Signals and Systems
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Lecture - 36
Ideal Sampling

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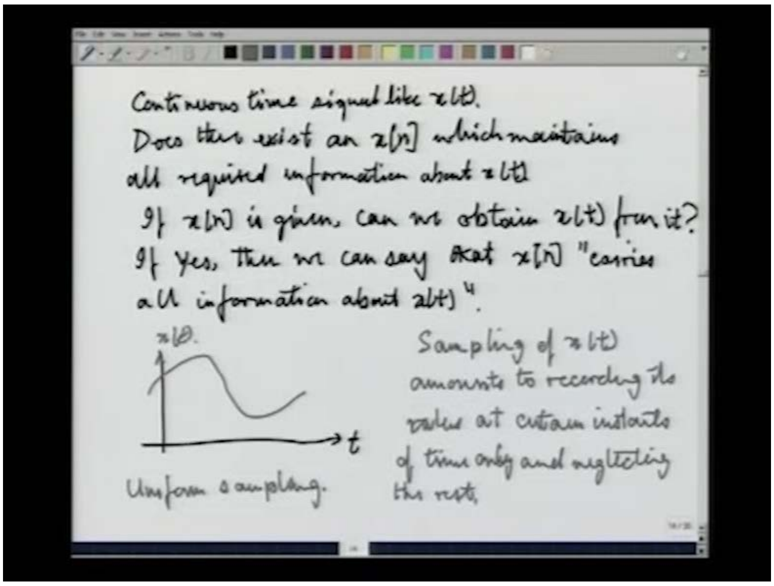

We now start upon an altogether new topic. And this is the issue of relating discrete time signals to continuous time signals, suppose we have a continuous time signal like x (t), x (t); is it possible to find a discrete sequence time sequence x (n), such that x (n) carries within itself, all the required information about x (t), does there exist, and x (n) which maintains all required information about x (t).

Now, what do we mean by saying all required information, that seems a pretty vague thing to say, it simply means that well, if x (n) is given can we obtain x (t) from it, if yes then we can say that x (n) carries all information about x (t). So, this is, what we can call the proof of the ((Refer Time: 03:23)), if you can do this then we can say that x (n) carries all the information. So, we want to go from a continuous time function to a discrete time sequence, and see if all the information about x (t) can be preserved. Now, the first step in this process is to converted into a sequence of so called samples of x (t), and in the second step of the process, we will replace I mean we will represent the sequence of samples of x (t) by the sequence x (n).

So, let us I mean this entire discussion is going to be very graphics intensive. So, we will draw or make a few plots to illustrate the idea. So, suppose this is time continuous time, and we have some function x (t) does looks like this, x (t); if x (t) looks like this, then the first thing we do is to sample x (t). The way one samples x (t) is the following sampling of x (t) amounts to recording its value at certain instance of time certain distances of time only, and neglecting the rest. So, you have to choose certain instance of time, we will choose what is called uniform sampling? Where the chosen instants of time are at uniform distances from each other.

Now, the samples can be taken in the form of impulses whose weights are determined by the value of x (t) at that point of time. In short what we would get after sampling would be a train of impulses. This train of impulses would be such that the strength of an impulse at a particular point n T s would be equal to the value of x (t) at that point of time.

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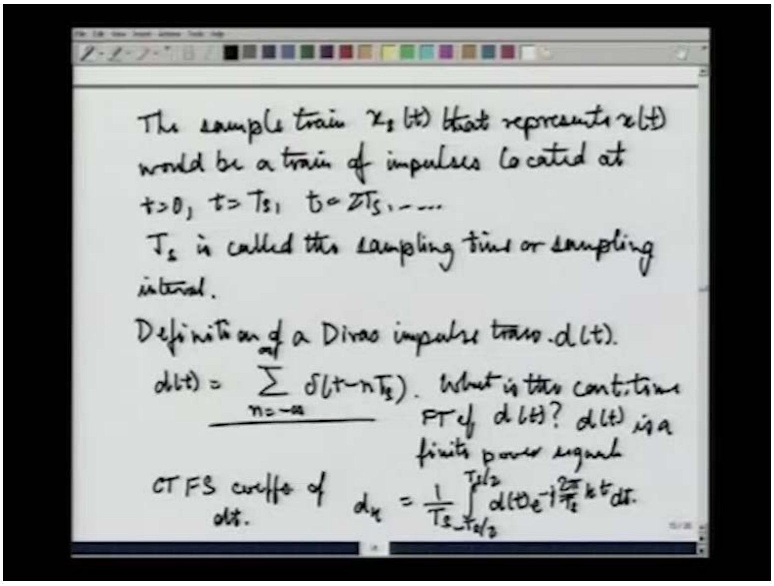

The sample train that represents excess of t, that represents x (t) would be a train of impulses located at t equal to 0, t equal to T s, t equal to 2 T s, and so on; of course also at t equal to minus T s minus 2 T s, and so on. T s is called the sampling time or sampling interval, there are several questions which we have not even begin to answer, for example, what determines T s and so on, but first let us see what an impulse train is and how it is defined the definition of an impulse train, which is essential to the continuation of the discussion of a Dirac impulse train, it is a train of impulses uniformly spaced with respect to each other at a distance T s and all having the same magnitude.

So, I will denote this by impulse train Dirac impulse train d (t), where d (t) equals summation over n for all n, that is to say n equals minus infinity to infinity delta, where delta of course represents Dirac delta t minus n T s, this is d (t). One of the first things we will require is to find the Fourier transform of d (t), what is the continuous time Fourier transform of d (t), continuous time Fourier transform of d (t)? To find the continuous time Fourier transform of d (t), first note that d (t) is a function of continuous time and that is why we are looking for a continuous time Fourier transform of it, second it is a periodic signal, because it is been defined like this, it is a periodic signal.

Now, if we wish to find the continuous time Fourier transform of a periodic signal which is not a finite energy signal, but a finite power signal we have to go through a certain set of steps all this follows on the fact that it is not a finite energy signal. So, d (t) is a finite power signal d (t) is a finite power signal. So, in order to find its Fourier transform the first step is to find its continuous time Fourier series coefficients CTFS coefficients of DTI will call them d k given by 1 by T s integral say minus T s by 2 to T s by 2, you have d (t) e to the minus j 2 pi by T s k t d (t), this is what we have what does this evaluate to we have to recognise.

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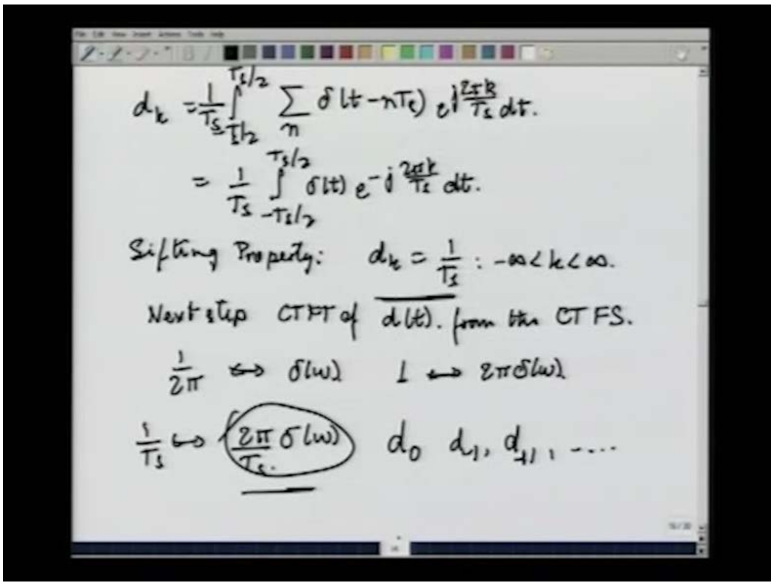

Let us substitute for d (t) in the expression d k equals integral minus t by 2 T s by 2 to T s by 2, of course one by T s times of this, and d (t) is a sequence of impulses delta t minus n T s e to the j 2 pi by 2 pi k by T s d (t), this is what we have. Now, note that this is equal to 1 by T s integral minus T s by 2 to T s by 2 of simply delta t e to the minus j 2 pi k by T s, I really think that this there is a mistake over here, this should be n make that correction.

Now, 2 pi k by T s d (t), this is what we have, why do we, how is this manipulation justify simply, because of this train of impulses delta t minus n T s only one lies within the interval minus T s by 2 to T s by 2, and that one is delta of t, the others are outside the scope of integration. And there are therefore not relevant to the integration, now in order to evaluate this integral, we have to simply apply the shifting property, and that gives us d k equals 1 by T s for minus k less than minus infinity less than k less than infinity. So, it is all the discrete time Fourier series coefficients are equal to 1 by T s, this is what we have.

So, now we have the DTFS coefficients sorry CTFS coefficients continuous time Fourier series coefficients an infinite array of coefficient values. From here how do we proceed to finding the continuous time Fourier transform from the continuous time Fourier series, we want the continuous time Fourier transform remember, that the Fourier series coefficients; each Fourier series coefficient gave rise to 1 impulse in the frequency domain representation of the periodic waveform, you will essentially have to recall our discussion in which we extended the scope of the continuous Fourier transform to also encompass finite power signals such as d (t) there, we said that if you have delta omega as an impulse in the frequency domain its inverse Fourier transform would be 1 by 2 pi 1 by 2 pi had a Fourier transform of delta omega, fine.

So, what we have is that one has a Fourier transform of 2 pi delta omega fine, hence 1 by T s will have a Fourier transform of 2 pi delta omega by T s. So, this is what we get a 2 pi delta omega by T s is for this first coefficient. Now, what we have is a series of this coefficients at different frequencies k omega naught or k 2 pi by t naught k omega naught with k 2 pi by T s, and each of those is independently going to give us one expression like this, one came out of d 0; this is the result of just considering d 0. Similarly, we have d minus 1 d plus 1, and so on and so for which gives us complete Fourier series representation.

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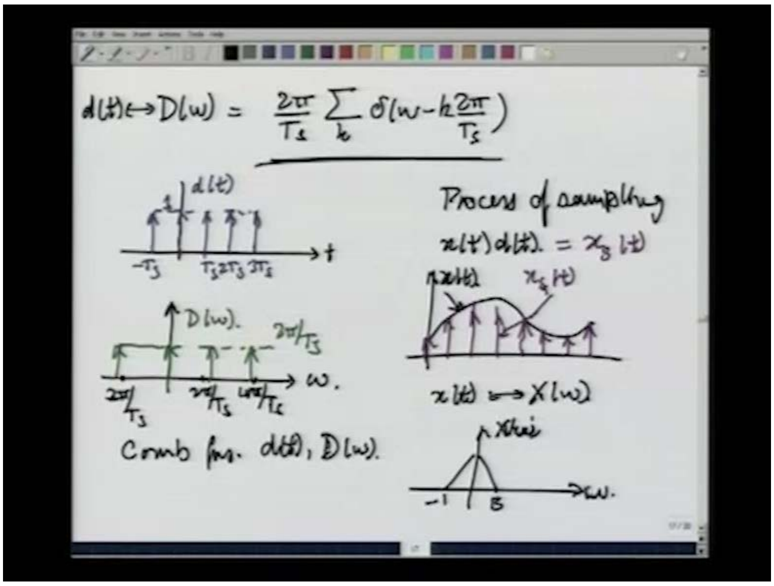

And this therefore, yields what I call d omega which is the Fourier transform of d (t) as given by 2 pi by T s summation overall k delta omega minus k omega s or k 2 pi by T s, this is the complete transform of d (t). So, let us keep this in mind an important result. Now, we have the important Fourier transform of the discrete time sequence of the impulse train in continuous time, and the very important property of this, you see is that even in the frequency domain it is an impulse train.

So, we this is one of the rare cases, where you have a function whose Fourier transform looks somewhat similar to itself, what do we, how do we depict the Fourier transform or how do we depict the d (t) itself, this is d (t), fine. Now, what is the distance between the one sample and the next is T s 2, T s 3, T s minus T s and so on, this is the spacing and the height s of distance of all this samples is unity. Now, let us look at d omega? This is the omega axes, sorry omega axes and what you have is d omega? d omega is also an impulse train, but what is the spacing between the impulses here, it is 2 pi by T s. So, you have over here an impulse at 2 pi by T s, then you have at 4 pi by T s and so on.

Minus pi by minus 2 pi by T s and at each of these places you have impulses, what is the height of this the height is also equal to 2 pi by T s, both these are called comb functions. So, d (t) is a comb function d omega is a comb function, these are both comb functions and its interesting as I said to see that the Fourier transform of the comb function is also a comb function with a slight difference in the values along the horizontal, and vertical axes, otherwise they are both comb functions; comb function goes to comb function. Now, we have d (t) with us, and let us say we have x (t), what we do is to multiply d (t) and x (t) to get X s of t, this is called the sampling. The act of sampling or the process of sampling; the process of sampling is to carry out this product x (t) d (t), when you do x (t) d t, let us make a sketch for it, let us see this is x (t). Now x (t) d (t) is to take this d (t) sequence that we have over here, and multiply it with this to get an impulse train which is x (t) d (t). So, d (t) is already given its all of uniform height equal to unity, but because of multiplication the x (t) distance of the impulses will change at different places. So, you will get something like you.

Have a train of impulses of height s determined by the current value of the x (t), the function x (t) and this x (t) d t is what we will call X s of t, this is what we have plotted over here, x (t) is this, x (t) is this function; and X s of t is, this all this impulses X s of t all this impulses. So, now we now have X s of t, let us first find Fourier transform of X s of t before we can find it, we should assume some kind of a spectrum for x of t. So, let us say that x of t transforms to x omega, let us say x (t) transforms to x omega, and x (t) is this original black curve, we have over here. If you transform it to x omega, you will get the Fourier spectrum the magnitude spectrum, let us put it as which let us say for simplicity sake looks like, this the highest frequency value at which x omega is non-zero, we will call as B and at the other extreme we will call it as minus b. So, we are assured of course we are making an assumption.

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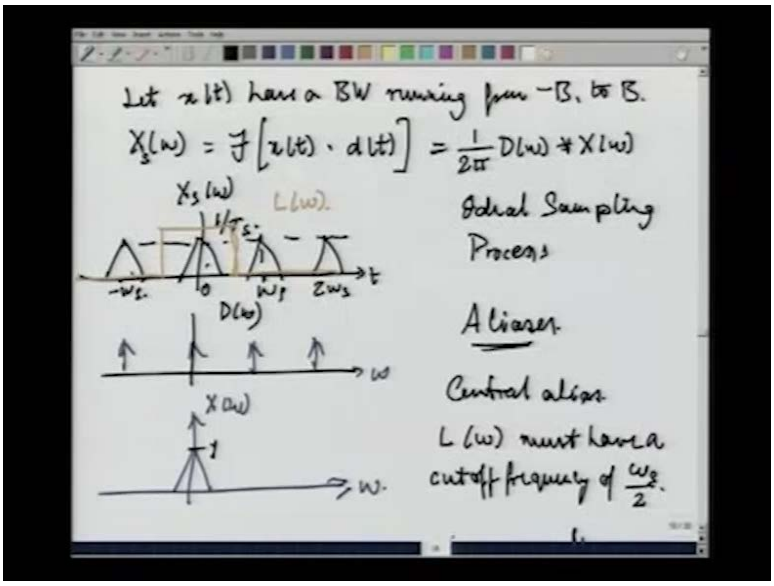

There x is real or otherwise, we would not have had a symmetric spectrum, but in any case the point is that it has a bandwidth from minus B to B. So, let x (t) have a bandwidth running from minus t minus B to B, if we have this. Then let us look at the expression for X s of omega X s of omega is the Fourier transform of the product, is the product is the Fourier transform of x (t) convolved linearly with d (t), because it is a linear convolution what we are going to get is in the frequency domain, sorry it is not convolved its multiplied over here, and hence the Fourier transform will be convolution of the Fourier spectrum of d (t) and the Fourier spectrum of x (t), what you will get is 1 by 2 pi d omega convolved with x omega, now what is d omega?

We made a plot over here, d omega - d omega is, this is d omega you have d omega, and you convolve it with x omega convolution of an impulse sequence with x omega with Fourier transform of any continuous signal will yield a series of copies of original signal. In short you will get something short, that looks like this, X s of omega, this is X s of omega, and how it is obtained by a convolution of the impulse train d omega containing impulses over all, these places and the x omega; x omega is just this simple one time function when you convolve the x omega with d omega. This is what you will get? You will get X s of omega as shown over there; fine, this process of generating X s of omega is called the ideal sampling process.

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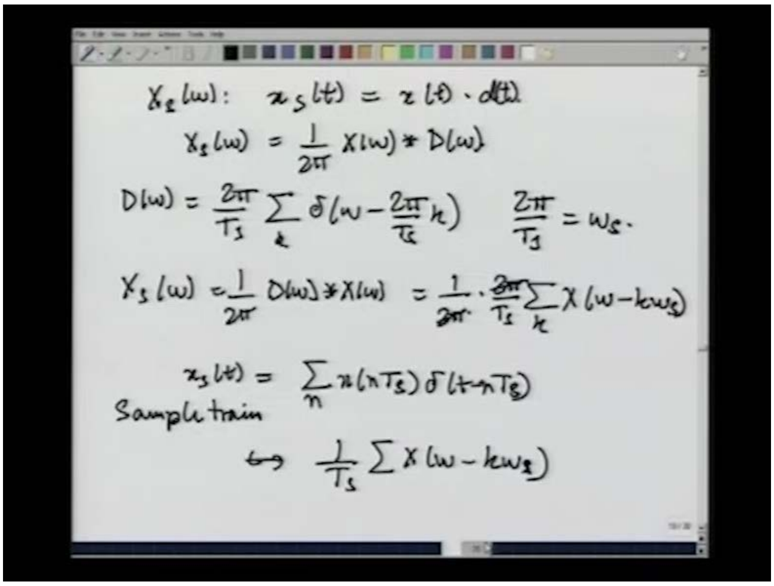

Let us now evolve an expression for X s of omega, first what is X s of t X s of t equals x x of t multiplied by d of t, hence X s of omega must be equal to 1 by 2 pi times x omega convolved with d omega. Let us get back let go back, and find out what is d omega, now d omega is equal to 2 pi by T s summation over all k delta omega minus 2 pi by T s times k for convenience, let us call 2 pi by T s as omega s in future. So, this is d of omega and we, of course have x omega.

Now, when you convolve these two using this delta functions over here at the location of each Dirac delta in the frequency domain, what we will actually get is a copy of x omega translated to the position k omega s. So, you get that X s of omega equals one by 2 pi d omega convolved with x omega equals 1 by 2 pi times t pi by T s multiplied by the summation over all k of x omega minus k omega s this is what we get? So, 2 pi cancels the 2 pi in the numerator. And we finally get X s of t which is equal to the sample train over all n of x (n) T s delta t minus n T s this is X s of t having a transform of 1 by T s summation x of omega minus k omega s, the sketch of this was already shown in the previous slide over here.

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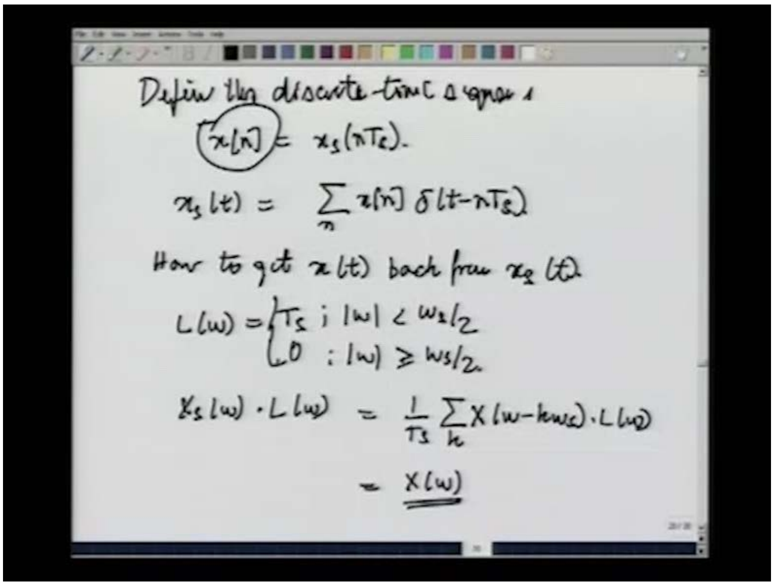

This is the sketch it has a constant repetition of these copies of x omega, where x omega is originally shown over here. Now, that we have X s of omega, let us see if it is possible X s of omega is the Fourier transform of X s of t. So, if we have the sampled train alone, then we are already one step towards generating the discrete sequence x (n), which carries our information or which represent s the information about x of t. So, from X s of n T, we will define the discrete sequence, the discrete time sequence x (n) equal to X s at n T s, this is what we are going to do.

So, having done this, this is the only thing we will carry with us the discrete sequence x (n), suppose we have the discrete sequence x (n) is not hard to reintroduce X s of n T s, that is it is not hard to get X s of t. How do we get X s of t given x (n) simply as follows X s of t equals summation overall n x (n) delta t minus n T s, this gives you X s of t back. So, this much is straight forward, if you have X s of t; you can go to x (n) and then with the knowledge of T s of course with the knowledge of what value T s takes you can get X s of t back, if you did not have the knowledge of what T s is?

Then of course you would have a problem of not knowing what is the space between the consecutive samples and X s of t, for suppose you know T s and you know x (n), then you can get X s of t does the real challenge is how to get x (t) back from X s of t that is the real hard question. How do we do this? If we go back and take a look at Fourier spectrum of X s of omega, it does give us some hints on, how to do it you see this has a spectrum identical to x omega except for a scale of, if this was equal to 1. Then this has a scale of 1 by T s we know this, but apart from this matter of scale which is not a serious matter the shape of X s of omega in the neighbourhood of omega equal to 0 is pretty much identical except for the scale factor to x of omega.

So, one way to get something back to get back x omega from X s of omega is to just get rid of these other copies that have horizon on all sides, these copies are called aliases they are duplicates photocopies, if you like of the original x omega. So, you want to get rid of all the aliases except this, which is called the central alias, you only want to capture the central alias and get rid of the others, and evidently that is not hard to do all, you have to do is to set up an ideal low pass filter, an ideal low pass filter is the one which allows all frequencies up to a certain lower than a certain frequency and rejects everything outside.

So, now, you see this is after all omega s this is 2 omega s, this is minus omega s and so on, and we really need a low pass filter such as this to be imposed, this low pass filter I will call as l of omega. Now, what should be the bandwidth? What should be cut off frequency of this low pass filter l omega must have a cut off frequency of omega s by 2 right, it should have a cut off frequency of omega s by 2. If you have an cut off frequency of omega s by 2, then applying l of omega to X s of omega would imply multiplying the two Fourier transforms, and this process would yield only the central alias now you can even make sure that the scale error in the scale of one by T s is cancelled off by giving an appropriate gain of T s to l of omega.

So, let us define l omega properly l omega must be equal to T s for mod omega less than omega s by 2, and it must be equal to 0 mod omega greater than or equal to omega s by 2, this is what constitutes the specification of l omega. Now, let us see what happens? Theoretically at least for the figure that we have shown over there, what should happen is X s of omega gets multiplied by l omega, and therefore this equals one by T s summation over all k x omega minus k omega s times l omega which by the fact, that it has a cut off at omega s by 2 will simply give you x omega, this means we have gone through.

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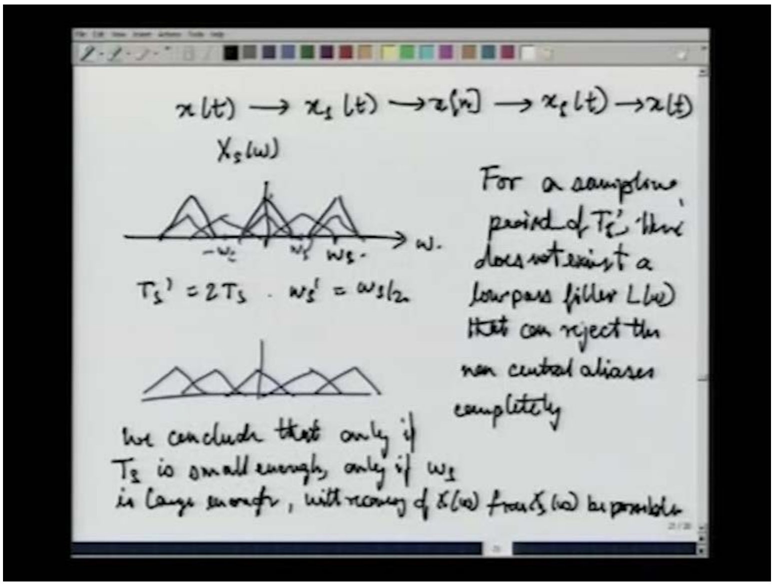

We have successfully gone through the entire sequence of steps starting with x (t), we got X s of t from which we got x (n) from which with the knowledge of T s, you can again get X s of t and using l omega we got x (t) again all seems to be working fine, that we can start with a continuous time function get X s of t, then represent it by x (n) a discrete time sequence and from the discrete time sequence. And with a knowledge of the sampling time, and the sampling period interval of X s we can recover X s t from x (n) and from X s of t by doing, what we just did ideal low pass filtering we can get x (t), but there are some catches in this whole thing, the catch arises from the fact that we have not ensured by any over d statement, that x (t) must maintain certain standards in order to be represented in this manner. Let us go back to the sketch of the Fourier transform of X s of omega, this is what we had as X s of omega, and this resulted from the fact that omega s was equal to this much, but suppose omega s was less than this, suppose omega s suppose we had chosen choose T s dash equal to 2 T s.

So, that omega s dash becomes equal to omega s by 2, supposing we did this, then I will plot it on the same axes for convenience, we would get this as omega s dash exactly half of this omega s dash, this would be minus omega s dash. And the first alias would of course, remain the same, but since T s dash is 2 times T s we would also get half the height. So, let us do that we would get half the height. So, you will get we would get half the height, now apart from getting half the height, there are other things which happen, since this is omega s dash the second alias the original or central alias is what we have? Just drawn the second alias would not be. So, close or would not be where the previous alias was, but it would be centred about omega s dash.

So, you get the second alias like this, the third alias like this, the third alias would be like this, and so on. The left side you would get the second alias like this, third alias like this. Now, you see that the aliases are overlapping, and if you can just redraw this part over here for separately, you get an alias like this, a second alias like this, a third alias like this.

Now, the central alias that was there earlier is still there, now it is scaled by a different amount scaling is not an issue here, the serious issue is the fact that while there we realise that eliminating the non-central aliases, the second and higher aliases on the left, and the third side would yield or would allow to remain behind only the central alias. And we could reject them earlier, because they are sufficiently far away from the central alias. So, you could place a low pass filter which discriminated the central alias from the remaining aliases.

Now, we cannot do it, because they are overlapping and hence there does not exist for a sampling period of T s dash, there does not exist a low pass filter l omega that can reject the non-central aliases completely, clearly this is happened, because we chose T s too large. So, T s should not be very large we chose T s dash too large and therefore it is important for the sampling time or the sampling period not be too large from a time domain point of view, this simply says that the longer the time interval the more scope the signal has to make changes in itself. And therefore, a sufficiently short time sampling time interval is required to track all possible changes of the signal from one sampling instant to the next, so because of this problem that we have faced here. We conclude that only if T s is small enough, that is to say only if omega s is large enough will recovery of x omega from X s B sorry, X s is not right from X s of omega B possible only when it is large enough will the recovery be possible.

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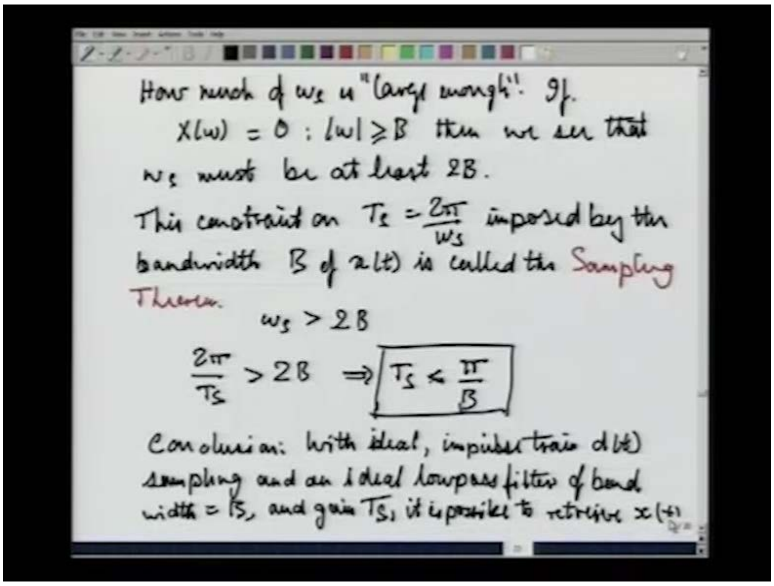

So, how much is large enough of omega s is large enough, this is clear from the figure, omega s must be at least equal to twice the bandwidth of the signal x omega, if x omega equals 0 for mod omega greater than equal to B. Then we see that omega s must be at least 2 B, if it is at least 2 B. Then at least theoretically an ideal low pass filter will be able to extract x omega from x of s this constraint on T s equal to 2 pi by omega s imposed by the bandwidth B of x (t) is called the sampling theorem.

Thus we finally get that omega s must be greater than two times B, that is to say 2 pi by T s must be greater than 2 B. So, that T s must be less than 2 pi by 2 B, that is pi by B, this is the sampling theorem. As long as the sampling theorem is followed that is as long as T s is chosen to be no larger than pi by B, which clearly means that the greater the value of B the smaller should be the value of T s.

The greater the value of B will simply means that x (t) has higher and higher frequency component s; that means, it is capable of very rapid variation, and as x (t) varies more and more rapidly to capture all its moments, one requires a smaller and smaller T s that is it has to be sampled more and more frequently in time. So, we conclude that with ideal sampling ideal impulse train sampling, d (t) sampling, and an ideal low pass filter of bandwidth equal to B, it is possible a bandwidth equal to B and gain of T s, it is possible to retrieve x (t), it is possible to retrieve x (t) exactly, but then this is just half the story rather less than half in fact, there are so many problems with ideal sampling.