Signals and Systems

Prof. K. S. Venkatesh
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 42
Properties of Laplace Transform

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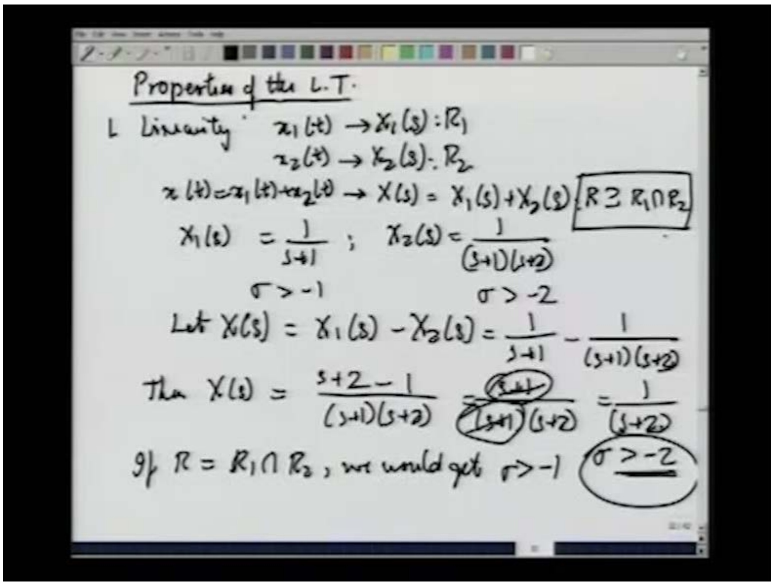

Now, we consider some properties of the Laplace transform. The first property that we always like to speak about is the linearity; we have been only dealing with linear transforms throughout this course. So, the first property of linearity is pretty obvious, it simply says that if you have x 1 t going to X 1 s, x 2 t going to X 2 s. This with an R O C R 1, and this with an R O C - R 2. Then x t equals x 1 t plus x 2 t will go to X s equal to X 1 s plus X 2 s that normally should have an R O C that is equal R 1 intersection R 2. But there can be peculiar situations where cancelations of poles and zero will leads to an R O C that is greater than this. So, the more general answer to this is not this, but to say that the overall R O C will contain R 1 intersection R 2. It might have more points than R 1 intersection R 2 sometimes. As an example, we will consider the following suppose we have x 1 t equals 1 by s plus 1, X 1 s equal to 1 by s plus 1. And X 2 s is equal to 1 by s plus 1 times s plus 2. Let us say this one has an R O C sigma greater than minus 1 this has an R O C sigma greater than minus 2.

Now, if R was R 1 intersection R 2, we should get R as sigma greater than minus one, but let us see what we get. What is X s? Let X s be given by X 1 s minus X 2 s which is equal to 1 by s plus 1 minus 1 by s plus 1 times s plus 2. Then X s equals with a numerator of s plus 2 minus 1 which equals s plus 1 over s plus 1 times s plus 2, which amounts to 1 by s plus 2. Now given that both the input signals have rightward R O Cs. They both represent right sided functions, and hence one expects X of s which is the difference of 2 right sided functions to be also right sided. Hence, we will say we can see from this that the R O C of this must be sigma greater than minus two.

This surprises us because if R had being equal to R 1 intersection R 2, we would have got the intersection of sigma greater than minus 1 and the intersection and sigma greater than minus 2 would simply be sigma greater than minus one. But what we are actually getting is this sigma greater than minus 2 which is more than the intersection of the 2 R O C's. And that has evidently happened because of the fortuitous cancelation of this of this with this which has knocked off upon to the right of the s equal to minus 2 pole. It is this that has expanded R O C. I just wanted I presented this example to show you that these things can happen, and that is why we say that the R O C simply for the sum of two expressions is given by R super set of R 1 intersection R 2.

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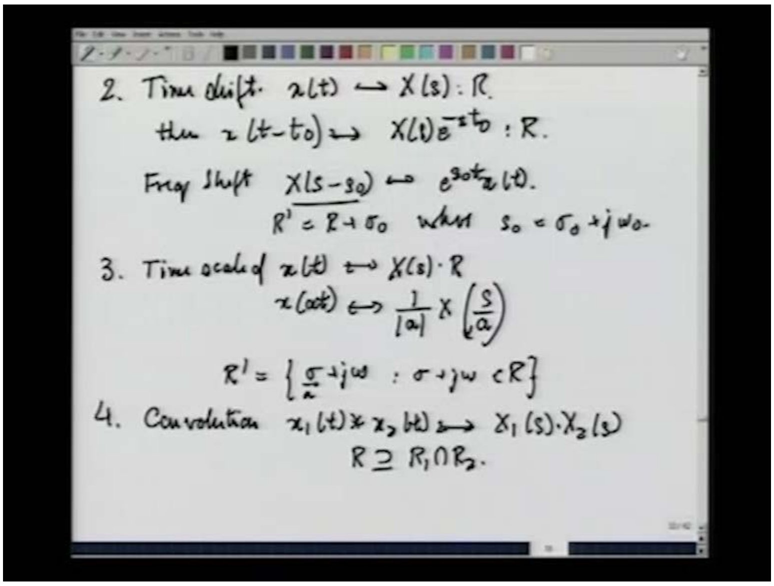

Now, let us move on to sum other properties quickly time shift. If x(t) has a Fourier transform Laplace transform sorry X(s) with an R O C of R, then x t minus t naught will have a Laplace transform X(s) e to the power minus s t naught. The way you prove this is exactly the way that we did for the continuous time Fourier transform, the steps are pretty much the same. We only have to worry about the R O C. Now what can we say about the R O C here. You see X(s) is known to converge over a region R of the X plane that is a set of the points R on the s plane at which X(s) is finite.

Now, we are multiplying X(s) by e to the minus s t naught; now e to the minus s t naught for any finite s is a finite number and so if X(s) is already finite then e to the minus s t naught X s will also be finite. If X(s) is not finite, then e to the minus s t naught times X(s) will also not be finite. In sort multiplication by a finite valued function such as e to the minus s t naught will not at all effect the R O C. So, this still has an R O C equal to R.

Next frequency domain shift, we can consider this under the same number. So, let us say frequency shift. What we want to ask is what is the Fourier transform of X, what is the time function corresponding to X s minus s naught, for sum s naught. Now again following the usual steps you will find that this yields e to the s naught t x(t), this yields e to the s naught t x(t). Next thing we have to ask is what is the R O C of this. Now all that has happened to the R O C is it has got the entire function has got shifted by an amount s naught; X of s is now shifted to X of s minus s naught. And hence what will happen is that every point in the R O C that was there earlier will get also correspondingly shifted. So, if R was the original R O C sorry if R was the original R O C as we written over here then the new R O C which I will call R dash is just equal to R plus sigma naught where s naught equals sigma naught plus j omega naught. So, the new R O C is obtained by shifting all the old R O C points by the amount sigma naught.

Next property - time scale of x(t), time scale of x(t). So, if x(t) has a Laplace transform X(s) with the region R what will happen to X a T, going by a process to what we used for the continuous time Fourier transform, we will get a very similar expression 1 by mod a X of s by a. Now what is the R O C of X of s by a. It terms out that again look at the expression X of a is what we now have earlier we had the X of s with the region of points R. Now X of s by a will simply yield a new set of points, which relate to the old set of points by the following means. So, you take R dash equals the set of points sigma plus j omega by sigma by a plus j omega where sigma plus j omega was in R.

Next, we speak of convolution. Along lines very similar to what we have earlier followed it terms out that x 1 t convert with x 2 t will have a Laplace transform given by

X 1 s times X 2 s, where the R O C, R of the convolution will contain R 1 intersection R 2. It should normally be equal to the intersection of R 1 and R 2, but it could exceed it for reasons similar what happened with the discussion of linearity that is the cancellation of some poles of X 1 with those with the zeros of X 2 or vice-versa. This would free up the R O C a little and you could have an R O C larger than before.

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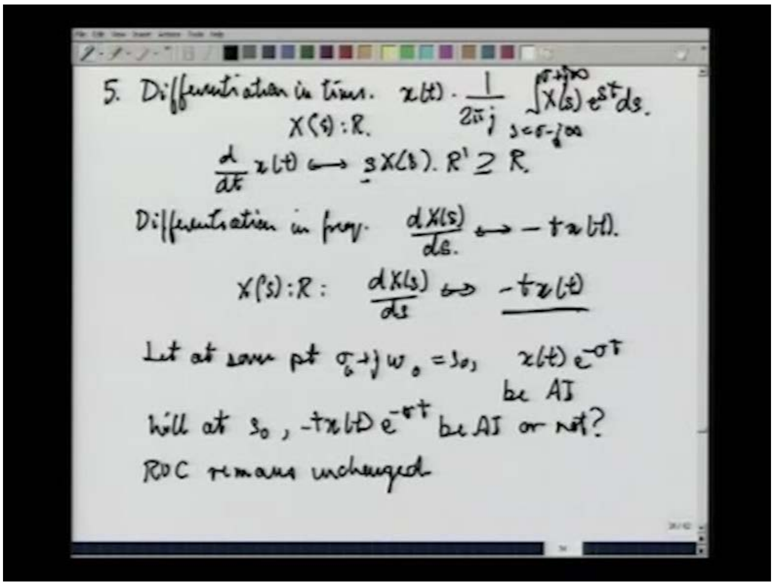

Next, we consider differentiation in time. We have the inverse Fourier transform expression that we had inverse Laplace transformation expression that we had already obtained. And that was that x(t) was 1 by 2 pi j integral over a vertical line line in the R O C s equal to sigma minus j infinity to sigma plus j infinity of X(s) e to the s t ds, this is what we have. Now if we differentiate this against time, the only if thing that differentiated is e to the s t, so you just get s X(s) as the answer that is to say that d by dt of x(t) will have a Laplace transform X(s). Now what do we except to be the R O C of the new function. If X(s) had an R O C of R, what would be the R O C of s X(s). We see that there is a factor s that multiplies X(s). Now factor X(s) is known to be finite for all the finite X plane. So, if X(s) is finite in some place f X(s) will be finite in that place if X(s) not convergent at some place then now will be s X(s).

So, on this argument, we can say that the R O C of s X(s) must be the same as the R O C of X(s). Except for the usual provision that if there is a pole at s that is a pole at the origin in the expression for X(s) then that pole will cancel with s that has now appeared as a factor over here. And the result will be an expansion of R o c. So, we can only say that R dash is a super set of R, the new R O C is the super set of R, probably the same. But if X(s) has a pole at s X equal to 0 then it will that pole will cancel with the factor s that has just appeared, and you will have an expansion of the R O C.

Next case differentiation in frequency that is we want ask what happens if you consider d X(s) by ds this turns out to have a time function, counterpart time function counterpart given by minus j t minus t x(t) there is no j; minus t x(t) that is all. Now the question is what is the R O C if X(s) had an R O C of R then what is the R O C of d X(s) by ds which transforms to minus t x(t). In order to determine this, we have to argue along slightly different lines, we have to argue in terms of the time domain function x(t) and not with respect to the Laplace transform as we have been doing up to now. There as for example with differentiation in time we looked at the R O C of X(s) and said that if a point lies in the R O C of X(s) then s X(s) will also lie in the R O C that is that point the same point will also lie in the R O C of s X(s).

Now, we look at the time function. Now the argument is this, we are looking at points sigma plus j omega in the s plane for which X(t) e to the minus sigma t is absolutely integrable. So the fact that at some point sigma, let at some point sigma plus j omega, let us say sigma naught plus j omega naught equal to s naught x(t) e to the minus sigma t be AI be absolutely integral. Then we have to ask if at that same point will at s naught t x(t) or if you write minus t x(t) e to the minus sigma t be AI or not that is what we have to ask, and the answer is very straight forward. You know that t the new factor in the time function increases linearly with time, whether e to the minus sigma t attenuates exponentially.

So, if you have an outright war between something like t and e to the power minus sigma t, e to the minus sigma t will win hands down. In sort, we can be in certain that if for any sigma e to the minus sigma t x(t) converges that is is absolutely integrable. Then so also will minus t x(t) e to the minus sigma t there is no problem there. Further more we can say that even if instead of t you had t squared or t cubed or t to the power n even those when multiplied by x(t) and e to the minus sigma t would still absolutely integrable. In sort differentiation, one time in frequency or more than one time in frequency does not bother the R O C. The R O C will remain unchanged.

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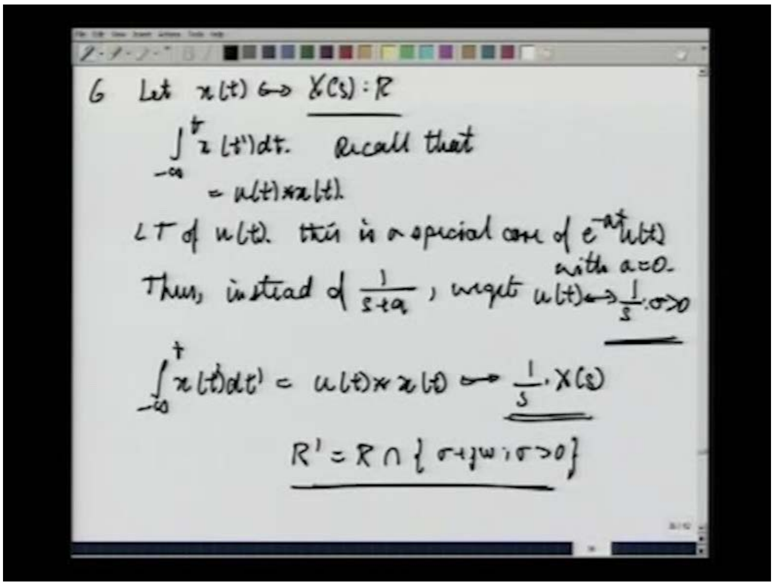

Next, let us consider the running integral the Laplace transform of the running integral of x(t). Let x(t) transform X(s) with an R O C R. What happens to integral minus infinity to t x t dash dt dash which is the running integral. Well in order to answer this, let us recall that this simply equals u(t) convolved with x(t). This is an old regard, which we have been carrying around for a while. So, if you want to answer this, we already have a theorem on convolution. So, we are ready to use it expect that we do not know the Laplace transform of u(t). So, what is this L t of u(t). Well u(t) is nothing but a special case of e to the minus a t u(t) with a equal to 0. Thus instead of 1 by s plus a, we get u(t) transforming to 1 by s, just put a equal to zero. So, u(t) transforms to 1 by s with an R O C of sigma greater than 0, because this the pole here 1 by s has a pole at the origin. So, sigma greater than 0 is the R o c. So, this is what we know about u(t), u(s) you can call it now, because it is the Laplace transform.

So, now, we say that integral minus infinity to t x(t) dash dt dash equals u(t) convolved with x(t) which is equal to which is not equal to but which transforms to 1 by s times X(s). Now what is the R O C? The R O C is the intersection of the R O C of X s with the right half of the s plane, which is the R O C for this. So, R dash equals R, because X had a Laplace transformation with an R O C of R, R intersection the set sigma plus j omega for sigma greater than 0, that is the R O C of the running integral. And that gives me the expression for the Laplace transform of the running integral. This completes most of the important properties of the Laplace transform. In fact, we have come to the close of the

Laplace transform as a topic for discussion at the level of this course. I will now conclude with the few final remarks, the Laplace transform is a very powerful means of understanding the behavior of signals. More importantly, it gives us powerful tools to discuss the stability of systems, for which it can take the Laplace transform function of the impulse functions.

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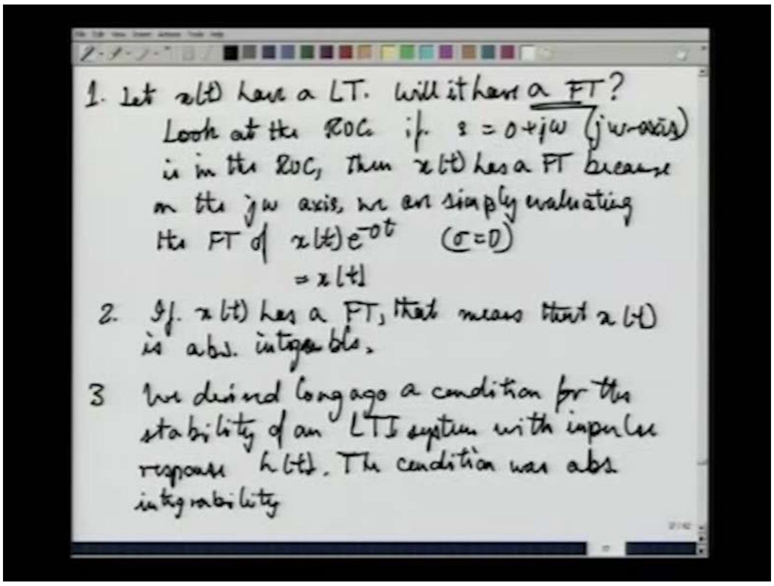

Now let us understand a few things. Let x(t) have a Laplace transform, then will it have a Fourier transform continuous times Fourier transform. How do we answer this question? To answer this, we will look at the R O C, if s equals to 0 plus j omega which is the what you call the j omega axis is in the R O C then x(t) has a Fourier transform, because on the j omega axis. We are simply evaluating the of x(t) e to the minus 0 t that is sigma equals to 0, which is simply equal to x(t); that means, that whenever the imaginary axis lies within the region of convergence, clearly x(t) will have a Fourier transform. Now, if next step, this is the first step of the argument, the second step of the argument. If x(t) has a Fourier transform; that means, that x(t) is absolutely integrable. Finally, we derived long ago a condition for stability, for the stability of an L T I system with impulse response h(t) the condition was absolute integrability. Putting all these three points together, we conclude the following.

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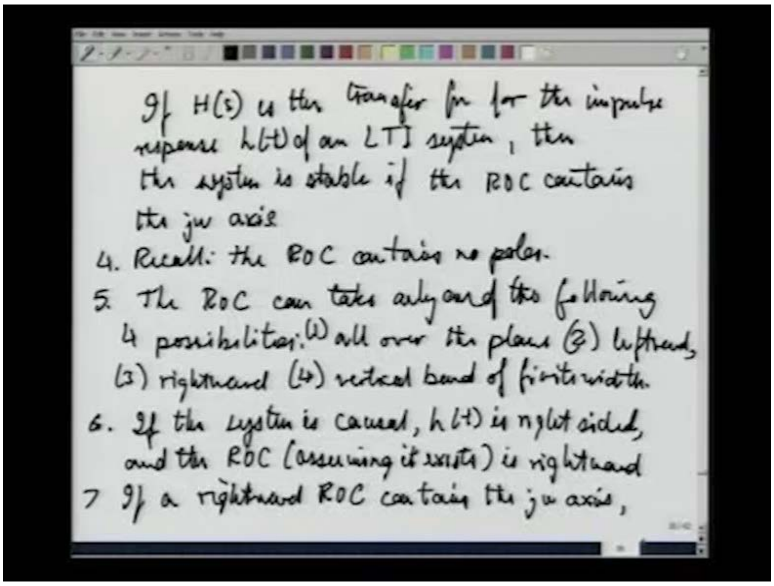

If H(s) is the transfer function for the impulse response h(t)of an LT I system then the system is stable if the R O C contains the j omega axis in h of s, well j omega axis. Further, next argument, the R O C contains no poles, we just recalling the argument, recall the R O C, R O C contains no poles. Now there are only a few possibilities for R O C. The R O C, we found can either be leftward or it can be a rightward or it can be everywhere on the x plane or it can be in the form of narrow patch. These are the only four things that can result by intersection of elementary R O C. All over the plane, this is possibility number one, the second possibility is leftward, third possibility is rightward and the fourth possibility is is vertical band of finite width; these are the only four possibilities we have. Now if the system we are concerned with is causal H(t) is the right sided, and the R O C will be rightward, assuming it exists of course is rightward.

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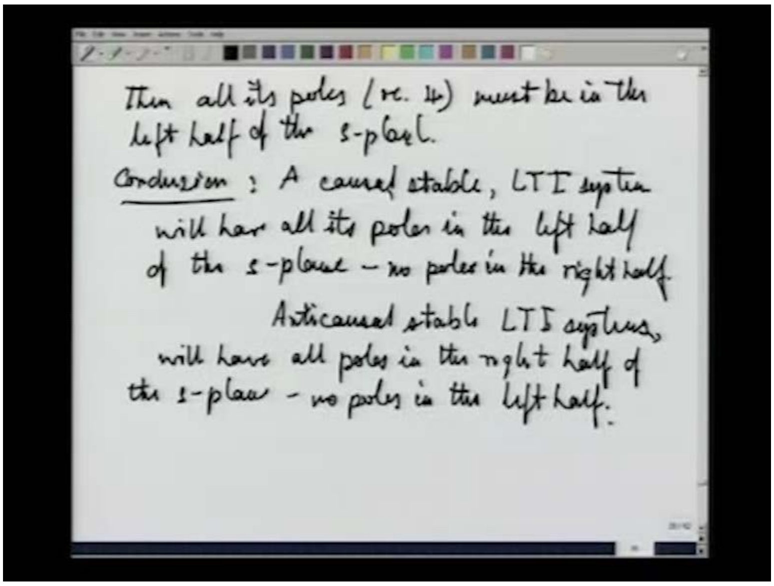

If a rightward R O C, R O C contains the j omega axis then all its poles refer to point number four just given above. All its poles given must be in the left half of the S-plane. Summary a causal stable L T I system will have all its poles in the left half of the S-plane; no poles in the right half that is the summary that is the conclusion of all the arguments, maybe I should call it conclusion. And along exactly similar lines if one argued for anti casual systems, anti casual stable L T I systems will have, I do not need to go through a very symmetrical set of arguments once again all poles in the right half of the S-plane; no poles in the left half. So, these are pretty advanced conclusions that one can come to about the pole and zero placements pole placements in particular of L T I systems which are stable.