Signals and Systems
Prof. K. S. Venkatesh
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 34
Properties of Discrete Time Fourier Transform

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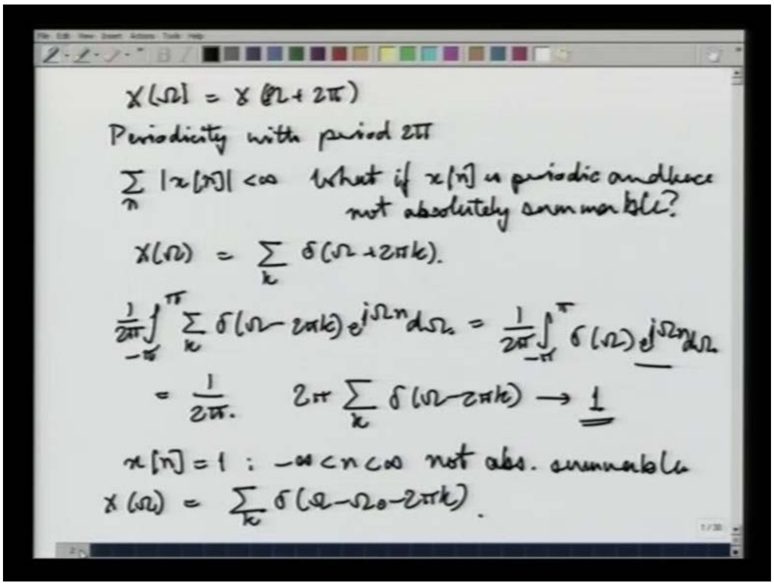

To all the examples we have considered one thing stands out, and that is that all the x omega we have at hand are periodic with a period 2 pi, x omega equals x omega plus 2 pi, this can be verified for all the 3 examples that we have presented. Thus this is one of the most fundamental properties of the DTFT, the periodicity with period 2 pi, this is one of the most fundamental properties. Now, given this periodicity we will have to ensure that any signal in the frequency domain that we wish to find the inverse transform for inverse v t f t for must first of all be periodic with period 2 pi, otherwise it is not even a valid continuous function of capital omega, that is to say it is not a function which is available to the inverse DTFT.

Next keeping this in mind let us move on, and see if this DTFT which is known to exist only for absolutely summable signals, that is to say signals such as signals which meet this criterion when summed over all n, this is the kind of signals we have considered so far in the three examples. What happens? If these signals are not absolutely summable, but are periodic, periodic and hence not absolutely summable, what happens then? This is relevant of the similar situation that we faced in the context of the continuous time period transform, which was originally meant for finite energy signals, but which got extended to the case of finite power signals as well to give a unified formulation at a later stage.

Let us see we can repeat that performance over here, and the way we do that is to guess that direct impulses must have something to do with it, suppose we place a direct impulse rather as this is really required in the case of the DTFT a train of direct impulses placed 2 pi apart from each other on the omega axis and see what its inverse transform will be… So, we have let us say x omega equal to summation over all k delta omega plus 2 pi k, it is only a signal that is periodic with period two pi that qualifies to be the DTFT of some discrete sequence, so now this does meet the criteria.

Let us now try to find the inverse DTFT of this sequence, which is 1 by 2 pi minus pi to pi summation over all k delta omega minus 2 pi k e to the j omega n d omega, clearly out of this summation, only one impulse namely delta omega lies within the scope of the integral within the range of the integration. So, this becomes equal to 1 by 2 pi integral minus pi to pi delta omega e to the j omega n d omega, we now apply the shifting property of the direct impulse on this, and find that the entire integral will evaluate to the value of the companion function at the point of occurrence of impulse namely omega equal to 0.

So, that we get that this equals 1 by 2 pi, this means that 2 pi delta sorry 2 pi times summation over all k delta omega minus 2 pi k is a function whose inverse DTFT equals one, that is x n equals one for minus infinity less than n less than infinity which is clearly not absolutely summable, because it will blow up to infinity when you try to sum it. Here then is a clear case of an non absolutely summable signal, which has shown to have a DTFT which has appeared to have a DTFT. Now, let us take a more general example by not taking an impulse or a train of impulses located at 2 pi k, but at some arbitrary position omega equal to omega not minus 2 pi k. So, we will take let us say x omega equal to summation overall k delta omega minus omega not minus 2 pi k, what do we get when we find the inverse DTFT of this function.

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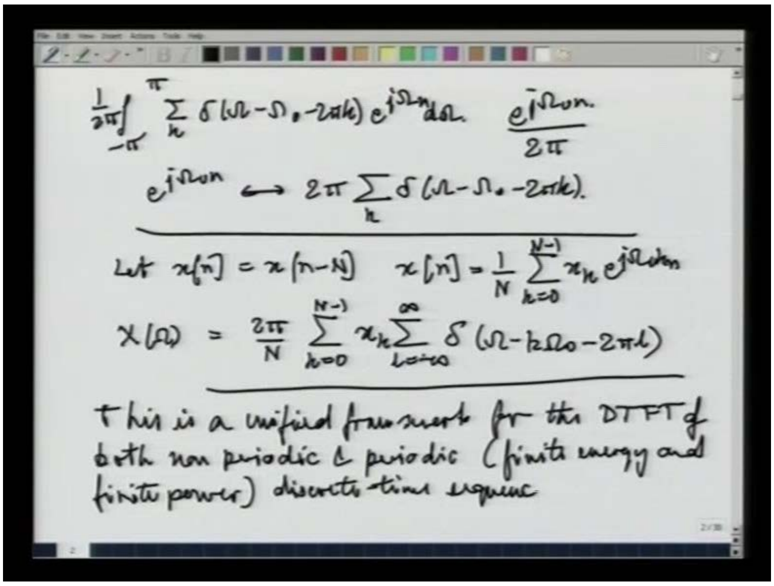

So, we get integral minus pi to pi 1 by 2 pi of the summation overall k of delta omega minus omega not minus 2 pi k e to the j omega n d omega, this if evaluated just as in the previous example will give you that its inverse transform is e to the j omega not n, which is a periodic complex exponential which is clearly not absolutely summable. In short we can say sorry, this it is actually equal to this divided by 2 pi, because this there is a factor of 2 pi outside. So, we can say that e to the j omega naught n transforms to 2 pi summation overall k delta omega minus omega naught minus 2 pi k. So, this is another Fourier transform pair for a periodic signal, a periodic complex exponential.

Now, we are equipped to take the next big jump, and find the Fourier transform for any arbitrary periodic discrete time signal, if it is a periodic with period n discrete time signal, then it has a DTFS representation or DFT representation whatever we like to call it, of the form given by this synthesis equation. Thus let x n d equal to x n minus n, so that we have x n equals 1 by n summation k equals 0 to n minus 1 x k e to the j omega naught k n, this is what we have?

Now, what we get? This is a sum of exponentials waited by the respective x case, now what do we get when we try can we try the DTFT of this sequence. This can definitely be done, because all we have to do is treat each term in this expression in the same manner that we have handled a complex periodic exponential in the previous example. So, we will finally, get x omega equals 2 pi by n, which is a common factor that lies outside summation k equals 0 to n minus 1 x k summation l equal to minus infinity to infinity delta omega minus remember that this k th component will occur at frequency omega naught, where omega naught is the fundamental frequency minus 2 pi l, which takes care of the repetition with period 2 l with period two pi.

This is the expression for the discrete time Fourier or the extended discrete time Fourier transform for an arbitrary periodic sequence which has a set of DFS, DTFS coefficients x k; k going from 0 to n minus 1. So, this has now created a unified frame work, this is a unified frame work for the DTFT of both non-periodic and periodic, that is to say finite energy, and finite power discrete sequences - discrete time sequences.

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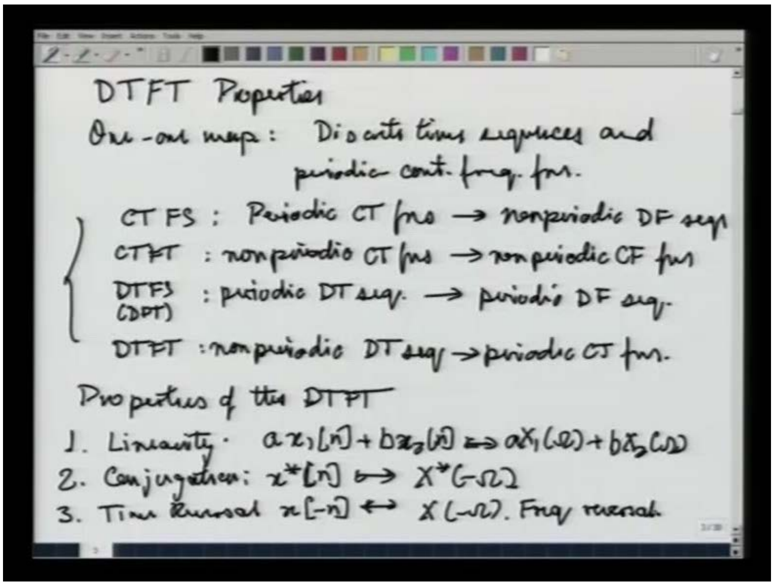

Next item and our agenda is to explore the mathematical properties of the DTFT. Fundamentally the DTFT takes infinite discrete time sequences into periodic continuous frequency functions. So, one-one map between discrete time sequences and periodic continuous time or continuous frequency functions.

Let us look at the various varieties that we have considered so far, let us make a little table, we had the CTFT then we had no we first had the CTFS continuous time Fourier series, then we had the continuous time Fourier transform followed by the discrete time Fourier series, and now the discrete time Fourier transform. Each of them is related to others in some way, and they all tie up together very nicely. The CTFT the CTFS mapped continuous time periodic functions to discrete sequences which were the sequences of the discrete continuous time Fourier series coefficients, non-periodic discrete sequences.

So, CTFS map periodic continuous time functions to non-periodic discrete sequences, discrete frequency sequences discrete frequency sequences. Next the CTFT mapped non-periodic continuous time functions to non-periodic continuous frequency functions, next the discrete time Fourier series also called the DFT mapped periodic discrete time sequences to periodic discrete frequency sequences. So, all possible combinations all possible pairs of continuous discrete periodic, non-periodic are manifesting themselves in this story. Finally, you have the DTFT which maps non-periodic, discrete time sequences to periodic continuous time functions.

So, you see all possible combinations have occurred in one example or the other. So, with this summary of the relationships between the various transforms and the series representations, that we have considered. Let us go and look at these specific properties of the DTFT. The first property that we consider is linearity is linearity, which says that a x 1 n plus b x 2 n will transform to a x 1 omega plus b x 2 omega, that combines both super position and homogeneity, that is to say additivity and homogeneity, and that is the end of that hardly any effort required to prove that.

Next conjugation I am going to skip the proofs, now because whatever I do not prove is not proved only because it is extremely straight forward, and can be drawn analogically, the proof can be evolved analogically with the corresponding proofs for the continuous time case. Conjugation x star n transforms to x star of minus omega time reversal is x of minus n which yields x of minus omega. Since x of minus n goes to x of minus omega, we automatically taken care of frequency reversal as well, because we simply means that x of minus omega goes x of minus n x of minus n goes to x of minus omega. So, this covers frequency reversal.

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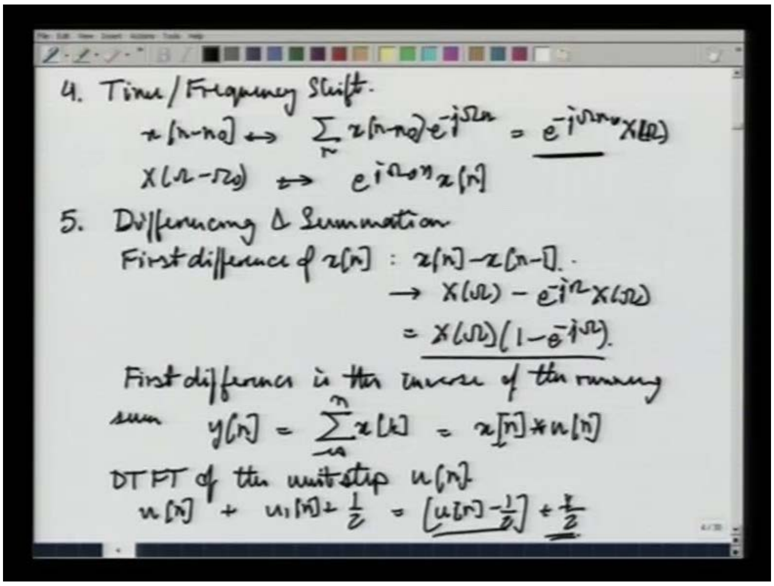

Next of course is time and frequency shift time and frequency shift, and here we have let us say x n minus n naught which will transform to a summation overall n of x n minus n not e to the minus j omega n, which can be evaluated by just replacing n minus n naught with n dash and making all the appropriate modifications and manipulations that normally go with it to yield e to the minus j omega n naught x omega, that is the multiplication by a linear face factor e to the minus j omega n. Frequency shift would be where we had x omega minus omega naught what would x omega minus omega naught do to us, it turns out that this transforms to e to the j omega naught n x n, all these are very, very easy to prove, so this no need to spend time on this in the middle of the course.

Next differencing and summation five differencing and summation, we know what is differencing? The first difference of x n of x n is defined as x n minus x n minus 1 combining this with the time shift property we already have, this simply will have period transform given by x omega minus e to the minus j omega x omega. So, that this is equal to x omega times 1 minus e to the minus j omega, that is it that gives you the first difference. Now, the opposite of the first difference are the inverse operation of the first difference is the running sum, is the inverse of the running sum where the running sum is defined as y n is the running sum of x n, if this equal to summation of minus infinity to n of x k, this is called the running sum as with the case of the continuous time running integral this terms out to be nothing but x n convolved with u n the unit discrete step.

So, in order to evaluate the Fourier transform of the running sum y omega, we first have to evaluate what is the Fourier transform of the unit step. Once you have the Fourier transform of the unit step, we just need to multiplied as will turn out later on when we do the convolution theorem, it will turn out that it is just multiplying the Fourier transform of u n with the Fourier transform of x n, but let us first find out the unit step and its Fourier transform. Fourier transform are DTFT of the unit step discrete unit step u n as we did very similar to what we did in the case of the continuous time Fourier transform u n is going to be expressed as the sum of a 0 average part u 1 n minus or other plus half, which is equal to u n minus half plus half, that is what u n this is the 0 average part, and this is the average value.

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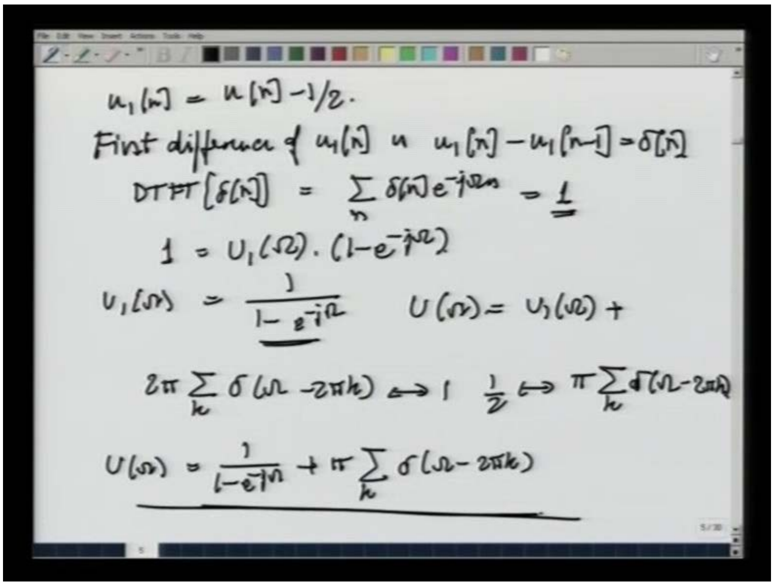

Now, what is it transfer function of the 0 average part, the 0 average part is u 1 n equals u n u n minus 1, this is what we have? u n minus, sorry u n minus half. Now, in order to find this, let us first find the Fourier transform of the first difference of u 1 of n, the first difference of u 1 of n is u 1 of n minus u 1 of n minus 1 which is nothing but equal to delta n. Now what is the Fourier transform of delta n we seem to be going astray, but we really are not, we are going from one problem to another, but we really are not what is the Fourier transform of delta n DTFT of delta n equals summation delta n e to the minus j omega n as n varies from minus infinity to infinity.

And since delta equal to 0, everywhere else this will have only one term it survives and that therefore, makes this become equal to one nothing more than one. So, DTFT f delta n is known is equal to one. Now, if its DTFT is known then we can say since delta n is the first difference of u 1 n delta n must be equal to the transform of delta n one must be equal to u 1 of n, sorry u 1 of omega multiplied by 1 minus e to the minus omega, no e to the minus j omega, fine.

So, u 1 n gets multiplied by this, because it is by multiplying by this in the frequency domain that we carry out the equivalent of differentiating or rather differencing u 1 of n in the time domain. So, you have this, therefore u 1 of omega equals 1 by 1 minus e to the minus j omega you have this, finally then u of omega equals u 1 of u 1 of omega, sorry plus the DTFT of half, since the DTFT of one was the DTFT of one is not yet been found, so let us just find it. So, we had that 2 pi summation overall k delta omega minus 2 pi k this had an inverse transform of 1.

So, we now have that half is transformed to pi summation overall k delta omega minus 2 pi k, this is half is the transform of half. So, finally u omega equals u 1 omega which standout to be equal to one by 1 minus e to the minus j omega this plus the other part the constant part half which is equal to pi summation overall k of delta 2 pi minus, sorry delta of omega minus 2 pi k, this is the Fourier transform of the complete unit step, with this in hand which I will call u omega.

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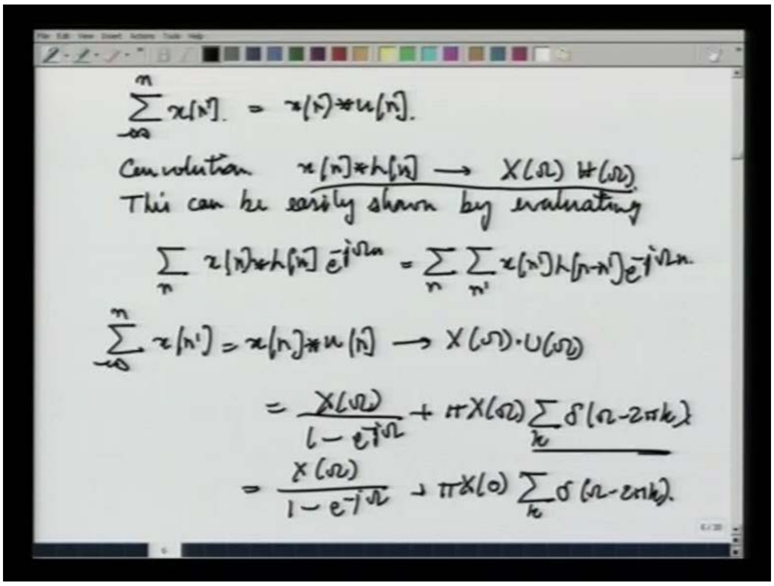

Since we know u omega now, and we have x omega as the Fourier transform of x t of x n then we can find, what is the Fourier transform of minus infinity to n of x n dash which is the running sum in terms of x omega. Now, this equals as we said x n convolved with u n, now we have the Fourier transform of u n, discrete time Fourier transform of u n, we have the discrete time Fourier transform of x n, but we still do not know what the convolution does. What is the Fourier transform of convolution of two sequences? This we can very easily state this is another digression the convolution of two sequences x n h n transforms to x omega h omega, this can be easily shown by evaluating summation overall k of x n summation overall n, sorry of x n convolved with h n e to the minus j omega n, which is equal to summation overall n of the convolution itself now is summation overall n dash x n dash h n minus n dash e to the minus j omega n.

I am not going to carry out the remaining two steps that will lead us to this result, but it is not very hard you just have to use the property of time shift, and then repeat the exercise of finding the Fourier transform, so you will get this particular result. With this result in hand we are now in a position to evaluate the running integral, because it is the convolution of x n with u n.

And hence finally the running integral from minus infinity to n of x n dash equals as we said x n convolved with h n, sorry u n u n is equal to or other transforms to x omega times u omega, which is equal to x omega by 1 minus e to the minus j omega plus, we can see the expression if we go back for the second term, it is pi summation overall k delta omega minus 2 pi k pi x omega summation overall k of delta omega minus 2 pi k, this is what it comes true, but if we note that this x omega that this expression this summation will be 0 at all points in minus pi to pi except for omega equal to 0. This can be simplified to x omega by 1 minus e to the minus j omega plus pi x times 0 x of 0 summation overall k delta omega minus 2 pi k. So, this has yielded the Fourier transform of the running integral of a function. So, having delta with the differencing an summation effects or the effects of differencing an summation on the DTFT.

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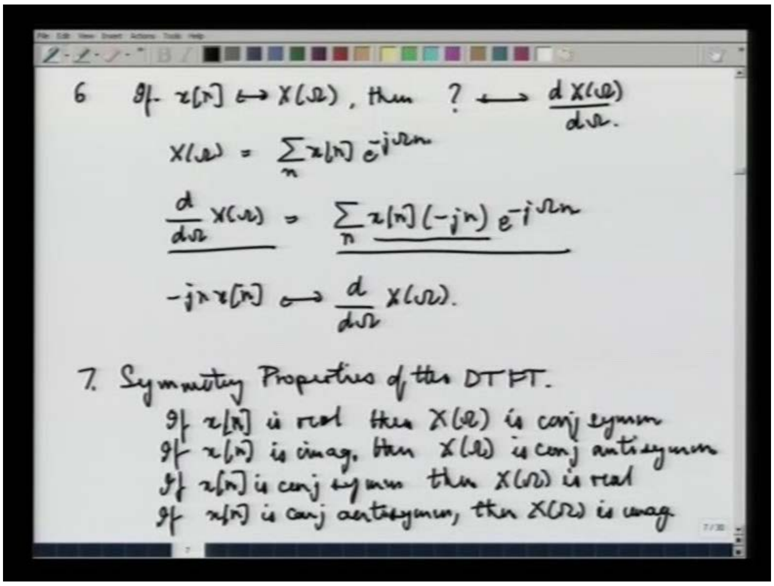

We will now see something else, what happens to x n if x omega is differentiated in omega, that is to say if x n maps to x omega under the DTFT, then what maps to d x omega by d omega, that is the question we ask. And the answer is easily found by looking at the synthesis equation, the analysis equation; the analysis equation is x omega equals summation overall n x n e to the minus j omega n. Now, if we differentiate both sides with respect to omega, we find that the only quantity that awaits differentiation on the right side is inside the summation e to the minus j omega n.

So, this gives us d by d omega of x omega equals summation overall n x n minus j n e to the minus j omega n. Now, while the left side is this is the frequency domain derivative of x omega, the right side still looks exactly like the analysis expression for the new function namely this for this function. Thus this would be the inverse transform or the time domain sequence for which d by d omega of x omega is the DTFT, thus we get the pair that minus j n x n transforms to d by d omega of x omega, fine; with that having being crossed we will discuss something that is a little over due by now, but which is relatively obvious to all of us, since we have already gone through it in considerable detail in the case of the continuous time Fourier transform.

And that is the symmetry properties the symmetry properties of the DTFT, well; as I said I will not spend time and deriving this properties or proving this properties I will just list them out to you and way you prove them is exactly the same as you would do in the case of the continuous Fourier transform barring the minor differences that would come. So, the first is that if x n is real, then x omega is conjugate symmetric, if x n is imaginary then x omega is conjugate anti symmetric, conversely if x n is conjugate symmetric then x omega is real. And finally, if x n is conjugate anti symmetric then x omega is imaginary, as before it yields interesting and present results like if x n is both real and even then x omega will also be even and real, and several other similar results that one can derive at once leisure. So, what for these symmetric properties?

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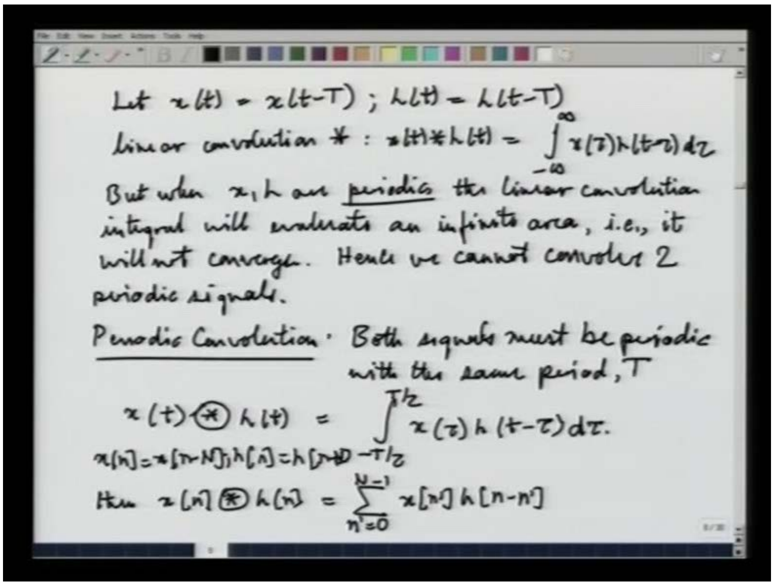

Now, we introduce a notion which is avoidable at this stage and which could have probably been introduced earlier, a notion of convolution of different kind, we know how convolution is defined, but let us see what happens if you try to convolve two periodic non-zero signals. So, let x t, I am presently going to deal with continuous time signals, and then I will move onto discrete time signals in both cases the same point is being met. Let x t be equal to x t minus T h t be equal to h t minus T be two periodic signals, now the conventional definition of what is called linear convolution.

Let we denote by this is x t convolved with h t as given by integral minus infinity to infinity x tau h t minus tau d tau, as far as non-periodic signals is concerned there is no problem, but when x h are periodic an integral, the linear convolution integral will evaluate an infinite area, in short it will not converge. Hence carrying out the linear convolution of two periodic signals is out of the question. We cannot convolve two periodic signals, what we therefore introduce is something called periodic convolution, this is applied between two signals both of which must be periodic with the same period. Let us say with the same period T as we have above with x t and h t justified, then what is the definition of periodic convolution? Periodic convolution which will define as x t periodically convolved written by the convolution symbol enclosed in the circle with h t is given by an integral not over an infinity interval, because that is what will make the convolution shoot of to infinity, but over one period it does not matter over which period it is, but over one period. Let us say for example, minus t by 2 to t by 2 of x tau h t minus tau d tau, now this quantity is certain to remain finite for all values of t, because the area the interval of integration is only one period.

So, while tau varies from minus t by 2 to t by 2 essentially you can say that one of the functions remain static, and the other function is time reversed an shifted by the amount tow as tau varies from minus t by 2 to t by 2, and inner products are evaluated at each value of tau and these inner products essentially give you the value of h t. So, that is the periodic convolution.

Now, why it is of interest to ask all of us sudden? Before I even answer that question let us just dispose of the periodic convolution of discrete functions, discrete time sequences. We have just now provided the formulation for continuous time sequences for discrete time sequences, suppose you have x n equals x n minus n h n equals h n minus n then x n periodically convolved with h n is given by a similar definition, you have to sum over one interval it does not real matter with it is minus n by d to n by 2 or 0 to n minus 1 or whatever, we will do it from 0 to n minus 1 n dash equals 0 to n minus 1 of x n dash h n minus n dash. This is the circular convolution in continuous time in discrete time.

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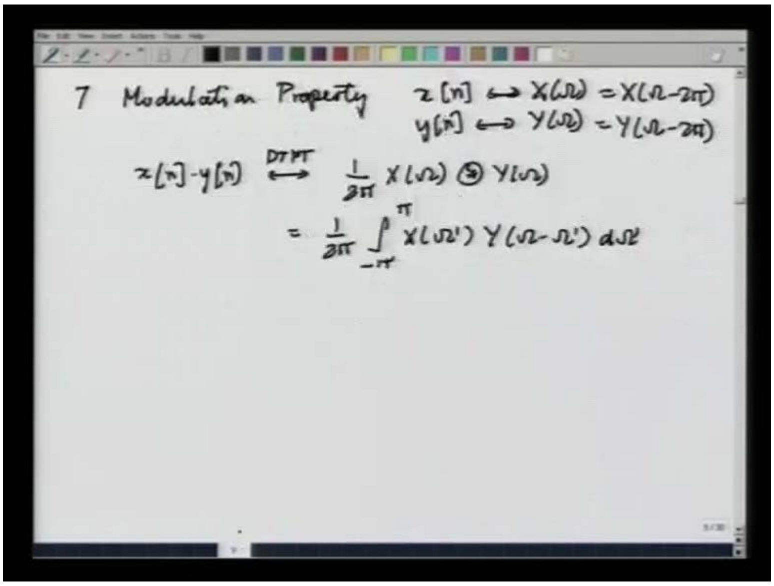

Now where this becomes interesting to us is when we consider the next important property of the DTFT called the modulation property, now the modulation property is all about multiplying two discrete time sequences and seeing what happens to the DTFT of the product. So, if x, sorry x n transforms to x omega, y n transforms to y omega we know that both y omega and x omega are periodic that is to say this equals to x omega minus 2 pi, this is equal to y omega minus 2 pi. Taking the q from the modulation property of the continuous time Fourier transform, what we except here is that the Fourier transform of the product x n, y n must be the convolution of the Fourier transforms 1 by 2 pi times the convolution of the Fourier transforms, but unfortunately here you really cannot carry out the usual or linear convolution of the two Fourier transform, because both x omega and y omega are periodic functions with the period of two pi.

So, how on earth can one carry out their convolution. So, the answer is obviously, we carry out the periodic convolution of x omega and y omega and therefore it can be shown that x n times y n has a Fourier transform given by as a DTFT given by 1 by 2 pi x omega periodically convolved with y omega, where of course we fall up fall back up on the continuous variable version of the transform as given over here, this is what we are going to use here, and this therefore amounts to 1 by 2 pi integral remember that period is 2 pi, so you can say 0 to 2 pi or minus pi to pi. Let us say minus pi to pi for no of particular reason x omega dash y omega minus omega dash d omega dash.

This is the Fourier transform of the product of the two discrete sequences x n y n having Fourier transforms x omega y omega respectively. So, that ends the set of most of the important properties of the DTFT