of the problem we have solved where a was the certain complex number and we found that the ROC corresponded to mod z greater than a let us say that a is over here. Let us say that this is a this point which we have marked with a crones is a. Now, this has some magnitude whose value is given by this and to say that the region of convergence includes all points on the z plane for which the magnitude is more than the magnitude of a is to be able to draw a circle of the radius of a and then to mark by shading all points outside the circle as the region of convergence.

So, the green area is the ROC. Now, if you have 2 functions like we had in the last example where you had x 1 n and x 2 n, lets let us make that example a little more congregate let us say that x 1 n equals say 2 to the power n u n and x 2 n equals 1 by 5 to the power n u n then clearly we have x 1 z equals 1 by 1 minus 2 z inverse and 1 by sorry, x 2 z.

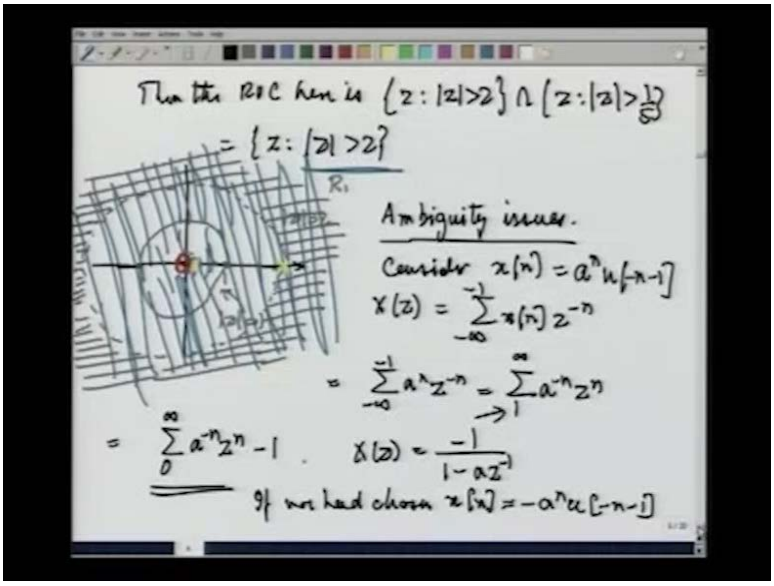
So, this is equal to z by z minus 2. x 2 z equals 1 by 1 minus 0.2z inverse equals z by z minus 0.2. We can see 2 thinks, first that the individual transforms are ratios of polynomials in z that is the first observation to make. The second observation is that exponential term the expression for x z contains a 0 numerator root at z equal to 0 and a pole.

(Refer Slide Time: 34:02)



Which is a denominator root at z equal to a, whatever is the value of a 2 for x 1 of s and its 1 by 5 equal to 0.24 x 2 of s, there is always a pole and a 0 both these things are there and it is in polynomial rational form just like was the Laplace transform right now with these two things, we will now look at the ROC s the ROC of the x s must be obtained with a knowledge of the ROC s of x 1 z and x 2 z. Now, for this we have r greater than 2 here, we have r greater than 0.2 this is the ROC for this ,this is the ROC for this right.

(Refer Slide Time: 35:49)



Now, we will see thus then the ROC here is intersection set of all points is the set which is the intersection of the set of all points z for which mod z is greater than 2 with the set z for the set of all point mod z greater than 1 by 5 which equals the set containing those line for which mod z greater than 2. Thus if we go back and draw the ROC well there is no place over there. So, we will make a fresh diagram we will draw the poles the zeros as well as the ROC s of the individual terms and the ROC of the overall term over all expression. Let us draw the unit circle first. So, that is the unit circle mod z equal to 1 then you has a pole at z equal to 2.

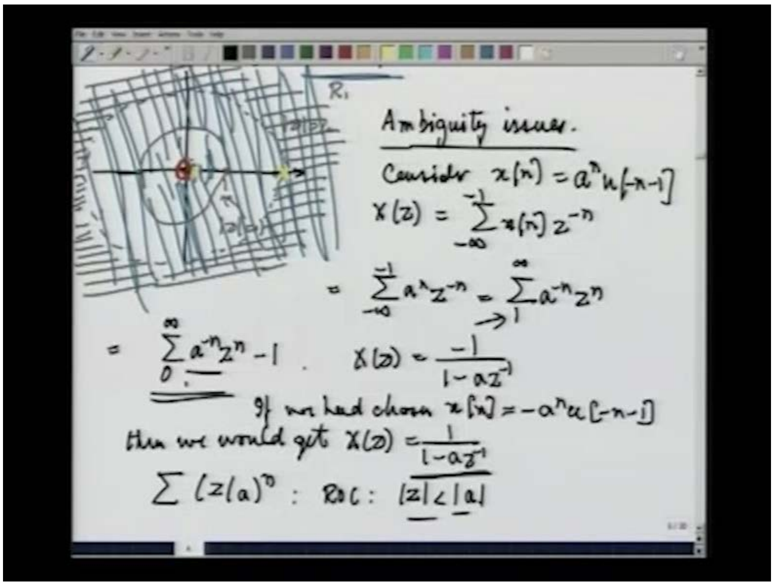
So, you have a pole over here. The pole is represented with the cross there is another pole at 1 by 5 sorry let me just redraw this diagram the scale is not convenient. The unit circle mod z equal to 1 this is 1, and this is 2. So, you have this is mod z equal to 2 the ROC for the second expression extends beyond this for not for the second expression for the first expression r x 1 s this is for x 1 s which is therefore called r 1. So, this is for r 1 now if you try to plot r 2 will be this at 0.2 you have to draw a circle and it will be all things radiating out from here, clearly the ROC of the combination is the intersection of the individual ROC s which comes to this is what we have just proved and this explains a lot of things now, we have a pole over here I mark it with a different color there is a pole over here in yellow there is another pole at 2 over here this is the second pole and then there are 2 zeros which I will mark in red right at the origin 2 zeros at the origin well.

So, much for the figure it is a little. So, so becomes of a rather poor choice of values of the pole positions we should have had more moderate values rather than such large and such small values. Next thing we introduce is also similar what we did in the case of the Laplace transform issues of ambiguity of the z transform. Consider x n equal to a to the power n u minus n minus 1 fine, this is now a non casual sequence non casual exponential sequence.

Now, let us see if we can find the z transform of this x z equal to summation because there is u minus 1 minus n minus 1, which is summations of minus infinity to minus 1 of x n z to the minus n which equals summation minus infinity to minus 1 of a n z to the minus n which can be written as taking n dash to be minus n this will become 1 to infinity of a to the minus n z to the n, which is equal to summation this is rather is equal to 1 sorry, just one second, summation 0 to infinity a to the minus n z to the n minus 1 because for n equal to 0. We actually do not have the term in this expression over here, but we are including that in this expression over here thus you get after simplification this expression we know how to expand. So, after we simply everything this comes to x z equal to also minus 1 by 1 minus a z inverse which seems to suggest that if we had chosen x n equal to minus a n u minus n minus 1.

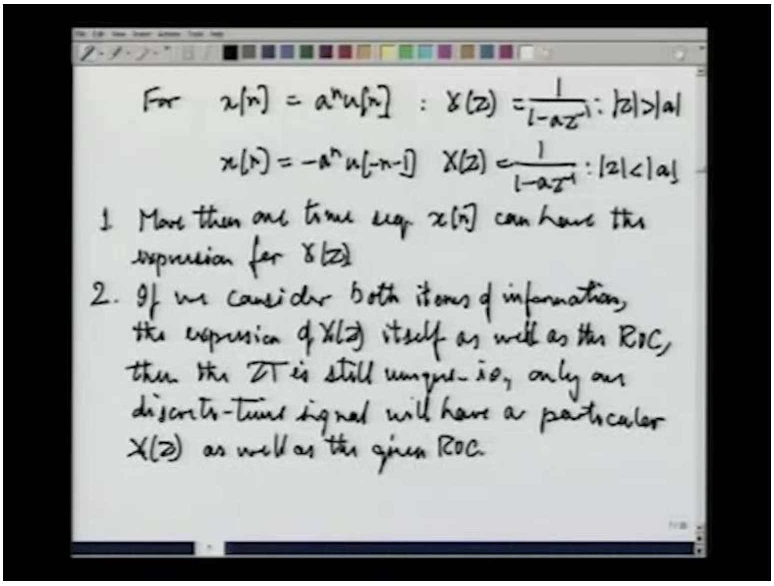
Then we would get the negative of this as x z we would get x z equal to 1 by 1 minus a z inverse, which is identical for what we got a different expression in a little while ago; however, remember that whenever we get an algebraic expression form for x z it inherently says within it that this is valid only for those point z where it converges, now let us see for what points z this will converge, it will converge where the summation z by a as seen from this part over here will converge what we are essentially summing is summation z by a to the power n.

(Refer Slide Time: 44:26)



Now, when will z by a to the power n converge for positive n, because we have now change variables, so here and positive n it will converge only where mod z is less than mod a. That is to say the region of convergence is mod z less than mod a. So, the expression is the same as for the earlier term, but the ROC is different lets collect the 2 examples together and write it down.

(Refer Slide Time: 46:14)



So, for x n equal to a to the n u n you get x z equals 1 by 1 minus a z inverse, z greater than mod a, which is what we plotted a little while ago and for x n equal to minus a to the n u minus n minus 1 which is anti causal sequence exponential sequence you get x z equal to the same thing as before 1 minus a z inverse, but now we have mod z less than mod a. So, summary is as fallows more than 1 time sequence x n can have the same expression for x z the exact. Second, if we consider both items of information the expression itself as well as the ROC we consider both this then the z T is till unique that is to say only one discrete time signal will have a particular x z as well as the ROC given. So that put things in perspective in the perspective turns out exact to same as for the Laplace transform.