Signals and Systems
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Lecture - 39
Interpolation

So, much for our discussion of alternatives to ideal sampling, since the ideal sampling was not possible we had to resort to all these devious approaches to sample the signal using rectangular pulses rather than impulses, but beyond this there is the problem of retrieval. Retrieval using an ideal low pass filter as we had mentioned briefly earlier is also an impossibility, because an ideal low pass filter cannot be constructed in the real world.

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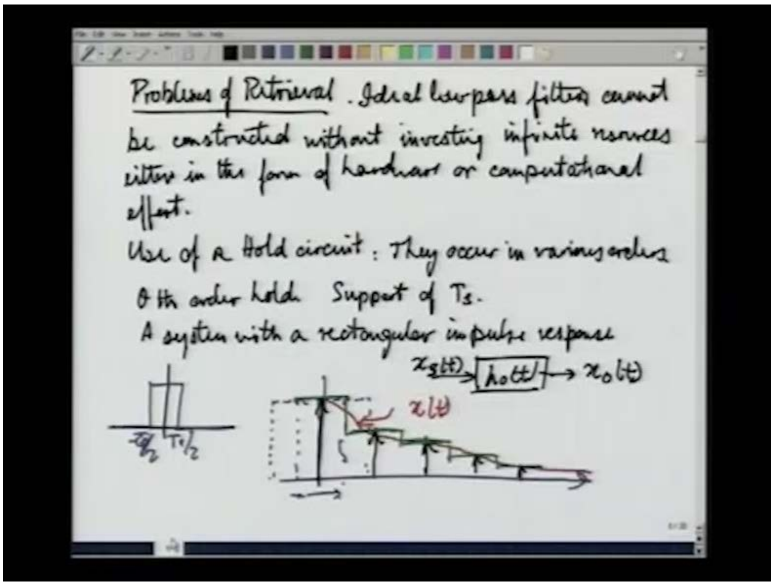

So, we will talk about problems of retrieval. Ideal low pass filtering a ideal low pass filters cannot be constructed without investing infinite resources either in the form of hardware or computational effort. So, ideal low pass filtering is out, what can we do instead of ideal low pass filtering. We can use what is called a hold circuit, use of a hold or a hold circuit is what you often called a hold circuit is of various kinds and in fact, the hold circuits are characterized as zero th order hold, first order hold, and so on hold circuits are found in various orders.

They occur in various orders. So, zero th order hold is the easiest to implement, but is unfortunately capable of considerably distorting the signal. What is a zero th order hold? A zero th order hold is nothing but a system with a rectangular impulse response, all it does is to smooth out the impulse train excess of tin in what follows we will assume that the system has ideally sampled the input signal and so the input to our hold circuit will be x s of T the ideally sampled sampled train.

Now, this ideally sampled train will be subjected to various kinds of holds, and we will see the results. So, a system with the rectangular impulse response, so it is actually like this, the width of this should be T s by 2 on this side and minus T s by 2 on this side. So, its total width is T s.

Now, let us see what it does, when it is applied to a sample train. What we really have is let us say a train of impulses, suppose we have this train of impulses, the output of the hold circuit will stay at a value that is equal to the impulse; the input impulse which lies within the support of the hold as the hold moves from left to right. Hence what will happen is, you will get a stepped kind of output as may be drawn from here seen from here. So, you will get output of constant equal to this when the hold moves from all of here to all of here.

So, that its centre is here as more from here to here, and the centre is more from here to here all you have got over here is this green line. Now as well as it crosses this impulse falls out of the support of the hold, and the second impulse falls into the support of the hold. And so the output abruptly drops to this value, it stays at this value until the next midpoint between two impulses and then jumps down like this and goes over to here, then another jump yet another, and this is finally the shape that you see.

So, if we denote by x r of T or by x 0 of T the output of the 0 th order hold, then you have x s of x s of T going in and we have what I will call h 0 of omega or if you o is to write it in time domain terms we can write h 0 of T for a 0 th order hold, and the output will be what I call x 0 of t. You can see that x 0 of T is a far cry from the original input signal x of t, which might probably have looked like this, x of t would have look like this, but what we get is some kind of a staircase that attempts to approximate the original x of t, a lot of destruction, a simple amount of analysis or a small amount of analysis will tell us what is the nature of the destruction as viewed from the frequency domain, and we shall do that right away.

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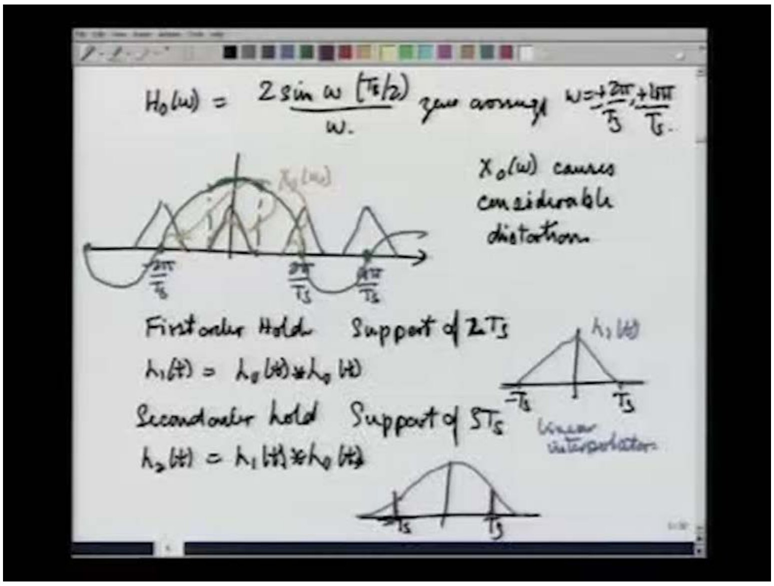

What is the spectrum of h 0 of omega, what is h 0 of omega? Remember that h 0 of T is a rectangular pulse, which is equal to one from minus T s by 2 to T s by 2. So, we just get h 0 omega equal to 2 sin omega T s by 2 by omega, this is the Fourier transform of the zero th order hold. Now, where will be the minima of this the minima the 0 crossings of these will be at 0 crossings at omega at T s by 2 equal to pi; that means, at omega T s equals 2 pi which is to say that omega equal to 2 pi by T s, then again at omega T s by 2 equal to 2 pi that is 4 pi by T s and so on, these are the 0. Now let us go back to the spectrum of excess of T excess of omega, and look at what we are to expect.

Let us say this is excess of omega, this is 2 pi by T s this is 4 pi by T s, this is minus 2 pi by T s and so on and with this 0 crossings of these at T s by 2 equal to pi; that means, at omega T s by 2 equal to pi which means that omega equal to sorry omega equal to 2 pi by T s plus or minus 2 pi by T s comma plus or minus four pi by T s and so on. So, where is 2 pi by T s is right over here 2 pi by T s is here this 0 crossing over here and here, then there is another 0 crossing here and here.

So, what we have is a spectrum of a hold function that goes like this, and then goes back like this goes on like this here to it goes back like this and so, on there is within the scope of minus B to B, where B could be as large as omega as by 2, you have a considerable amount of distortion in this. So, stop and also one cannot ignore the fact, that this is also going to allow in a little bit of these two alias on the left and the right.

So, that you actually have a very terrible function which we will now plot the output x r rather x 0 of T will have a spectrum that looks like this, and the problem does not end there you will have a little bit over here, you will have a little bit, you will have contributions of all over the place, which is easiest described by saying it is a complete mess. So, this is what a zero th order hold does, of course if that is heavy consolation it is the central alias which contributes the maximum amount of energy to x 0 of T, and all the others contributes smaller and smaller amounts of energy, but certainly not negligible hence x 0 of T or x 0 of omega, which is what I just plotted this what I plotted was x 0 of omega consisting of all this little bits over here, and here, and here, and so on.

This is x 0 of omega is an extremely distorted version of what we really wanted it is nowhere near x of omega it is a complete mess. So, this is what the zero th order hold does we will say that it causes considerable destruction alright, now we can improve on these by using what is called a first order hold you see a zero th order hold has a support of T s and a first order hold has a support of 2 T s.

Let us make a note of that going over here where we started the zero th order hold and say support of T s, this has a support of 2 T s and the first order hold is simply obtained by convolving to 0 th order holds. So, x 1 rather I will call this h 1 of T is nothing but h 0 of T convolved with h 0 of T, similarly you would have a second order hold h 2 of T equal to h 1 of T convolved with h 0 of T, that is a second order hold support of 3 T s and so on, you can have finite order holds of all these values.

And these are the standard second first, and second order hold that we are considering one can of course make modifications of this. Now, what is the shape of the impulse response of h 1 of T, because it is the convolution you can see that h 1 of T will be just this, this is T s this is minus T s this will be of this form a triangular pulse this is what we called h 1 of T, and because it is a triangular pulse, we call it a linear interpolator. Now, this is a convolution of this triangle with another rectangular pulse h 0 of T of width T s. So, that the overall support is given wider T s minus T s and so you will get a function that looks like this, which goes to 0 exactly at 1.5 T s.

On the right side and minus 1.5 T s on the left side, now the first order hold is fine and it gives a better response then the zero th order hold, but the second order hold onwards troubles, begin you see the first order hold has very desirable property, that while it is nonzero at T equal to 0, it is 0 at T s at minus T s and that for all values k T s greater than k greater than one mod k greater than 1. But the second order hold unfortunately has a nonzero value at these places at these places T s and minus T s it has a non zero value, now that is trouble, because if you convolved x t with a second order hold such as x 2 of T.

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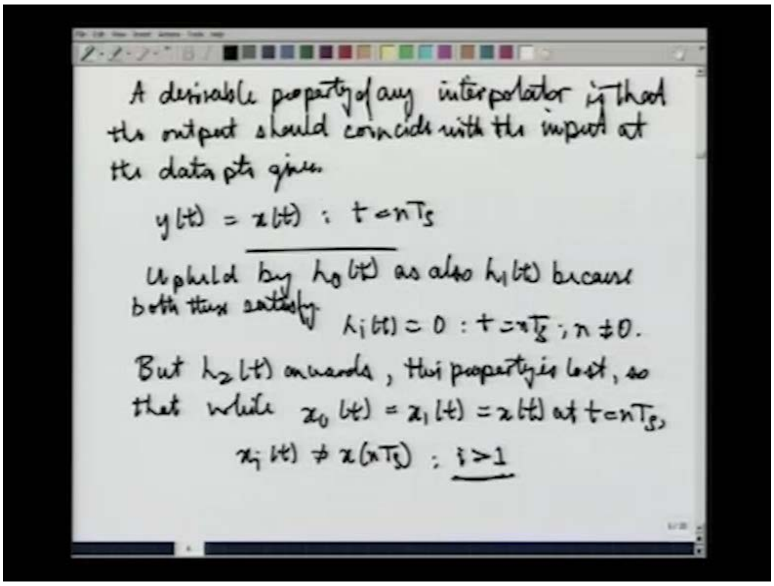

Then the output which we will call x 2 of T does not even match x of t at the points T equal to n T s, let us put this down at desirable property of any hold circuit or any retrieval circuit, any retrieval system is that the output should. Instead of calling it a retrieval system which it is of course, we will call it an interpolator is that the output should coincide with the input at the data points given thus, if y t is the result of an interpolation, then y t must be equal to x t at T equal to n T s though. Of course, at other places where x t is 0 it ought to fill up fill in values by computation in some manner.

Now this property is upheld by h 0 of T as also h 1 of T, because both these satisfy h I of T equal to 0 at T equal to n T s, and not equal to 0; this is a very, very important property, but h 2 of T onwards this property is lost. So, that while x 0 of T equals x 1 of T equals x t at T equals n T s x I of T is not equal to x of n T s for I greater than 1.

Hence there is no point in concerning ourselves with anything more than a first order hold, what the first order hold itself does, we will just illustrate by a diagram a figure and then we will leave our first order hold, and have a final last discussion about how the ideal low pass filter is also a hold circuit, but of infinite order. So, the first order hold let us say this is the original x t it has been ideally sampled

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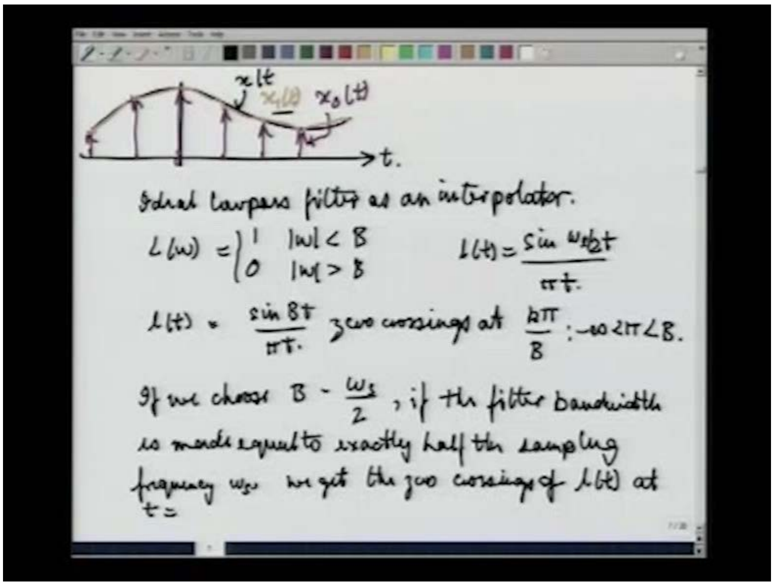

So, you have a sample train that looks like this, and on the basis of the information available in the sample train, this is x s of T sample train we have to carry out an interpolation and what the first order hold does the 0 th order hold produce a staircase what this produces is the series of ranks and it achieves this by simply connecting the tops of these samples with straight lines clearly, this is x 1 of T this x 1 of T is sometimes above the curves sometimes below the curve and so on, and so one can never be really sure, whether it exceeds or is less than the values of x t.

However, at least we know that it joins the tops of the samples with the straight line as it is doing here, fine. So, this is what is done by a first order hold x 1 of T, the output of first order hold is x 1 of T. Now the ideal low pass filter as an interpolator, the ideal low pass filter seen as an interpolator must be viewed in the time domain remember that l omega was given by the expression which equal to one for mod omega less than B, and which equal to 0 for mod omega greater than B, fine. Now, what is l of T, well for a rectangular system like this l of T is given by w B by pi sin B T by T, the time domain expression for l omega that is l T is sin B T by pi T, this clearly has 0 crossings at B T equal to k pi, that is h T equal to k pi by B minus infinity less than pi less than B right. If we choose B equal to omega s by 2, that is if you tailor the filter.

So, that its bandwidth is equal to half the sampling frequency, if the filter bandwidth is made equal to half exactly, half the sampling frequency, frequency omega s. We get the 0 crossings of l of T at T equal to k pi by k into 2 pi by T s divided by 2, that is B equal to half of this. So, that let just see what that comes to we write B equals omega s by 2. So, l of T will be equal to sin omega s by 2 into T divided by pi t. So, that the 0 crossings will be at omega s by 2 T omega s by a omega s T by 2 equal to k pi. So, 2 k pi omega s is 2 pi by T s.

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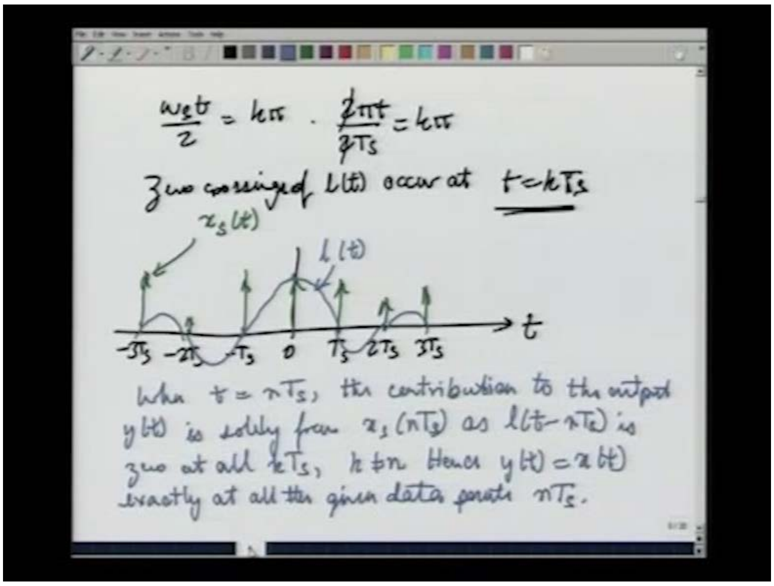

Then let us just go to the next page. So, you have omega s T by 2 equals k pi that is to say that omega s is actually 2 pi by T s 2 pi by 2 T s equals k pi. So, you get 2 pi T by 2 T s equals this. So, you get at the 0 crossings crossings of l T occur at k pi by T s, just a minute at T equal to k T s right at T equal to k T s; that means, let us just super pose the impulse response of the low pass filter, ideal low pass filter whose bandwidth is equal to half of omega s with this sample train itself, let us make a sample train here is the some sample train. So, this is actually x s of T this is the sample train that you have, and then let us make the impulse response of l T sit at T equal to 0 that is centred about that origin.

And let it look like this, it has 0 crossings at T equal to k T s. So, here you have T s, sorry T s 2 T s 3 T s 0 minus T s minus 2 T s minus 3 T s and so on, and we are assured that except for T s equal to 0 expect for T equal to 0, you have 0 crossings at all this other places. So, the impulse response would look like this, and then it would go up like this is l of l of T, now what is? So, special about l of T, you can be seen that when T equal to n T s the contribution to the output y of t is solely from x s of n T s as l of T l of T minus n T sis 0 at all k T s k not equal to n.

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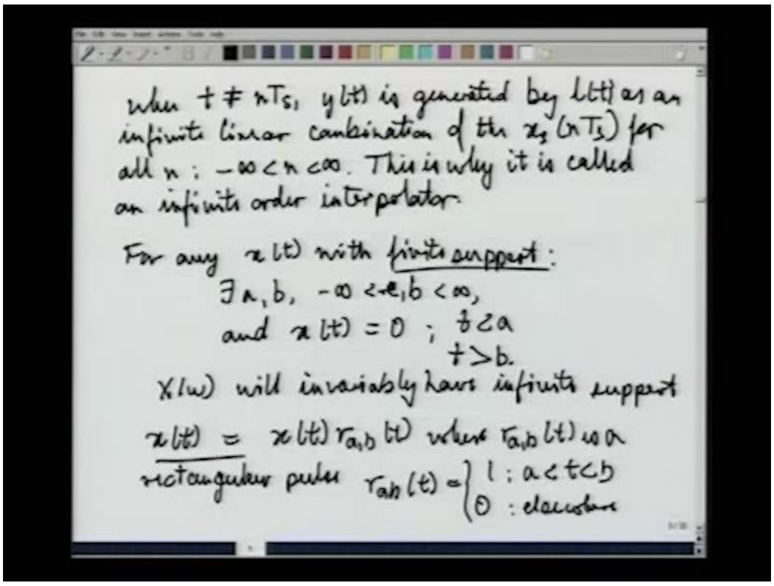

Hence the reconstructed signal y of t equals x of t exactly at all the given data points n T s, when T is not equal to n T s y of t is generated by l of T as an infinite linear combination of the excess at n T s. For all n minus infinity less than n less than infinity, that is why? It is an infinity order interpolator it is called an right from the time, when we learnt about the continuous time period transform, it is not hard to show that if a function has finite support in the time domain. Then its frequency response will have an infinite support, and that is the case here also because it is just the dual of that statement in the frequency domain l of T is a rectangle. And hence l of omega is a rectangle, and hence l of T is the sink function, which never completely goes to 0 for however large values of T it is…

In fact, this property of the rectangle which enforces that for any symbol, this is an important theorem for any x t with finite support, that is to say x there exist a B, such that minus infinity is less than a and B less than infinity, and x t equals 0 T less than a T greater than b. This is what we mean by finite support, whenever x t has finite support x omega will invariably have infinite support x omega will have invariably will have infinite support to prove this, let us recognize that x t can be written as equal to x t multiplied by r a comma B of T, where r a comma B of T is a rectangular pulse R a b of T equals one for a less than T less than B 0, elsewhere once we recognize that we can write x t like this.

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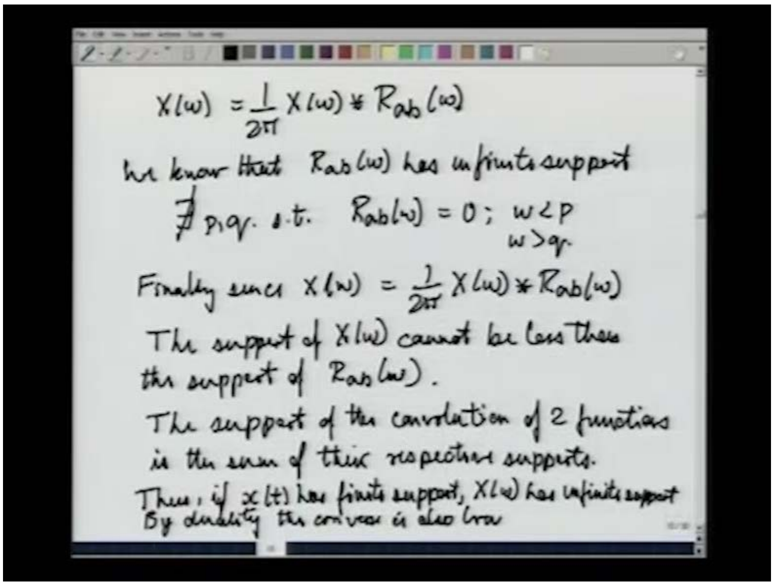

It is another simple step forward to see that x omega equals x omega convolved one by 2 pi, of course times this convolved with R a b of omega. Now, we know that a rectangular pulse like R a b of t will have an infinite support in the frequency domain R a b of omega will have infinite, support that is to say that does not exist say p q, such that R a b of omega equals 0 for omega less than p omega greater than q, that is why we say R a b has infinite support.

Now finally, since x omega equals 1 by 2 pi x omega convolved with R a b of omega, the support of x omega x omega cannot be less than that of R a b of omega, this last remark follows from the fact that under convolution, the support of the convolution of two functions is the sum of the supports of the respective functions; their respective supports it is on this principle. We finally, claim that x omega has infinite support thus if x t has finite support x omega has infinite support, and by duality the converse also holds the converse is also true, finally some concluding remarks about the theory of signals sampling and interpolation.

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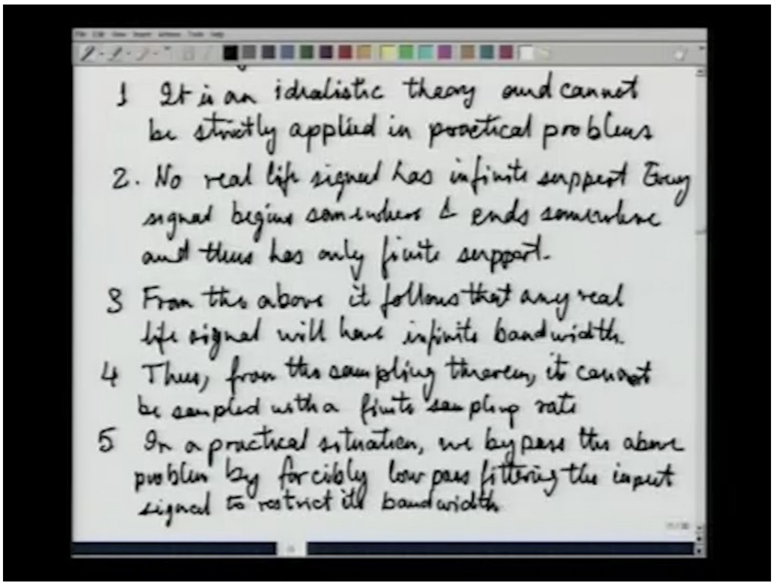

The main thing is that it is an idealist theory, and cannot be strictly applied in practical problems, this is because we will go to the next point. No real life function of time, no real life signal has infinite support has infinite support, every signals begin somewhere and ends somewhere. So, it has only finite support, every signal somewhere begins, somewhere and ends somewhere, and thus has only finite support, from this it follows that it has infinite bandwidth. From the above it follows that any real life signal will have infinite bandwidth, thus from the sampling theorem, it cannot be sampled with a finite sampling rate. In a practical situation we get by the above problem, we bypass the above problem problem by forcibly low pass filtering the input signal, the input signal to restrict its bandwidth.