Signals and Systems
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Lecture - 40
Laplace Transform

We are now going to start discussion on an entirely a new topic; and this will be called Laplace transform.

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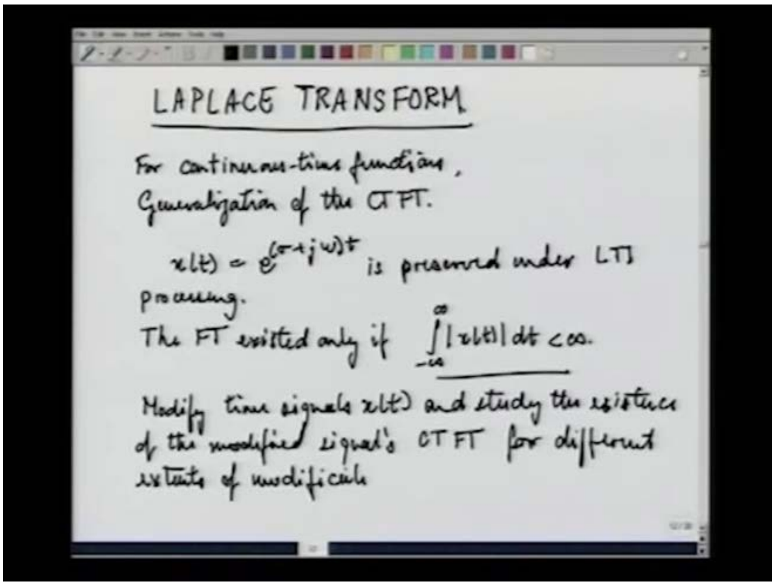

Having studied so many transforms already one might wonder why we need to look at one more transform. Now, this transform is for continuous functions, continuous time and in certain way can be understood as a sort of generalisation of the continuous time Fourier transform. Both, I mean the continuous Fourier transform attempted to represent both periodic and non periodic continuous time functions using representing signals, which were only periodic complex exponentials. So, the representing signals are always periodic complex exponentials in the CTFT case.

We know that the general complex exponential given by X t equal to e to the power sigma plus j omega t, where the parameter of the exponential is a complex number is also preserved a preserved signal under the linear time invariant processing. But so far we have not given any opportunity for a real component of the exponential to play any useful role, this is what we are now going to do. Now, so one way of looking at the

Laplace transform is that. It is a generalisation of the Fourier transform in which the frequency is allowed to be complex.

Earlier it was allowed to be only imaginary that is to say we had only j omega as exponent. Now, we are allowing sigma plus j omega, so that is one way of giving the Laplace transform as the generalisation of full wave transform. But more importantly we know that the full wave transform had certain limitation the full wave transform existed only for signals which are absolutely intergrable or summable, integrable. So, the full wave transform existed only if this was true, so if the absolute integrability was not assured then probably the Fourier transform would not convert, this is what we were faced with.

What we are now going to do is, to see is by modifying extreme some manner we can make an extreme, which was inherently unable to meet this criteria into one which can meet this criteria. This is the second way of looking at in the Laplace transform as the means of generating a Fourier transform of a modified signal and studying at four different levels or different degrees of modification. So, modify time signals X t and study the existence of its of the modified signals continuous time Fourier transform for different degrees of modification extends of modification, so let us give the definition the Laplace transform of X t is written as X s.

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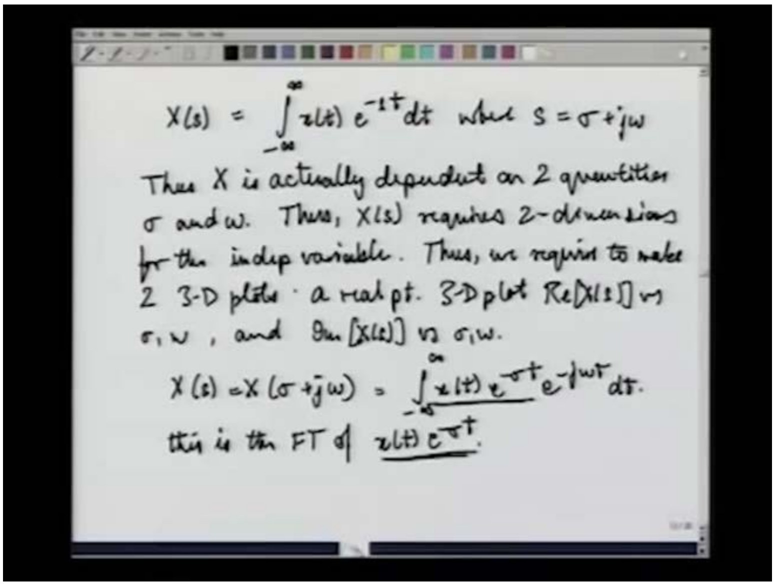

It is defined as integral minus infinity to infinity X t e to the minus s t d t where s equals sigma plus j omega. So, what we get in the transform domain is a function of a complex variable earlier we had only a function of an imaginary variable. If we looked at j omega or a real variable, if we look at only omega, but being s now being a complex number there are two variables, which can change the dependent sigma and omega thus X capital X is actually dependent on two quantities sigma and omega. This means something drastically new has happened, we can no longer really plot the Laplace transform of a function x t like we plotted the Fourier transform spectrum of the same function, because now it depends on two variables.

So, now what we have is a plot of two independent variable a plot that depends on two independent variables it is what we call a 3 D plot thus x s requires two dimensions for the independent variable further more x s itself can be complex. So, at each point s equal to sigma plus j omega x s will have a real part and an imaginary plot imaginary part or a magnitude part and a phase part. Hence, you would actually get two 3 D plots the real 3 D plot plus the imaginary 3 D plot or the magnitude 3 D plot plus the phase 3 D plot.

Thus we required to make two 3 D plots a real part 3 D plot real x s verses sigma and omega and imaginary x s verses sigma and omega. So, this gives us the picture of what we except with this introduction to the Laplace transform. Let us now see, what we can say about it next this much is about the Laplace transform we see if we write the expression for as we have done just now and try to interpret as the Fourier transform. What actually happens? X s equals X of sigma plus j omega and this equals the integral minus infinity to infinity.

X t e to the power minus s t is actually e to the minus sigma t e to the minus j omega t d t, which can we looked upon as the Fourier transform of this function. That is this is the Fourier transform of x t e to the minus sigma t. While this just seems like a interpretation this will form the basis of our perception of the Laplace transform throughout the discussion this will be the most convenient and easy way of trying to understand the Laplace transform. So, I will start the rest of the discussion by prefacing it with the brief review of how e to the minus sigma t behaves for different values of sigma.

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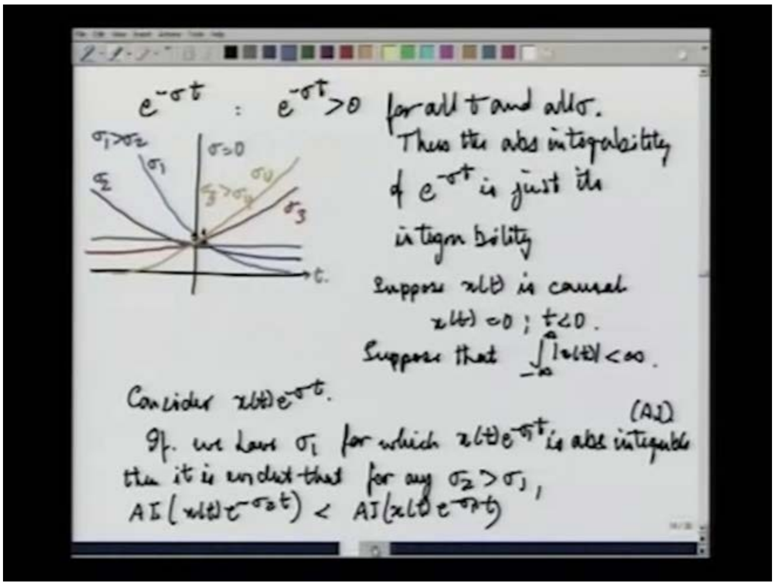

Let us make a plot one point is sure e to the minus is greater than 0 for all t and all sigma this is a point s where the observing thus the absolute integrability of e to the minus sigma t is the same as the integral integrability of e to the minus sigma t. Now, let us make some value make some plot for different values of sigma let us take this point as 1. Now, suppose sigma is greater than 0, if sigma is greater than 0 and you plot e to the minus sigma t for positive time we know that it is a decaying exponential, which means for negative time it is a growing up or growing exponential.

So, we can make a plot that goes like this is for some value of sigma if you choose a smaller value of sigma that is to say sigma is greater than 0, anywhere. But we are saying choose a value of sigma closer to 0 than this one then you will have a an exponential that is closer to flat that means you would have an positive time something like this. So, this we will call sigma 2 this will call sigma 1 then we will have sigma 1 is greater than sigma 2 this much we know.

Now, let us make another plot where sigma is actually equal to 0 just a flat horizontal line that is given as thus equal to 1 for all time then suppose sigma is less than 0, but small that is mod sigma is small. But sigma is negative then you will have a function e to the minus sigma t is raises for positive time and is decaying towards negative time. So, you get a curve that looks like this we call this sigma 3 next if you make a large negative when we are sigma that this sigma 4 less than sigma 3, then we will have something that rises faster in positive time and slower in negative time and falls faster in negative time.

So, we have so if this is sigma 4 then we know that sigma 3 is greater than sigma 3 is greater than sigma 4 we know this, so these are the various plots of e to the minus sigma t for different values of sigma. Now, the argument goes as follows suppose x t is causal it means to say x t equals 0 for t less than 0 only towards the right. It has non 0 values towards the left of x equal to 0 it has 0 values. Suppose that the x t of x exists that is to say suppose that this is thus it its absolutely summable, absolutely integrable when it is absolutely integrable.

Then let us see what we can say about e t minus sigma t x t if e to the minus sigma t x t is to be absolutely integrable, that means the product should have an area that does not grow up. Thus if we will have some value of sigma for which this is the case sigma 1 for which x t e to the minus sigma t sigma 1, t is integrable I mean of course is absolutely integrable. I think I will abbreviate this 2 a I absolutely integrable we will say is a I then it is evident that for any sigma to greater than sigma 1 the absolute integral of x t e to the minus sigma 2 t will be less than the absolute integral of x t e to the minus sigma 1 t correct.

Because this decays faster in pos positive time thus evident if you look at this pictures this function versus this function clearly that is this function is falling faster in positive time than this function. And x t is causal when it falls faster the area of the under the product will be less than it was for case of sigma 2 except that here we have chosen sigma 2 greater than sigma 1 here we have chosen sigma 1 greater than sigma 2. So, let us not confuse this sigma 1 sigma 2 with this sigma 1 sigma 2.

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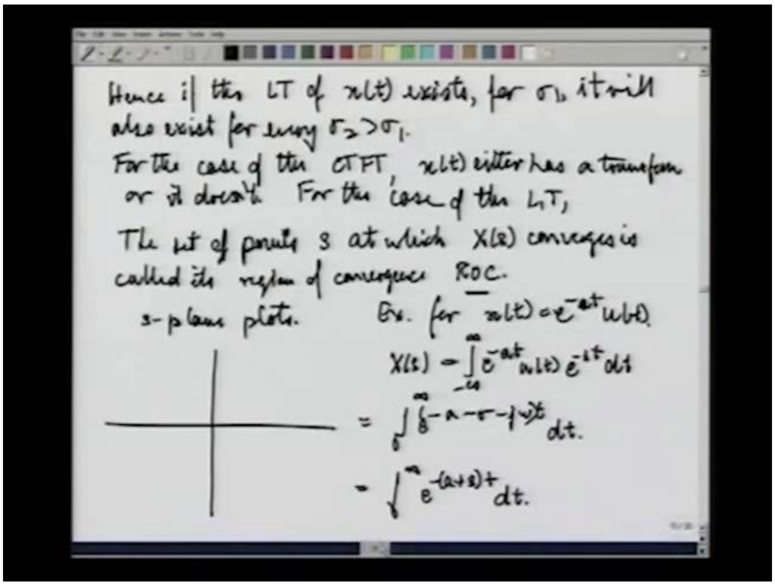

Hence, If the L T of x t exists it exist it exists for sigma one it will also exists for every sigma 2 greater than sigma 1 that is an interesting point to observe alright. So, this is the way we proceed we look at some function x t and see if there is any sigma for which x t e to the minus sigma t has a Fourier transform. If it has a Fourier transform then we will say that x t has a Laplace transform for that value of sigma, now the very terminology terms out to be different in the case of the Fourier transform.

It was a do or die kind of situation either x t has a Fourier transform or does not have a Fourier transform for the case we are continuing as time Fourier transform x t either has a transform or it does not or it does it was just make or break black or white. Whereas, in the case of the L T it turns out that there can exist certain values of sigma for which the L T exists other values of sigma for which the L T does not exist. Thus how it is the set of points of sigma for which the Laplace transform does exist is called its region of convergence it is called its region of convergence anyway we get R O C, all right?

So, this is where the Laplace transform fundamentally differ from the Fourier transform it has to make a record not just of what the Laplace transform of x t a is what value this, but also where those values are valid what points of s are those values are valid now. To make things more graphical and easy to picture we make what are called s plane plots s plane. Plots essentially are plots of the plane sigma omega where sigma is on the horizontal axis and omega is on the vertical axis and the s plane plot of array of x s divides s planes into regions over which x s exists or fails to exist. So, in some regions x s will exist and in some regions x s will not exists may not converge. So, this is called an s plane plot example for x t equals e to the minus a t u t, which is just like the causal example we have just done through.

This is also a causal example then e to the minus a t is some kind of occur e to the minus a t is some kind of occur which depends on the value of a. So, thus the point plotting next curve what we will do instead is to understand its Laplace transform e to the power minus a t u t have a Laplace transform x of s equal to integral minus infinity to infinity e to the minus a t u t e to the power minus s t d t. That is equal to because there is u t to over there 0 to infinity e to the power minus a minus sigma minus j omega t d t. This can be rewritten as summation from 0 to infinity e to the power minus a plus s into t d t.

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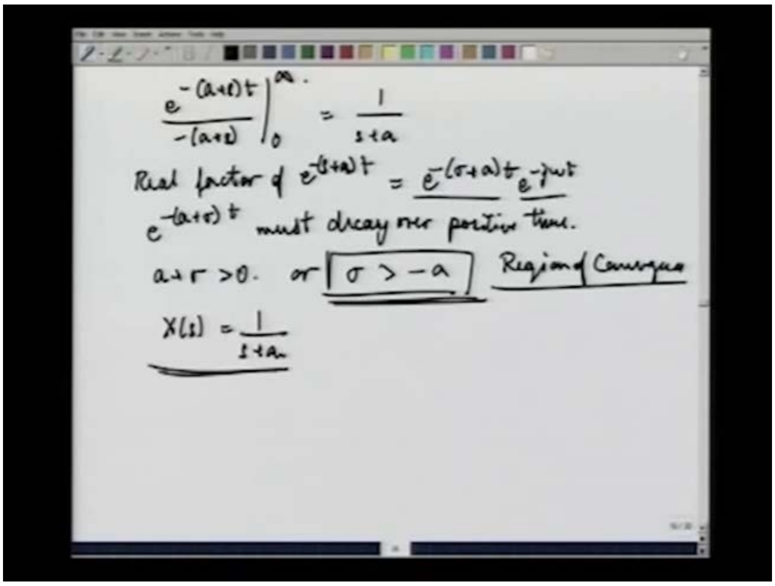

Let us assume that this integral converges if this integral converges then its integral will be e to the power minus a plus s times t divided by minus a plus s evaluated from 0 to infinity for t and this comes out to be equal to 1 by s plus a. But in order to assume that its convergence we have to see for what values of sigma this integral will actually converge. Now, the real part of into e power s plus a the real factor of e power minus s plus a times t is e power minus sigma plus a t e power minus j omega t this what we have.

Now, you see e power minus j omega t is not integrable because both is real and imaginary parts have finite power and not finite energy where not integrable that is the end of minus j omega t. So, any hope we may have of x s existing will rest with whether this is a decaying exponential over the interval of integration that means e to the power minus a plus sigma times t must decay over positive time. So, that when multiplied by something like e to the minus j omega t which has finite power and not finite energy the product will still have finite will still be absolutely integrable.

Now, let us see what this means, this means that a plus sigma must be greater than 0 or that sigma must be greater than minus a, in fact sigma must be greater than minus a it defines the so called region of convergence. This is the region of convergence of x s given by 1 by s plus a fine. So, this formula is valid only for this interval of s plane this part of the s plane thus it is clear that unlike the days of the continuous time Fourier transform where we nearly needed to take a record of the algebraic expression for x of omega. Now, every time we find the Laplace transform of some x t as x s we have to note the expression for x s as well as the region of the x s the region of the s plane, where it is valid. Thus in the example just concluded, we see that it is only for sigma greater than minus a that x s equal to 1 by s plus a is valid.

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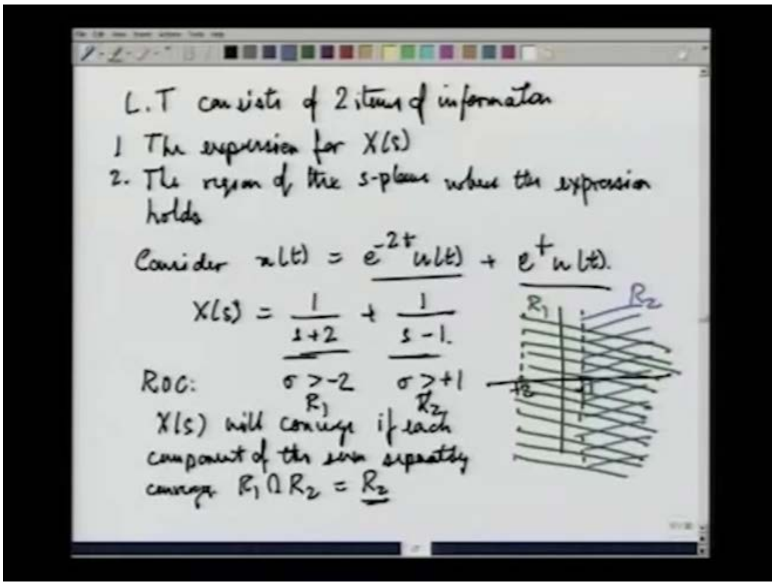

So, the Laplace transform consists of two parts, consists of two items of information; the expression 1 is the expression for x s into 2 the region of convergence the region of the s plane where the expression holds alright. Now, let us just see what happens if we consider the Laplace transform of 2 functions simultaneously that means of the sum of 2 functions. Consider x t equals e to the power minus 2 t u t plus e to the power t u t from the form of the Laplace transform equation it is clear that it is linear transform.

So, that the transform of the sum of two functions must be equal to the sum of the transforms, on this basis and knowing the Laplace transform as we just experimented with it for this expression. And this expression separately we can write that x s equals 1 by s plus 2 plus 1 by s minus 1. So, we can write this but then there is the issue of the region of convergence now since we cannot directly common the region of convergence of entire x s.

Let us look at the region of convergence of the individual parts for this part clearly if you go back to the original function e to the minus 2 t u t we find analogous arguing analogous with what we had earlier that 2 plus sigma must be greater than 0. That means sigma must be greater than minus 2 sigma must be greater than minus 2 is what this expression requires. Now, what does this expression require it requires with similar arguments that sigma must be greater than plus 1, so in the s plane suppose we make plots of the region of convergence of first component function and of the second component function. So, for one of them it is this for the second one it is this and for the first one it is this sigma greater than invert case.

So, it is this, this is what we get, so if we ask now what is reason for region of convergence of the combination then the answer is clear if x s will converge will converge. If each component of the sum of the sum separately converges there is to say we have to look at the interception of the convergence regions of the two terms, which comes outs to be which I call this r 1. If I call this r 2 then it comes out to be r 1 intersection r 2, which is just equal to r 2, so r 2 where r 1 is this let me just denote this is in a more friendly way using different colours. So, let us say that this is r 2 and this is r 1.

So, r 1 is in green and r 2 is in blue and r 1 intersection r 2 is the part which is has both by r 1 and r 2. So, this is what we get when the Laplace transform of a sum of functions for each of which the Laplace transform and the individual regions of convergence are known. Now, let us take let us introduce ourselves to a rather surprising fact, let us consider x t equal to e to the power minus a t u of minus t this is a function, which is what is called left sided that means that for t greater than 0 its 0 or t less than 0 or non 0, so this is e to power minus e t a t.

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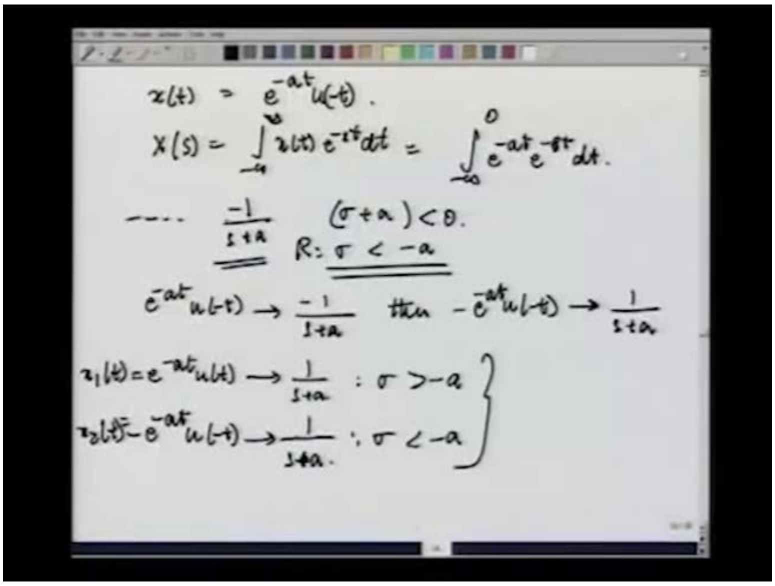

So, let us tried to find X s equals integral minus infinity to infinity x t e to the minus s t d t which simplifies along lines. Similarly, this earlier describe that its minus infinity to 0 thus the area of non zeroness of the argument e to the power minus a t e to the power minus s t d t, which after simplification comes to 1 by minus 1 by s plus a with a region of convergence that requires that sigma plus a must be negative. That means sigma must be less than minus k, so this is the definition of the this is the this describes of R O C for this function sigma must be less than minus a. Then the algebraic expression is that x s equals this, now this looks familiar the expression looks.

Familiar in fact since the Laplace transform is linear as is evident from inspection you will see that if e to the power minus a t u of minus t goes to minus one by s plus a then minus e to the power minus a t u of minus t must go to one by s plus a. But if we recall e to the power minus a t u t went to 1 by s plus a we solve this a few minutes ago this means that two totally different functions namely this and the second one v minus e to the minus a t u minus t also go to one by s plus a.

This seems to such as that unlike the continuous time Fourier transform x s is not unique different functions x t can have the same Laplace transform that is not really true. Because recall we have said right at right little while ago that the Laplace transform of function has 2 items of information. What we are now trying to do is to ignore the second item of information what we have written down. Here is only the first item of information, which is the algebraic value the expression for x of t for x of s we should also note that for this function the upper 1 sigma must be greater than minus a.

Whereas, here sigma must be less than minus a as we just found that means that their the expressions are same. The region of convergence are difference this is what helps us in distinguish between the first x t, which I will say I will call this as x 1 t and x 2 t x 1 t is this and x 2 t is this their expressions are the same. But their regions of convergence are different so as long as we take both items of information into consideration. The Laplace transform does remain a unique part as s t to x s.

Now, with this in mind with this in hand we will see what we can what more we can do to learn the Laplace transforms. Let us start with a few examples, now the fact is that for our purposes of use it is sufficient to consider Laplace transforms only of certain standard functions, which are related to the exponentials. Thus we will consider for example, we have already done e to the minus a t u t and e to the power minus e to the power minus a t u minus t for these functions we already have the Laplace transforms now we will consider I did not solve this in detail.

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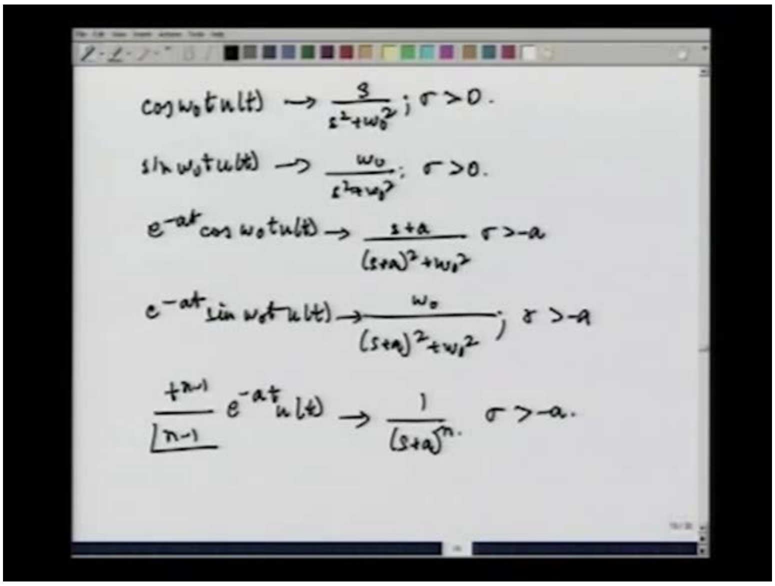

But we will consider cos omega naught t u t this comes out to have a Laplace transform of s by s squared plus omega not squared with sigma greater than 0. As the R O C co sine omega naught t u t comes out to have a transform omega naught by s squared plus omega naught squared for sigma greater than 0 again is are just a listing it is just a listing of the standard forms for which we look for Laplace transforms. Next suppose, you had e to the minus a t cos omega not t u t that would give you a transform of s plus a by s plus a whole squared plus omega naught squared sigma greater than minus a.

Similarly, e to the power minus a t sine omega naught t u t would have a transform of omega naught by s plus a whole squared plus omega naught squared sigma greater than minus a. Suppose, we had t to the n minus 1 by factorial n minus 1 e to the minus a t u t this would give us a Laplace transform of 1 by s plus a to the power n sigma greater than minus a. So, these are the general forms in terms of which most of the other simpler expressions for which we look for Laplace transforms will be created.