**Signals and Systems**
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**Lecture - 41**
**Inverse Laplace Transform**

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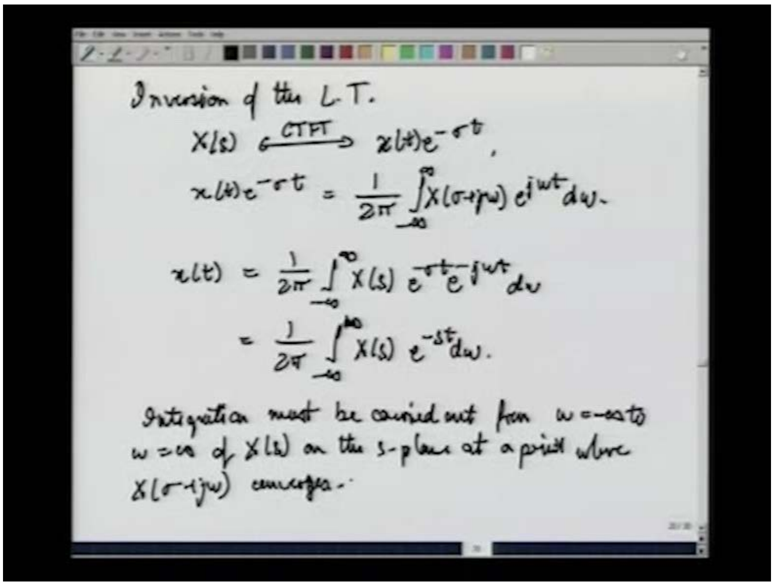


Now, we know some standard forms, but out of curiosity let us just examine the form of the inversion formula for the Laplace transform, if we have the Laplace transform of a function how do we find x of t. If you want to go through it by formal means, then we start by recognizing by it inversion of the Laplace transform. So, suppose we already know that if you have X s, then it is the continuous time Fourier transform of x t e to the minus sigma t. So, this is what we have. So, we can write that x t e to the minus sigma t equals the inverse Fourier transform 1 by 2 pi integral minus infinity to infinity X sigma plus j omega e to the power j omega t d omega. This is just the inverse Fourier transform of X s and as we said that will be equal to x t e to the minus sigma t, because the Fourier transform of x t e to the minus sigma t was the Laplace transform of x t.

If we have this now, then we just multiply both sides by e to the sigma t. So, that we have only x t on the left side and get x t equals 1 by 2 pi; that means we have to integrate x s over all values of omega, but do so at a place where the Laplace transform exist that is to say at a place where x t e to the minus sigma t is absolutely integrable that is in another words within the ROC. So, integration must be carried out from omega equals minus infinity to omega equal to infinity of X s on the s plane at a point where x of sigma plus j omega converges. That is the kind of place where we should do this.

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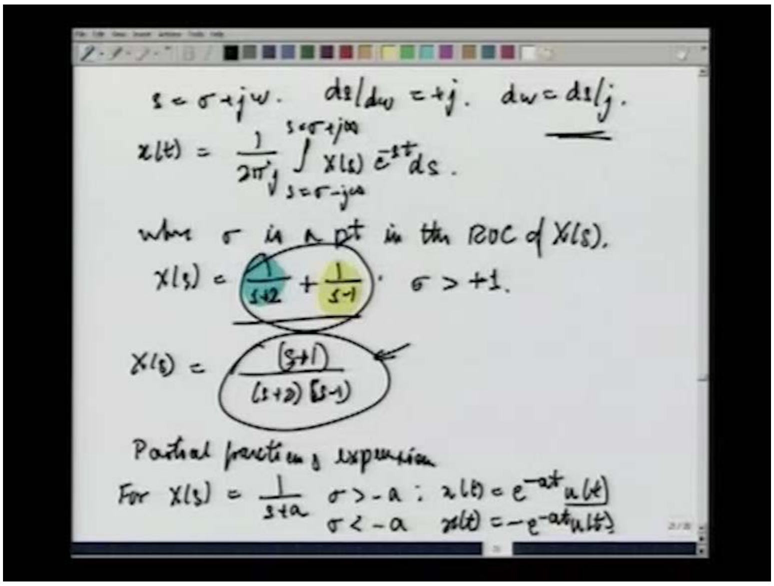


Now once we recognize this point, we can make the final changes to bring the expression into proper form s equals sigma plus j omega. So, d s by d omega equals plus j or d omega equals d s by j you have this. Furthermore we have that s as we just said must be chosen the points over which integration is carried out must be chosen to lie ROC. So, we will say x t equals 1 by 2 pi times j which has come from over here integral s equal to sigma minus j infinity to s equal to sigma plus j infinity of X s e to the power minus s t d s, where sigma is a point in the ROC of X s.

If you try to integrate it over a point outside the ROC; obviously, it would not converge. So, we have to do it inside the ROC and then it will be converge. So, this gives us the expression for the inverse Laplace transform of an arbitrary function X s. We can always do this, but as I said that we only deal with the few standard forms. So, what we can instead do is simply to reduce the problem given to us to 1 of the standard forms and then apply common sense, super position and 1 or 2 other things.

So, that is what we will now proceed to do in a couple examples. The general form of the Laplace transform of a function need not be straight forward. Because once you get even for example, in the previous case we had s plus 2 s minus1 zero z. Well wherever it was suppose you have X s equals 1 by s plus 2 plus 1 by s minus 1, thus sigma greater than plus 1, suppose we have this. Then what we will see is not a nice broken down expression like this we will unfortunately be given something like s plus 2 s minus 1 and in the numerator, we will have s minus 1 plus s plus 2 equals s plus 1 this is what we will be told is X s and we will be asked to get the solution.

So, the right way of going about it is to first develop a technique or recognize identify a technique that will break up an expression of this form to a form like this and that as we know is done very effectively by partial fractions expansion. There are still some more complications to worry about. Remember that if we really want to invert any of what we shall called the elementary expressions. Then we must have the elementary expression such as this or such as this we should have the elementary expression along with its own ROC. Only then we know how to invert it. If this problem is not clear recall that for X s equal to 1 by s plus a, which is an elementary expression. We do not know what is the inverse transform of this if we have told that sigma is greater than minus a then we will write x t equals e to the minus a t u t.

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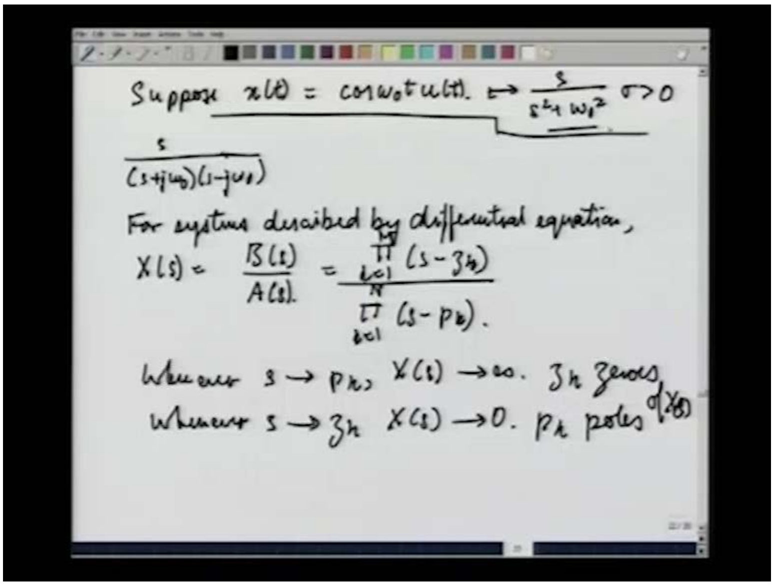


If we are told sigma is less than minus a then we will have to write x t equals minus e to the minus a t u t. So, it matters it is not sufficient to be able to expand using partial fraction’s expansion. The general form such as this into a sum of individual partial fractions that is not alone sufficient to solve our problem. We also required to know the

ROCs of each individual term. Now in order to answer this question let us see that the ROCs have a certain interesting relation with the algebraic expression. Let us look at a couple of examples and this will become clear.

Suppose we have x t equals cos omega naught t u t. We know that this has a Laplace transform expression of s by s square plus omega naught square sigma greater than 0 this is what we have. Now let us look at the expression for let us look at the way we can factorize this expression this can be written as s times s plus j omega naught s minus j omega naught. So, on factorization into first order terms the denominator has become this.

In general it is convenient to factorize in to first order terms. We are already aware that is for systems described by differential equations, the expression will be in the form of a rational a ratio of polynomials. So, X s will be in the form some B s by A s we make this claim on the basis of our experience with the continuous time Fourier transform. Now this can easily be buttress by actually looking at the Laplace transform of differential equation, but we will do that in a little while.

Now, so long as it is in the rational polynomial form then there will always exist a certain factorization k equals 1 to m of s minus z k over pi k equals 1 to n of s minus p k. Now just as with the Fourier transform, let us see what happens if s takes on values p k or z k it goes without saying that both p k's and z k's are either real. Because the coefficients of the differential equation will be real or the p k since that case will be founding complex conjugate pairs. This is a discussion we had already in different context and the same arguments hold here.

You see whenever, s approaches a p k, X s will approach infinity because a factor of the denominator is becoming 0. Likewise whenever, s approaches a z k then X s approaches 0, because a factor in the numerator it is going to 0. This is why the z k are called zeroes and the p k are called poles of X s. Now these poles and zeroes play a very interesting and important role in determining the ROC. This has been a diagration for the problem we were discussing over here. So, let us demonstrate the significance of the poles and zeros for these particular problems first. Lets plot the poles and zeroes of cos omega naught t u t

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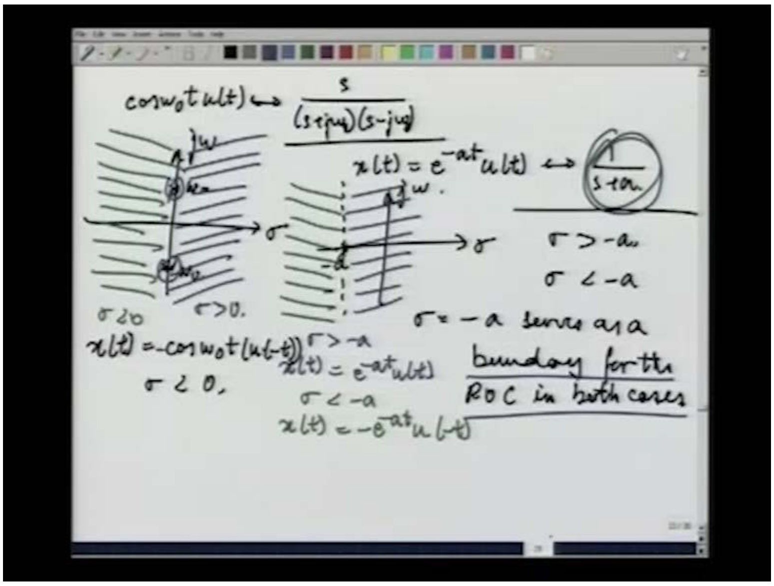


This transformed to s by s plus j omega 0 s minus j omega. Now, on the s plane, where you have sigma on the horizontal axis, as I said and j omega on the vertical axis, the roots are minus j omega naught and plus j omega naught. So, you have a root over here and omega naught and here at minus omega naught these are the 2 poles, if we consider earlier examples such as x t equal to e to the power minus a t u t which had a transform of 1 by s plus a. Then for that the Laplace transform this was sigma this was j omega, the root of this is already a first order factor. So, the root is simply s equal to minus a. So, at some place over here minus a you have a root. So, these are the poles. Now consider this expression, here we found 2 instances, 1 where the ROC was sigma greater than minus a and the other was sigma was less than minus a.

So, sigma equal to minus a serves as a boundary for the ROC in both cases. So, for sigma greater than minus a that is for x t let me just mark out the boundaries first. This was sigma greater than minus a and correspond to x t equals e to the minus a t u t or you could have and this was sigma less than minus a corresponding to x t equal to minus e to the minus a t u of minus t, all for the same expression 1 by s plus a right. So, this is what we observe.

Now, let us look at the cos omega naught t u t example once again. Here the ROC turned out to be this you have not 1 pole, but 2 poles over here. But again they are serving as a boundary poles generally serve as boundaries. Let us take the other example, where the left side is the ROC for this case, suppose you take x t equals cos omega naught t u of minus t. Then it is a function that will expand leftwards. Probably you will have to take minus cos naught t u minus t that something you can settle by just a solving the integration, but for this you get an expression that is similar to this.

But it would an ROC of sigma less than 0 and you therefore, get this as the ROC. This is the sigma less than 0, this is for sigma greater than 0. Again you see that the places where the poles are the vertical line containing the pole serves as a boundary. So, the reinforces, what we have just found over here poles serve as boundaries with this in mind. Let us try to get slightly more refined understanding of what is the problem of finding the inverse Laplace transform given the algebraic expression. First as I said, we will have X s as sum B of s by A of s.

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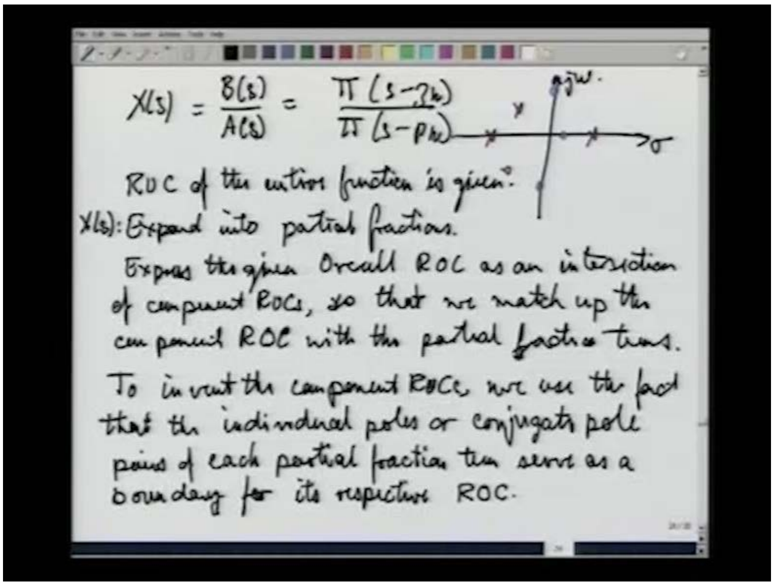


This we will express as a product of first order factors s minus z k by product of s minus p k this we will do. Then we will plot all the zeroes and the poles on the s plane, zeroes are marked a small circle like this and may be this and the poles are marked by a cross like this, this may be 1 more over here, 1 more over here and so on. May be you can add a couple of zeros as well. So, we have these zeroes and poles littering the s plane. Where this is sigma and this is j omega. So, we have these poles and zeros scattered and we can find their locations by factorizing the original expression of the Laplace transform given to us. Furthermore we will have the ROC of the entire function is given. And it is left to us to decompose the ROC into components and match them against the individual partial fraction parts.

So, on the one hand we will expand X s into partial fractions. Then express the given overall ROC as an intersection of component ROCs. So, that we match up the component ROCs with the partial fraction terms. To invent the component ROCs, we use the fact that the individual poles or conjugate pole pairs of each partial fraction term serve as a boundary for its ROC.

So, given a particular term it has only 2 possibilities ROC towards the left or ROC towards the right. There are other possibilities in some rare cases, but we will come to them eventually. So, what we have to do is to consider all the possibilities for each term and see which the intersection of which combination of ROCs yields the given overall ROC. This help us to determine the appropriate component ROCs.

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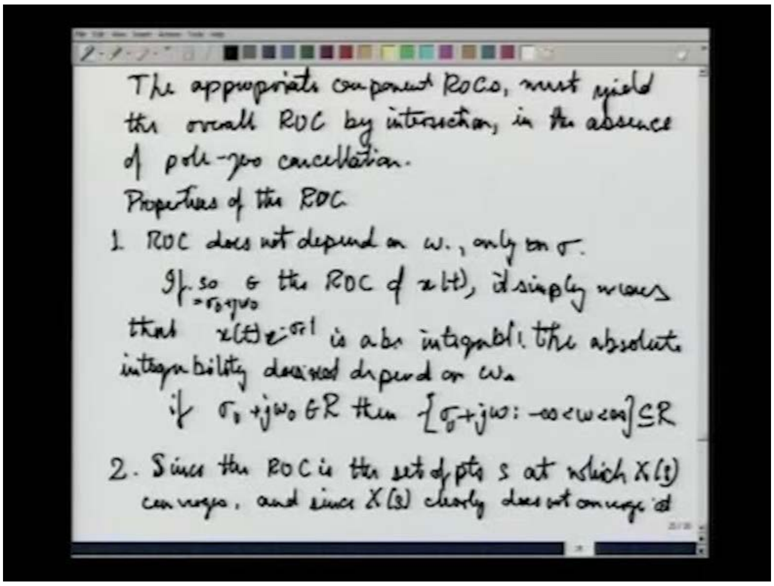


So, the appropriate component ROCs must yield the overall ROC by intersection in the absence of pole zero cancellation, fine. In order to get further inside into the behavior of the ROCs and their properties, we shall now investigate some properties of the ROC. A self evident property almost is something that we will be an observing all along. That the ROC are the region of points on s at which X s converges depends only on a value of sigma that is it depends upon whether s is greater than some value a or sigma is greater than some value a or less than some value a, it does not depend upon the value of omega.

So, the ROC does not depend this is property number one does not depend on omega, only on sigma this is easy to prove. What we are saying is if s naught is in the ROC of x t it simply means where s naught equals sorry if s naught equal to sigma naught plus j omega naught is in the ROC of x t, which simply means that x t e to the minus sigma naught t is absolutely integrable. And the absolute integrability of x t e to the minus sigma naught t is very; obviously, naught affected by the choice of omega naught, because omega naught does not even appears in this expression it does not depend omega naught, the absolute integrability does not depend on omega naught. Thus the upshot of this entire thing is that if sigma naught plus j omega naught is in the ROC which we write by saying is in R. Then sigma naught plus j omega the set of point sigma naught plus j omega minus infinity less than omega less than infinity is a sub set of R.

That means all knows other points are also in that is one property of the ROC. Another significant property is that since the ROC is the set of points s at which X s converges. And since X s clearly does not converge at s equal to p k for any k, that means, s the X s does not converge at the points of the poles it follows that no poles can lie within the ROC as that would violate the definition of ROC. So, let us consider that point to be clear no poles must lie within the ROC. In other words we can clearly see that when we look for the ROC for some function we should find the areas which are not inter pulsed with poles.

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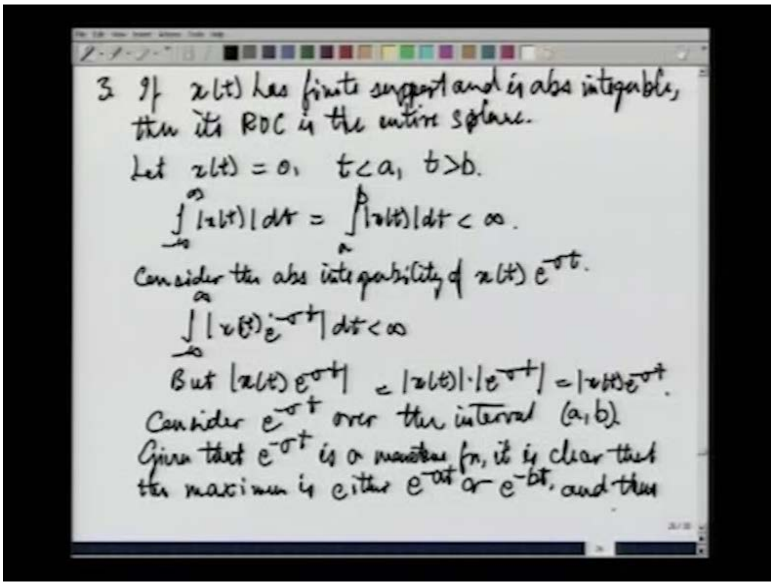

Next, if x t has finite support and is absolutely integrable then its ROC is the entire s plane. Now how do we prove this, x t has finite support and is absolutely integrable. Since it has finite support let say that the boundaries of the supporter a and b. So, let x t be equal to 0, t less than a, t greater than b. Suppose we have this x t given to us, then we also know it is absolutely integrable. We know that minus infinity to infinity x t d t which is just equal to integral a to b mod x t d t is finite we know this. If this is finite then all that we are worried about is the absolute integrability of x t e to the minus sigma t.

Now, here what do we find that we are concerned with whether minus infinity to infinity mod x t e to the minus sigma t d t we are concerned with whether this is finite. This is what we are concerned with, but mod x t e to the minus sigma t d t this equals x t mod times mod e to the minus sigma t. The second factor in this product is always non negative anyway, even without the mod sign. Because e to the minus sigma t is always greater than 0.

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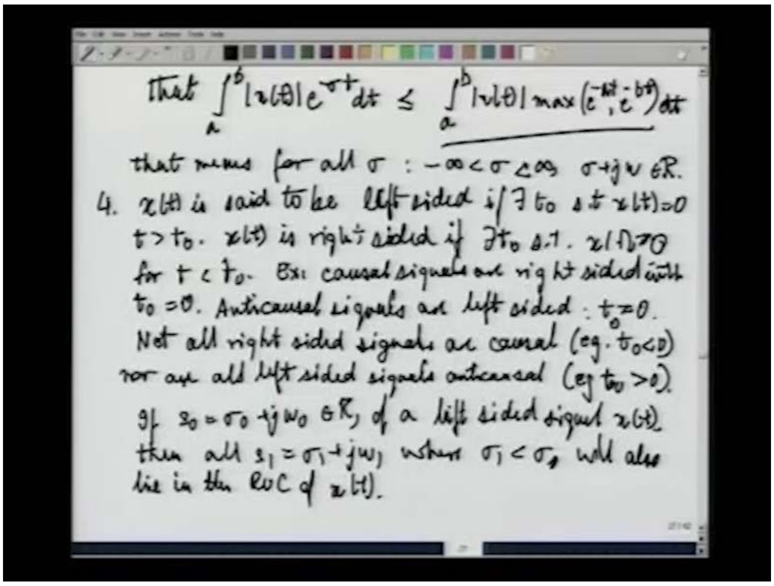


So, this is just equal to mod x t e to the minus sigma t fine. And this function, if you look at it and compare it with the area integral of the original function. Then you find the following well, consider e to the minus sigma t over the interval a b. Given that these are monotone functions that is its either increases monotonically or decreases monotonically depending upon the value of sigma. It is clear that the maximum is either e to the minus a t or e to the minus b t.

And thus that integral minus infinity to infinity mod x t e to the minus sigma t d t is less than equal to integral minus infinity to infinity mod x t times max e to the minus a t comma e to the power minus b t d t. That is to say that, this acts as an upper bound for this integral given that mod x t is absolutely integrable. So, since mod x t is absolutely integrable over the interval a to b and all these intervals can be replaced by a and b. And we will do that integral from a to b and here also integral from a to b fine. So, this is what we have this simply means that for all sigma, minus infinity less than sigma less than infinity, sigma plus j omega is inside the ROC. So, this completes the third property it pertains the signals with finite support.

The next property we consider is about the ROC of one sided signals. One sided signal can be either left sided or right sided and you will recall the definition of one sided signals from previous discussions in this course. That x t is said to be left sided, if there exists t 0 such that x t equals 0 for all t greater than t 0. Likewise it is right sided, x t is right sided if there exist t naught such that x t equals 0 for t less than t naught that is why these are right sided signals. So, for example, causal signal are right sided with t naught equal to 0. Anti-casual signals are left sided with t naught equal to zero.

Now, this does not mean that the converse is true. Not all right sided signals are casual, example t naught less than 0. Nor are all left sided signals anti casual, example when t naught is greater than 0 fine. So, that is just a review of the understanding of what constitutes a left sided or a right sided signals. Now, we will go to the actual claim about the ROC. It is claimed about the ROC that if s naught equal to the sigma naught plus j omega naught is in the ROC of a left sided signal x t. Then all s 1 equal to sigma 1 plus j omega 1, where sigma 1 is less than sigma naught will also lie in the ROC of x t. The argument is simple, we are saying that sigma 1 is less than sigma naught.

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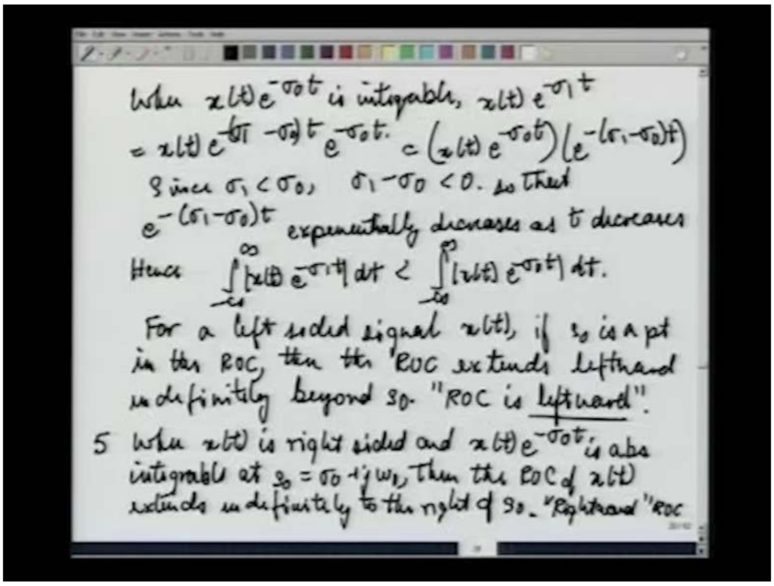


And we already know that x t e to minus sigma naught t converges that is absolutely intergrable, when e to the minus sorry, when x t e to the minus sigma naught t converges or is intergrable. x t e to the minus sigma 1 t which is actually equal to x t times e to the minus sigma naught minus sigma 1 t times e to the minus sigma naught t sorry just a minute, this is equal to this, fine. Now, you see this is equal to e to the minus sigma naught t multiplied by this term e to the minus sigma 1 minus sigma naught t. Now since sigma 1 is less than sigma naught, sigma 1 minus sigma naught is less than zero.

So, that e to the minus sigma 1 minus sigma naught t exponentially decreases as t decreases. Hence integral minus infinity to infinity x t e to the minus sigma 1 t d t is actually less than integral minus infinity to infinity x t e to the sigma naught t this guarantees convergence. So, summarizing we found that for a left sided signal x t, if s naught is a point in the ROC. Then the ROC extends leftward indefinitely beyond s naught that is the claim, we will say that the ROC is left ward. By analogy and very, very similar arguments, but symmetrically deferent from the present from what I have just been made, we can say that when x t is right sided and x t e to the minus sigma naught t is absolutely integrable at s naught equal to sigma naught plus j omega naught, then the ROC of x t extends indefinitely to the right of s naught.

This is called a rightward ROC. In both the case of the left sided and the right sided signals for which some ROC exists. We have found that the entire ROC becomes a half of the s plane to the left of s naught or to the right of s naught depending upon left sided signal or right sided signal. So, left sided signals have leftward ROCs, right sided signals have rightward ROCs.

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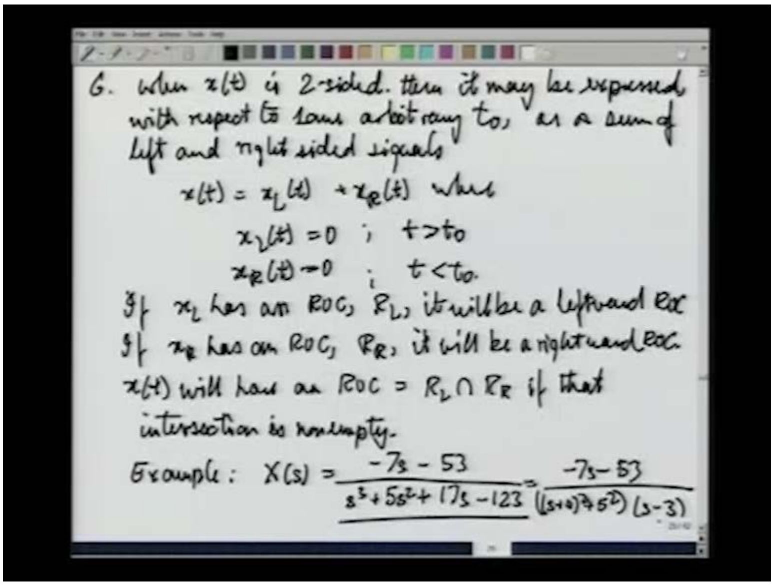


Finitely supported signals have ROCs everywhere as long as the signal itself is absolutely integrable. Finally, we come to the last possible case when is 2 sided; that means, it is neither left sided nor right sided it extends for all time. Then it may be expressed with respect to some arbitrary t naught as a some of left and right sided signals. x t equals x L t plus x R t where x L t equals 0, t less than t naught and x R t oh sorry, t greater than this is left sided function t greater than t naught and x R t equal to 0 for t less than t naught. Now x R t will have an right sided ROC. x L t will have an right sided ROC.

If not and we are not guaranteeing that x L and x R both have ROCs, we are only saying that if x L has an ROC it will be left sided R L, it will be a leftward ROC. If x R has an ROC R R, it will be a rightward ROC this follows from the previous results we have. Now if both this things are true, then we can say that x t will have an ROC equal to R L intersection R R, if that intersection is non-zero, non-empty is more precise. So, this is what happens with 2 sided signals.

So, this information helps us to find the ROCs for the individual component terms in an partial faction’s expansion. In order to illustrate this whole thing I propose to work out just one example though, I will skip some of the details the algebraic details in proceeding through the example. So, let X s be given as minus 7 s minus 53 divided by s cube plus 5 s squared plus 17 s minus 123. Now, that is look like terrifying expression and it indeed is and to be honest with you I know how to carry out the partial fraction's expansion of this. Straight away only because I started with the partial fraction's expansion and combined with the terms to obtain this. So, let me just go backwards through my notes I will tell you what this will expand to. Remember that to expand this expression. You will first have to factorize this; you have to factorize this expression. Factorize the denominator expression into separate factors. If you do that it turns out that this equals minus 7 s minus 53 as we said divided by s plus 4 whole squared plus 5 squared times s minus 3.

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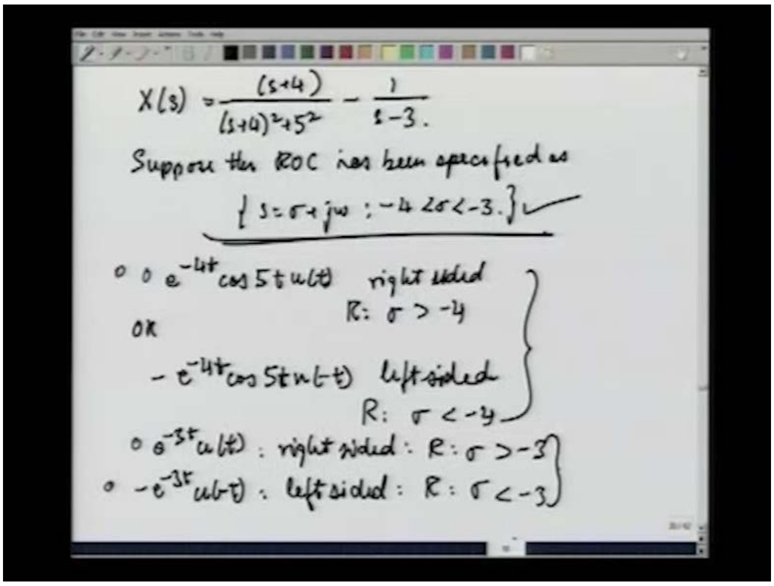


This is what it comes out to be. And then we do a partial fraction's expansion. To get 2 terms X s equals s plus 4 by s plus 4 whole squared plus 5 squared minus 1 by s minus 3 this is what you get. Now, from the expression we already presented for standard forms we could inverse both these directly. Let us say that we have been told that the ROC of this is greater than 0. Suppose the ROC has been specified as the set of all s equal to sigma plus j omega, where minus 4 is less than sigma is less than 3 suppose we have been told this. Remember that we now need to associate each term in the partial faction's expansion its own ROC.

Now, we know if we look at the first term that it will be of the form e to the minus 4 t cos 5 t. It is either minus this multiplied by u minus t in which case it would be a left ward signal or it would be this expression let me write both of them. It is either this times u t in which case it is right sided with R given by sigma greater than minus 4 or it is equal to the minus e to the minus 4 t cos 5 t u minus t in which case it would be left sided and R would be sigma less than minus 4. These are the 2 possibilities for the first term. For the second term which is 1 by s minus 3. We have very clearly that the time function is e to the power 3 t u t with which would be right sided. It would be for the second term the time expression would be e to the minus 3 t u t that is the first possibility right sided with R given by R is greater than minus 3 t. sigma greater than minus 3 t or you could have minus of e to the minus 3 t u of minus t. It would make it left sided and you would have R u n by sigma less than minus 3 that is what you would get.

So, these are the two possibilities for the second term. Now we can combine these 2 plus 2 possibilities in four different ways. But only one of them will yield by intersection the ROC that has actually be specified for the problem mainly this ROC is matched by only one of the four possibilities. So, what are the four possibilities. The first case is to take the first 1 of this with the first 1 of this right. And if we did that we would have an ROC that is greater than minus 4 sigma greater than minus 4 and sigma also greater than minus three

So, we would get a right sided ROC, because both functions are right sided. So, we expect the right ward ROC and that is what happens. So, that is fine that would be the first case, but that does not agree with ROC we have been given. Next you can consider the first possibility of this combined with the second possibility of the second one. If we did this we have a sum of a right sided function and a left sided function to get a 2 sided function, but what would its ROC. We would get an ROC which is bounded by minus 4 on the left side and bounded by minus 3 on the right side we would get a narrow band between minus 4 and minus 3.

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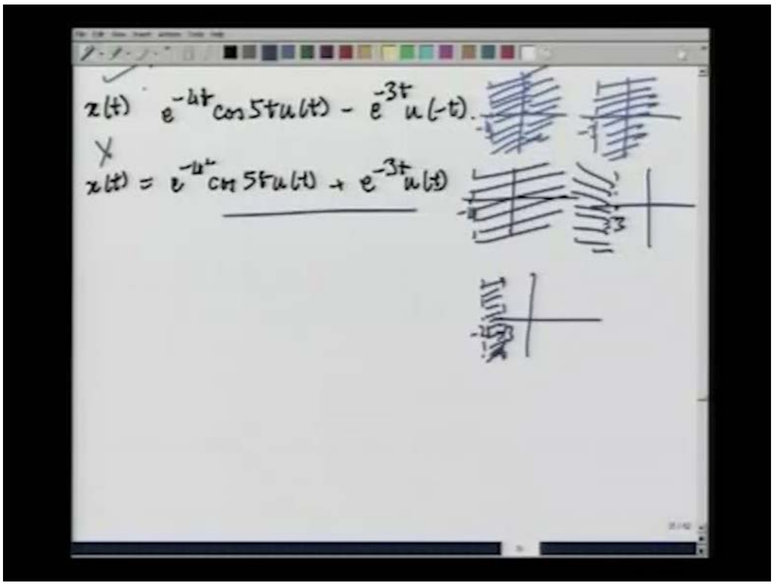


And that does agree with the specified ROC over here; that means, what we are looking for is the first possibility of the first term combined with the second possibility of the second term and we have the answer. The answer is e to the minus 4 t cos 5 t u t minus e to the minus 3 t u minus t this then is the x t we are looking for. Other possibilities that we would have got you can easily describe the possibility. We have already consider was this is the correct one.

The possibility we have just considered before was this one which was x t equal to e to the minus 4 t cos 5 t u t plus e to the minus 3 t u t which turned out to be not what we wanted, but what we just did now was the case. So, this was I will mark this with color and then there are 2 more possibility let just revert to the previous page and look we can take this second possibility of the first term and the second possibility of the second term in which case we would get sigma less than minus 4 sigma also less than minus 3 this would happen, because we have chosen both left sided signals. So, the some would be left sided and it would have a leftward ROC sigma less than minus 4.

This also does not agree with the specified ROC for the overall expression. Finally, we have the fourth possibility, where we take the right sided expression sorry, the second possibility for the first term which is the left sided expression minus e to the minus 4 t cos 5 t u minus t and combine with first possibility of the second expression e to the minus t 3 t u t.

Now, this would give again a two sided expression, sigma greater than minus 3 according to the second term, sigma less than minus 4 according to the first term. And this would leads to a null ROC; that means, to say that that expression would not have an Laplace transform at all. Because the basic idea of the Laplace transform which is to ensure that even though x t may not be absolutely intergrable x t e to the minus sigma t will at least be absolutely be intergrable fails in this case, because for no value of sigma will x t e to the minus sigma t ever be able to converge.

So, of the four possibilities we have discovered the correct possibility. We have rules out the other 3 possibilities and in fact, found one of them not evens a possibility. This example brings out the various possible cases and how one converges to the correct case. Now I will just draw a few diagrams for the first couple of examples over here and you get couple of examples possible solutions that we tried to give you an idea of how we combine the ROCs. So, for this case, what we do is we make ROCs for the first term. The ROC for the first term is sigma greater than minus 4.

So, this is minus 4 and you have an ROC like this going all the way across. Then you have the other ROC which is sigma greater than minus 3 and you have this as u t. So, when you take the intersection of these two ROCs you would; obviously, get an ROC which is also right sided rightward ROC. Now for x t as obtained in the second one which turned out to be the correct answer, we have a rightward ROC for the first term minus 4 yielding a right ward ROC and a second term minus 3 having a leftward ROC. If you combine these two together you get a narrow ROC like this a boundary at minus 3 another boundary at minus 4 this is minus 3 and this is minus 4 you would get an ROC that is just this much. And that is borne out by our earlier result; that means, you have an ROC which is a narrow band, which is the intersection of a leftward and rightward ROC it must correspond to a two sided signals this indeed a two sided signals.