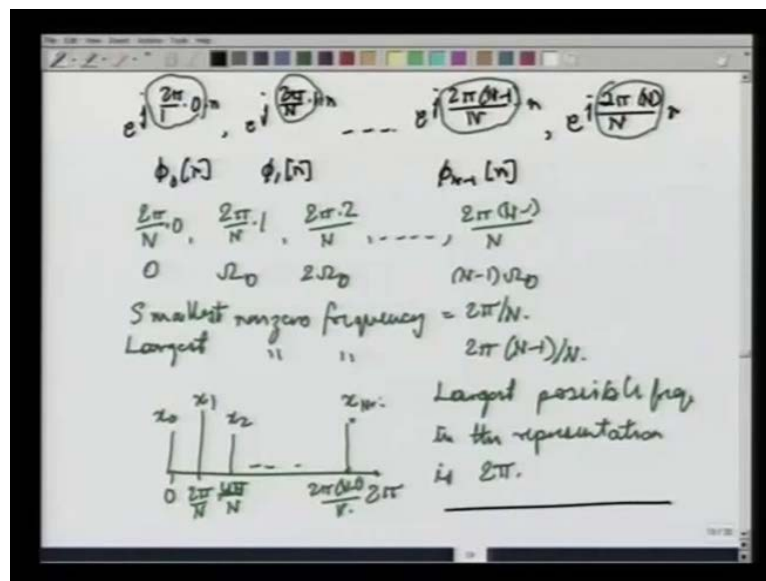


Signals and Systems
Prof. K. S. Venkatesh
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 33
Discrete Time Fourier Transform

It is essential for us, beyond recognising that there are only N distinct discrete frequencies in the $DTFS$, is still essential for us to understand what the values of these frequencies are and what range of values they lie within. It is with this in mind that let that we shall go and re-examine the different complex exponentials, that are associated with each of the discrete sequence. So, let us recall what the coefficients are. We have the component complex exponentials are.

(Refer Slide Time: 01:05)



The first one is $e^{j\frac{2\pi}{N} \cdot 0 \cdot n}$, this is the first number. This is what we have called ϕ_0 of n . Then we have $e^{j\frac{2\pi}{N} \cdot 1 \cdot n}$ which is what we call ϕ_1 of n and so on until we get $e^{j\frac{2\pi}{N} \cdot (N-1) \cdot n}$ which is ϕ_{N-1} of n . That is already given as N different frequencies, but just for explanation sake, let us see what the next number will be. The next number is $e^{j\frac{2\pi}{N} \cdot N \cdot n}$.

Now, what are the frequencies associated with each of these? The normal format of the complex exponential are of the purely imaginary exponential, as in this case, is there is

the time index which might be discrete or continuous. Then there is the frequency index. In this case, this is the frequency index. This represents the frequency. Here for example, this gives the frequency and so on until you have here, the frequency given by this and here by this.

So, that is what the frequency is supposed to represent. The frequencies, as you can see, run from 2π by N into 0 is ω_0 , 2π by N into 1 is what I will call ω_1 which is the fundamental frequency. This is the zero-frequency. Then you have 2π into 2 by N which we will call, say this is exactly twice the first the previous one. So, it is $2\omega_1$ and so on until you come to 2π N minus 1 by N which we will call N minus 1 ω_1 . Now, this is not common convention.

So, I will just make a slight change in what I have just said. I will just call this the zero-frequency and I will call this as ω_0 . This is the fundamental frequency. So, this becomes $2\omega_0$ and soon up to n minus 1 ω_0 . So, these are the frequency components. As you can see, the values vary from ω_0 to n minus 1 ω_0 where ω_0 is essentially 2π by N . So, the smallest non-zero frequency is 2π by N . The largest non-zero frequency is 2π times n minus 1 by N . So, they are all some fraction of 2π .

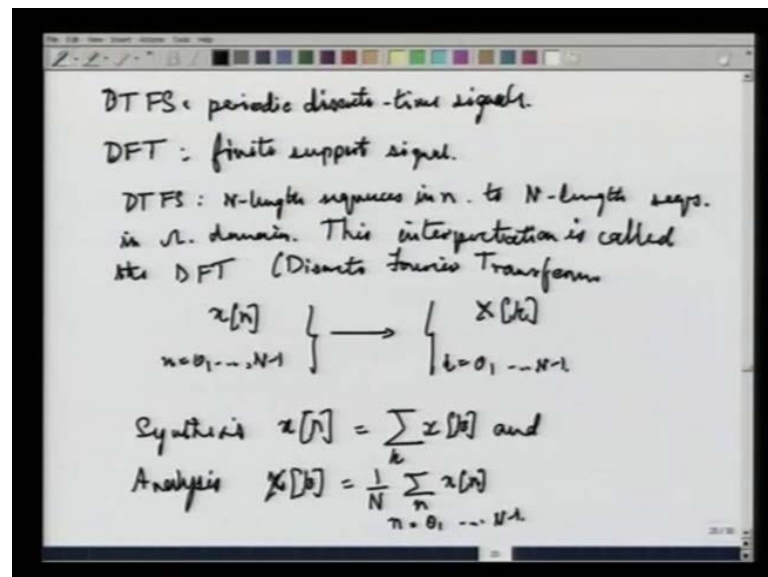
And they range from 0 to 2π times n minus 1 by N . For large n , you can see that the largest non-zero-frequency will become very close to 2π and this finally, tells us what the frequency axes should be like. It should be running from 0 to 2π and on this we will plot these coefficients of the $d_t f_s$ namely, the x_0 x_1 where this is 2π by N . Then we have x_2 occurring at 4π by N and so on until just before 2π , we have x_{n-1} corresponding to 2π n minus 1 by N . So, the largest possible frequency in their presentation is 2π and even this is really not achieved except for very large values of n , so much for giving a physical interpretation to the notion of frequency.

If you want an even more understanding of this then we must start with setting a certain analog frequency. A continuous time frequency value and then measure the rate at which this signal of continuous time is sampled to obtain a discrete signal and understand things from there. That is too much matter to get into at this stage. We will be doing it when we talk about the theory of sampling and reconstruction. Right now, we see that

discrete frequencies take on discrete values ω not, 0 ω not, 2π ω not up to n minus 1 by N ω not and all these lie within the range 0 to 2π .

With this understanding of discrete frequencies, we know how to make a spectral plot. It will be something like this. For example, for the frequency plot of the Fourier spectrum would be something like this. Line spectrum, this is what we have it is only a line spectrum. Next, we want to see what is the common way in which the discrete time Fourier series is used? You see the discrete time Fourier series, as I have been saying, deals with periodic discrete time signals.

(Refer Slide Time: 09:07)



But, another way of looking at periodic signal is to call it a finite support signal or what is called a signal of finitely many points and this is the nomenclature and terminology that is used in what is called the DFT. Finite support signal of a support of n points. Now, in the first case, we would say that the signal repeats itself an infinite number of times in all directions. The same n point of the signals are repeated. In the second case, we just say that now the signal is only defined for n points, we are not saying that it is 0 outside those n points, we are saying that it is defined only for n points.

Once we make that definition that is defined only for n points then lots of things become much simpler. For example, we can say since the Fourier transform the DFTs, the discrete time Fourier series, of this sequence has also n coefficients capital n coefficients and they too are periodic, it turns out even there you can take this time that it is only a

finitely supported sequence of transformed values. In short, you can say that the d f s maps n length sequences in n to n length sequences in the omega domain.

So, instead of periodic, we are saying it has a finite support. Now, this interpretation is given a different name, it is called the dft, standing for discrete Fourier transform. The dft has a slightly different set of notations. It takes sequences x_n for n equals 0 to n minus 1 and maps them to signals x_k as they are called discrete frequencies where k goes from 0 all the way to n minus 1. So, this is the discrete Fourier transform. As you can see it is only notationally different and if you wish to write the analysis and synthesis equations of the d f t, you would simply write synthesis and the analysis is X_k equals 1 by N summation over n of x_n , n equal to 0 n minus 1.

So, everything is of finite length now. The impulse response is of finite length, the signal is of finite length, the output is of finite length, everything is of finite length. The next thing we will move on to investigate is what happens, if we take a certain n periodic signal or n length signal depending up to whether we use the d t f s terminology or d f t terminology and pad it with an equal number of zeroes. What does happens to the Fourier series spectrum?

(Refer Slide Time: 15:32)

Handwritten mathematical derivations on a whiteboard:

$$\begin{aligned}
 x_1[n] &= x[n] : n=0, \dots, N-1 & x_1[n] &= x_1[n-N] \\
 x_2[n] &= x[n] : n=0, \dots, N-1 & x_2[n] &= x_2[n-2N] \\
 &= 0 : n=N, \dots, 2N-1. & 2N &= N'.
 \end{aligned}$$

$$\rightarrow Nx_k = \sum_{n=0}^{N-1} x[n] \phi_N^*[n]$$

$x_2[n]$ is of length N' .

$$Nx_{2k} = \sum_{n=0}^{N'-1} x_2[n] \phi_{N'}^*[n]$$

Coeffs of $x_2[n]$, $k=0, \dots, k=N'-1$
 $= k=2N-1$

Denote Nx_{1k} as $X_1(k\omega_0)$ where $\omega_0 = 2\pi/N$.

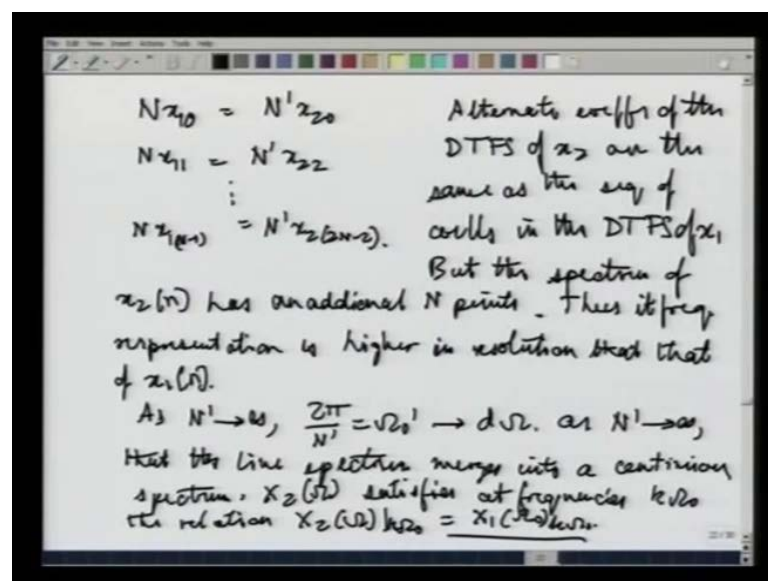
So, let us say that x_n or other x_1 n equals x_n for n equals 0 to N minus 1 and x_2 n equals x_n for n equals 0 to N minus 1 and its equal to 0 for n equal to N to $2N$ minus 1. Now, x_1 n is of length n or has period n . So, you can say, if we take d t f s terminology

which we are more familiar with, we can say therefore, $x_1[n]$ equals $x_1[n - N]$. Whereas, here $x_2[n]$ equals $x_2[n - 2N]$. Let us call $2N$ as N' . Now, I will rewrite the analysis equations in the slightly modified manner.

The analysis equations gave the value of the x_k for various values of k , k going from 0 to the number of points in the sequence. So, we will write first $N \times k$ which is the set of coefficients of $x_1[n]$ scaled by factor k , $n \times k$ equals summation $x[n]$ times $\phi_k[n]$ as n equal 0 to $N - 1$. So, there are N different coefficients available over here, but now, moving to $x_2[n]$, $x_2[n]$ is a $2N$ length sequence is an N' length sequence and hence we will write $N' \times k$. This we call $N \times k$, $N \times k$ will be equal to summation n equals 0 to $2N$ to the power k of $x_2[n]$.

Here it was x_1 then because x_1 equal was equal to $x_1[n]$, x_2 N $\phi_k[n]$ of $2N$. This earlier one was for this one, it was $\phi_k[n]$. That means there are N different complex exponentials which combine whose correlations we have to evaluate. Now, there are $2N$ over here. So now, the coefficients of these, the coefficients of $x_2[n]$ run from k equal to 0 to k equal to $N' - 1$ which is equal to k equal to $2N - 1$. So, these are the different coefficients you have. Now, we shall denote N times $x_1[k]$ as $x_1[k\omega]$ not where ω not equals 2π by N .

(Refer Slide Time: 21:30)



Then it turns out, using similar notation for x_2 also, it turns out that $N \times 2$ 0 or rather N' $\times 1$ 0 which is the zero-th coefficient of x_1 multiplied by N equals N' $\times 2$.

Here, then $N \times 1$ equals $N/2$ and so on until you get $N/2 - 1$ equals $N/2$. That is to say that alternate coefficients of the DFTs of x_2 are the same as the sequence of coefficients in the DFTs of x_1 . In fact, all the even coefficients of x_2 correspond 1 for 1 in magnitude and frequency position with those for x_1 , but the spectrum of x_2 has an additional N points. That is to say that its frequency representation is higher in the resolution than that of x_1 . We have doubled the frequency resolution by padding with n zeros.

If we had padded with three times n zeroes, we would have quadrupled the frequency resolutions and so on. In short, as we make N tend to infinity as N tends to infinity, $2\pi/N$ which I will call $\Delta\omega$ tends to $d\omega$ tends to an infinity symbol value. Of course, there are a large number of coefficients now in the Fourier transform. There will be coefficients far more in number than before until finally, as N goes to infinity, we find that the line spectrum merges into a continuous spectrum.

(Refer Slide Time: 28:23)

The Synthesis equation

$$x_2[n] = \sum_{k=0}^{N/2-1} X_2[k] e^{jk\omega_0 n}$$

$$X_2[k] = \sum_k \frac{X_1[k]}{N} e^{-jk\omega_0 n}$$

$e^{jk\omega_0 n} \rightarrow e^{j\Omega n}$. Summation becomes an integration and $1/N = d\Omega/2\pi$.

Summation over $k=0$ to $k=N/2-1$ is the same as summation over $\Omega = \Omega_0$ to $\Omega = (N/2-1)\Omega_0 \rightarrow 2\pi$

$$x_2[n] = \frac{1}{2\pi} \int_0^{2\pi} X_2(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_2(\Omega) e^{j\Omega n} d\Omega$$

This continuous spectrum which we will call $X_2(\omega)$ is that what I have called it satisfies at frequencies $k\omega_0$. The relation $X_2(\omega)$ at ω_0 equals X_1 of ω and ω equal to $k\omega_0$ equals X_1 of ω_0 at $k\omega_0$ and it is non-zero elsewhere whereas, this is 0 elsewhere. By means of this process, we can go on increasing the resolution until it becomes a continuum. When it becomes a continuum,

this implies certain modifications in the analysis and synthesis equations. Let us look at each of them, first of all analysis.

The synthesis equation says that $x[n]$ equals summation $x[k] e^{j k \omega_0 n}$ and for $x[k]$ it could be this and subscript of $x[k]$ over here. But, we have already made the notational change that $N - x[k]$ is what we are calling $X[k]$ at $k \omega_0$. If we use this notation, then $x[n]$ becomes equal to a summation, all the summation is of course over all k . So, I am just writing k over here, overall k of k going from 0 to $N - 1$.

So, you will get $X[k]$ of $k \omega_0$ by $N - 1$ because that is $x[k]$ into the $j k \omega_0 n$. Now, as N increases as I said there are more closely spaced line frequencies. So, $e^{j k \omega_0 n}$ simply becomes $e^{j \omega n}$ and the summation becomes an integration. Finally, you have on by N , summation becomes an integration and $1/N$ is nothing but $d\omega$ by 2π .

(Refer Slide Time: 33:48)

Analysis equation

$$N x_k = X_2(k\omega_0) = \sum_{n=0}^{N-1} x[n] e^{-j k \omega_0 n}$$

$N \rightarrow \infty, \quad e^{-j k \omega_0 n} \rightarrow e^{-j \omega n}$ as the sum runs from $n = -\infty$ to $n = \infty$

$$X_2(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \omega n}$$

With this the spectrum of the nonperiodic $x[n]$ has become a fn of the cont variable ω as it runs over an interval of length 2π . So we no longer have a line spectrum but instead a continuous spectrum.

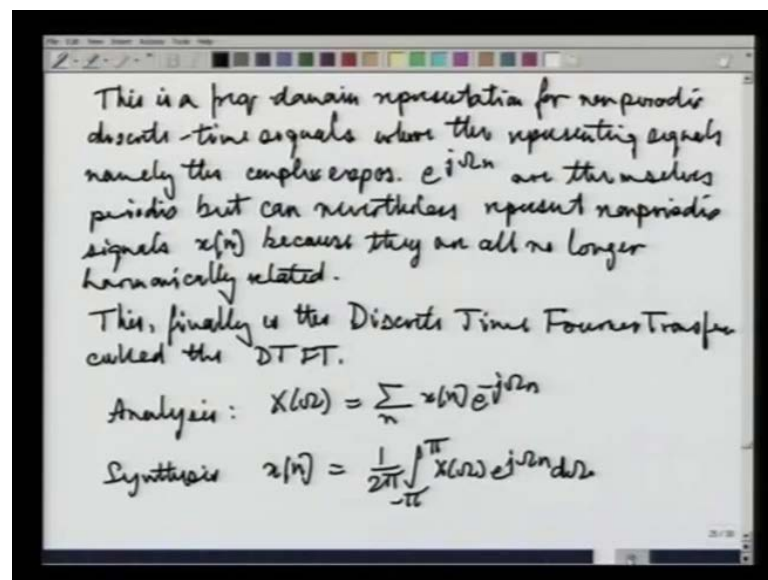
Making all these changes and recognising that integration from summation for k equal to 0 to $N - 1$ over k equal to 0 to k equal to $N - 1$ is the same as summation over ω equal to ω_0 to ω equal to $N - 1 \omega_0$ which tends to 2π . We can write with all these changes that $x[n]$ which is now a non-periodic signal because it is of infinity duration, equals $1/2\pi$ integral 0 to 2π $X_2(\omega) e^{j \omega n} d\omega$. But since, this $x[n]$ of ω is periodic

which is very easy to show, we could actually integrate over any contiguous period of length 2π . For example, we could rewrite this as $\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega n} d\omega$. It is equally valid to write this.

Now, the analysis equation. We have $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ was given by summation n equals 0 to $N-1$. Applying the same changes over here, we find that as N tends to infinity, $e^{-j\omega n}$ simply becomes $e^{-j\omega n}$ and n runs from $-\infty$ to ∞ . So, that you get $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$.

With this, the spectrum of the non-periodic $x[n]$ has become a function of the continuous variable ω as it runs over an interval of length 2π . So, we have a continuous spectrum. No longer have a line spectrum but instead a continuous spectrum. However this spectrum is periodic with period 2π , something we will demonstrate in a moment but let us understand at least what has finally, resulted.

(Refer Slide Time: 37:43)



What has finally resulted is a frequency domain representation for non-periodic discrete time signals, where the representing signals namely the complex exponentials, the periodic complex exponentials $e^{j\omega n}$, are themselves periodic. But, can never the less represent non periodic signals $x[n]$ because they are all no longer harmonically related. This is analogous to the similar discussion that we had for the case

of the continuous time Fourier transform. Even there the representing signals are all periodic signals.

But they manage to represent the non-periodic signals there and that again happens because we are actually combining by addition periodic signals, which are not harmonically related. So, this is called discrete time Fourier transform called d t f t whose analysis equation is given by $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ and synthesis is $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$. These are the analysis and synthesis equations, with this implies it is time to discuss issues like the convergence behaviour of convergence existence issues of the d t f t.

(Refer Slide Time: 41:47)

Convergence & Existence Issues

if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, then $X(\omega)$ will be finite for all ω : $0 \leq \omega \leq 2\pi$.

Whenever $X(\omega)$ exists, the reconstruction

$$\hat{x}_w[n] = \frac{1}{2\pi} \int_{-w}^w X(\omega) e^{j\omega n} d\omega : 0 \leq w \leq \pi$$

$$\lim_{w \rightarrow \pi} [\hat{x}_w[n]] = x[n] ; -\infty < n < \infty.$$

It can be shown that if the sequence, infinitely long discrete sequence the non-periodically long discrete sequence, $x[n]$ is absolutely sum able overall n . Then $X(\omega)$ will be finite for all ω $0 \leq \omega \leq 2\pi$ and whenever $X(\omega)$ exists, the reconstruction x at $w[n]$ I will say, integral from minus w to w $X(\omega) e^{j\omega n} d\omega$ for $0 \leq w \leq \pi$, this is X at n . Now, this turns out that limit as w tends to π of x at w of n equals $x[n]$ for minus infinity less than n less than infinity. That is for all n .

The reconstruction matches or converges to the original sequence. Whenever the discrete time Fourier transform exists, this convergence will happen. We only had to take care

that the existence is ensured, rest of it will follow automatically. So, this has brought us to a certain level. Now, what can we say next is that at this stage it will be worthwhile to look at examples of d t f t.

(Refer Slide Time: 45:26)

Examples of the DTFT.

1. $x[n] = a^n u[n]$: $X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{j\omega n}$

Note that $|a| < 1$ $= \frac{1}{1 - a e^{j\omega}}$

2. $x[n] = a^{|n|}$ ($|a| < 1$).

$X(e^{j\omega}) = \sum_{n=-\infty}^{-1} a^n e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$

$= \frac{1}{1 - a e^{j\omega}} + \frac{1}{1 - a e^{-j\omega}} = \frac{1 - a^2}{1 - 2a \cos \omega + a^2}$

Examples of the d t f t remember that d t f t exists for sequences which are absolutely sum able. So, very evidently we cannot consider sequences which keep blowing up without limit. They should be absolutely sum able. So, we will consider such examples of sequences and see what the d t f t comes to. First example, let x_n be equal to $a^n u[n]$. That means it is a sequence like this a sequence which exponentially decays n tends to infinity.

Now, what would be the Fourier transform of the sequence? The discrete time fourier transform of the sequence, $X(e^{j\omega})$, would amount to summation from 0 to infinity because for negative values of infinity this is 0 because it is multiplied by $u[n]$ and you have a to the power n $e^{-j\omega n}$. Now, note that a must be between minus 1 and 1, $|a| < 1$ for this kind of graph to exist. Otherwise it will be going up and not going down. So, $|a| < 1$. When $|a| < 1$ and $X(e^{j\omega})$ is this, $X(e^{j\omega})$ evaluates to $\frac{1}{1 - a e^{j\omega}}$ by $e^{-j\omega n}$ a^n $e^{-j\omega n}$.

Next, second example. We will now consider a bilateral signal, a signal which is non-zero for both negative and positive time and which is also even symmetric and see how it

comes out. Clearly for the first example, $1 \text{ by } 1 \text{ minus } e \text{ to the minus } j \omega$ is not really a real function. It is complex value, because there is complex numbers in the denominator and that is understandable for reasons of symmetry that we will come to know a little while.

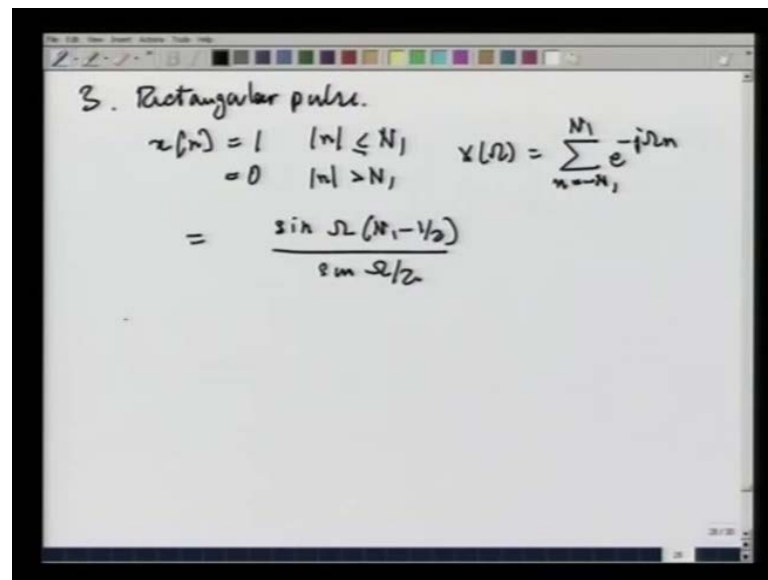
The second example is $x_n \text{ equals } a \text{ to the power } n \text{ mod } n$. Again, we are going to say that $a \text{ to the power } n \text{ mod } n$ where a is less than 1, $\text{mod } a$ is less than 1. Now, in this case, you will have $X \omega$ equal to summation from minus infinity to infinity, we can split it up into two terms, and get summation minus infinity to minus 1 $a \text{ to the minus } n \text{ e to the minus } j \omega n$ plus summation 0 to infinity $a \text{ to the } n \text{ e to the minus } j \omega n$.

Now, this expression is what we want to evaluate and this comes to finally, after simplification it comes $1 \text{ minus } a \text{ squared}$, actually let me write the intermediate steps as well. $1 \text{ by } 1 \text{ minus } a \text{ e to the minus } j \omega$ plus $1 \text{ by } 1 \text{ minus } a \text{ e to the } j \omega$. Going back to the previous problem there is just a small factor I have missed in the final expression in the $x \omega$.

If you look at this part, this part is wrong and we are just going to strike it out, I forgot about a , is equal to $1 \text{ by } 1 \text{ minus } a \text{ e to the minus } j \omega$. So, this can be knocked out now because it is wrong. This is the expression over here. Here, we have similar expressions, but separate expressions for the two parts, and now for simplification, this becomes $1 \text{ minus } a \text{ squared divided by } 1 \text{ minus } 2 a \cos \omega \text{ plus } a \text{ squared}$. So, this is the expression for the $d t f t$ of the second example.

Now, see second example is even symmetric it will turn out for reasons that we will describe when we discuss the properties of Fourier transform that they should be completely real as indeed actually is all terms, all factors in the expression for $x \omega$ can easily be seen to be completely real. There are no complex terms over here. Now, the third example is of the rectangular pulse.

(Refer Slide Time: 52:09)



3. Rectangular pulse.

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases} \quad X(\omega) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$
$$= \frac{\sin \omega (N_1 + 1/2)}{\sin \omega/2}$$

The rectangular pulse is given by $x[n]$ equal to 1 for $|n| \leq N_1$ and equal to 0 for $|n| > N_1$. So, you will get $X(\omega)$ as a finite sum from $n = -N_1$ to N_1 of $x[n]$ is anyway equal to 1. So, you just get the basis function $e^{-j\omega n}$. This is what you get and this comes to be equal to $\frac{\sin \omega (N_1 + 1/2)}{\sin \omega/2}$. So, these are a few standard examples of the dft of some simple expressions.