Signals and System
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Lecture - 32
Discrete Signals and System Representation

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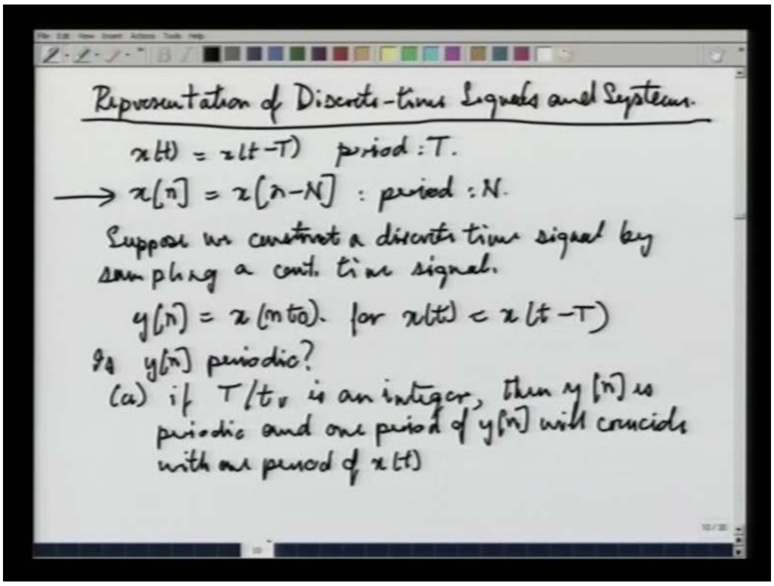

We shall now start a discussion of the representation of signals and systems for the discrete case discrete signals and systems and their representation one might argue that in the real world there are no discrete signal now if one takes that view then one is simply defining what one means by the real world, because in today’s world with digital cameras with digital recording of audio signals we do have discrete time signals and they are very much part of the real-world you might say that they are not present in the natural world, but then we will have to debate about what is natural and what is unnatural what is artificial and what is natural instead one can just confine oneself to the to a study of the advantages and problems that one associates that one can associate with a discrete formulation of the theory of signals and systems in contrast with the continuous formulation that we have spent some time on.

So, far we already have an introduction to discrete sequences which are called discrete discrete time signals we have an introduction to the discrete complex exponential in considerable detail all this has already been covered in this course what we are now going to look at is a parallel study a study along the lines we have already travelled, but this time for discrete time signals there if you recall we started with periodic continuous time signals of the form x T equals x T minus t. So, the period was T now in this case we start with discrete sequences which we will denote by x n equal to x n minus n. So, the period is now an integer that we call n we also generally indicate that the independent variable of the signals is a discrete variable by using brackets instead of parenthesis for n as we have done just now alright now given this let us try to see, if there is some relationship we can draw between discrete time sequences and continuous time functions which are both periodic suppose I construct, suppose I try to construct a discrete time sequence by sampling a continuous time signal suppose further that x T is periodic as been given over here I want to ask the question if we have sampled a continuous time periodic signal x T using a discrete set of samples which are placed say T naught seconds apart from each other under what circumstances will the discrete sequence that results be periodic as well. So, the samples we have are x let me call this new discrete sequence as say y n y n equals x of k T naught for x T equal to x T minus t.

So, capital T is the period of repetition of x T and T naught is what we call the sampling interval. So, at integer values of k you get k T naught points on time and at those instance the value of x T will be taken as the values of the sequence y n sorry it is not k we will call this n. So, y n equals x of n T naught this is what we have now we want to ask is y n periodic periodic now it is a discrete sequence. So, its periodicity or otherwise will be judged by whether it meets the criterion for periodicity of a discrete sequence as set out in this equation this is what is going to determine that now we will see is y n periodic now there are several cases to consider a if T by T naught is an integer than y n is periodic and one period of y n will coincide with one period of x t.

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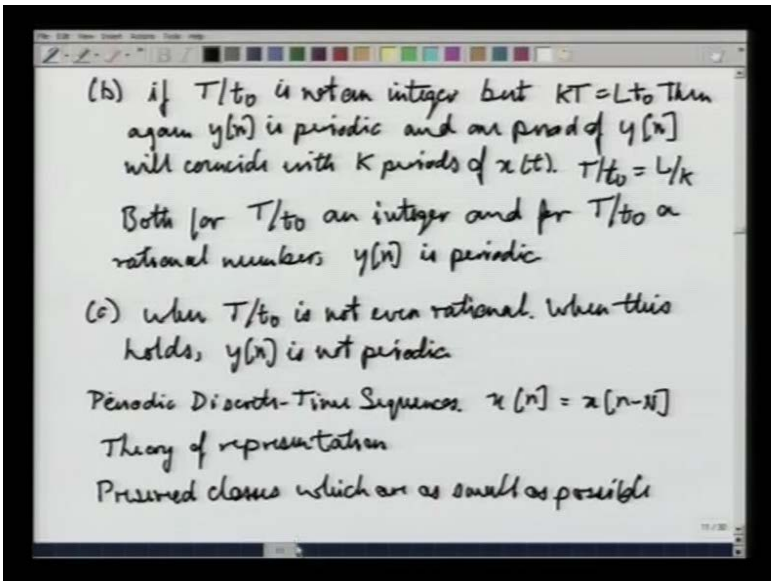

This is the first case now going to the second case b if T by T naught is not an integer, but k T equals L T naught than again y n is periodic and one period of y n will coincide with k periods of x T this is when we have T by T naught equals L by k where L and k are integers. So, both for T by T naught an integer and for T by T naught a rational number y n is periodic the third case is when T by T naught is not even rational when this happens even though x T is periodic y n will never be periodic y n is not periodic. So, this tells us that when you construct samples of when you elicit samples of a periodic continuous time function certain criteria have to be met for this sequence of samples to also be periodic and if the samples the sequence of samples has to have a period which coincides with the period of the original continuous time signal certain additional criteria namely that T by T naught must be an integer must be satisfied otherwise it does not happen right. So, with this much in hand we will completely disassociate ourselves from continuous time signals and just deal blindly with discrete time sequences which are periodic.

So, from now on we will only consider periodic discrete time sequences periodic discrete time sequences which satisfy x n equals x n minus n where capital n is therefore, called the period of the sequence we want a theory of representation for periodic discrete time sequences a theory of representation means we have to look for preserved classes which are as small as possible preserved classes which are as small as possible we could generate a set of preserved classes on the basis of finding harmonically related periodic signals that are harmonically related to x n.

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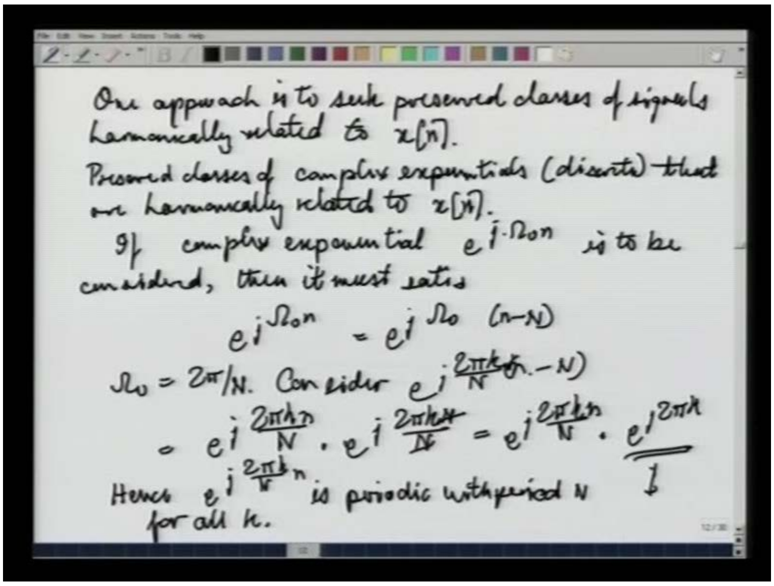

So, one approach is to seek preserved classes of signals harmonically related to x n this would give us signals which are not necessarily only complex exponentials, but because of this fact they will not have the special property in relation to linear time invariant systems that complex exponentials have these classes would also be larger in size because they would contain classes of periodic complex exponentials which are also harmonically related to x n. So, for reasons very similar to those that we discussed in the context of finding periodic preserved classes of complex exponentials to represent periodic continuous time signals in the context of L T I processing linear time invariant processing we will proceed to also look at preserved classes only of complex exponentials periodic complex exponentials for the representation of periodic discrete time sequences the reason of course, is that a complex exponential periodic or otherwise preserves its form under linear time invariant processing just as it did in the continuous time case it does.

So, in the discrete time case as well. So, we want to look at preserved classes of complex exponentials discrete of course, that are harmonically related to x n x n has a period n. So, if any complex exponential has to be harmonically related to it then it must satisfy the property that it also should repeat after every n members of the sequence that is to say if now we have a discrete complex exponential of the form e to the j 2 pi by n into n two pi by n into k n is to be considered for being harmonically related to x n then it must satisfy e to the j 2 pi by n k n must be equal to e to the j 2 pi by n k n minus n I have already let the cat out of the bag by saying that the fundamental frequency must be two pi by n if I had not told you this and if I had simply written these equations as e to the j omega naught n. And here also if I had written just e to the j omega naught n must be equal to the this then we would really have this job on our hands of finding out what choice of omega naught would yield this property of being periodic with respect to capital n and it does turn out that omega naught equal to 2 pi by n does yield this feature in order to verify this we will see the following consider e to the j 2 pi k by n times n I have not specified what k is I will leave it just as an integer for the time being now let us see what happens let us put n minus n over here.

So, we will consider this minus n we will consider actually this expression this is equal to e to the j 2 pi k n by n times e to the j 2 pi k n by n which is just equal to e to the j 2 pi k n by n plus sorry not plus times times e to the j 2 pi k which is always equal to 1. Hence e to the j 2 pi k by n times n is periodic with period n n for all k for any integer k since we said for any integer k it would seem that since k can take all values from minus infinity to infinity we have an infinite number of different periodic discrete complex exponentials which make the grade which are periodic with period n.

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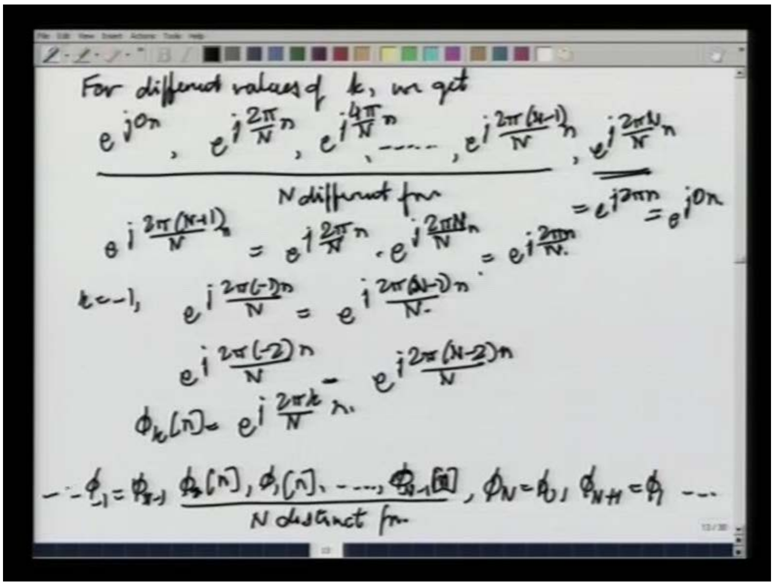

Now, while we have an infinite number of them in one manner of speaking they all of them do not turn out to be different from each other and this is what we will. Now, show by manipulating the variable k and see what happens if I choose k equal to 0 k equal to 1 and. So, on. So, for different values of k for different values of sorry small k we get e to the j 2 pi into 0 by n just gives you 0 e to the j 0 n this is the first sequence next you have e to the j 2 pi by n into k equal to 1 that is e to j 2 pi by n into n next you will get e to the j k equals two. So, you get four pi by n into n and going on us thus you will get e to the j 2 pi n minus 1 by n into n.

So, how many have we got over here we started with k equal to 0 then took k equal to 1 and so on up to k equal to n minus one. So, there are n minus 1 plus of course, the zero. So, we have totally n different functions now suppose we proceed further and write the next function in the sequence now k is not n minus 1 it has reached n minus 1 its going to become n. So, you will get e to the j 2 pi n by n times n, but this is simply equal to e to the j 2 pi n which is equal to e to the j e to the j 0 n because e to the j 2 pi n is the same as e to the j zero n.

That means, after the first n different functions that we got as listed over here the next member is the same as the first member there is no difference between the n th nplus 1 th member and the first member thus we find that we have only n different members you can go further and consider the n plus 1 th member n plus second th member and that will turn out to be the same as the second member for example, if you consider e to the j 2 pi n plus 1 by n into n this is nothing, but e to the j 2 pi by n into n times e to the j 2 pi n by n into n which is just equal to e to the j 2 pi n by n.

So, all these start repeating as you move further to the right in short we have found only n different discrete complex exponentials with period capital n though k can be given any value you like and you will end up with a complex exponential it is just a repetition of an earlier found complex exponential now let us move towards the left and see what happens we have been going only towards increasing k the first value we had was k equal to 0 let us take k equal to minus 1 what is this is equal to its equal to e to the j minus 2 pi just a second minus 2 pi n by n which is equal to 1 if you go further you will get e to the j minus 4 pi n by n which is sorry this is not equal to n equal to minus 2 pi which is nothing, but e to the j 2 pi n minus 1 n by n, if you go further to the left and examine e to the j 2 pi minus 2 n by n this turns out to be equal to e to the j 2 pi n minus

2 by n n minus 2 n by n, so even towards the left the sequences the the sequence of different discrete periodic complex exponentials repeats now suppose we denote by phi k of n to make our life a little less this expression e to the j 2 pi k by n into n suppose we write this than we have now found that we have phi 0 n phi 1 n phi n minus 1 n as n distinct periodic complex exponentials with period n.

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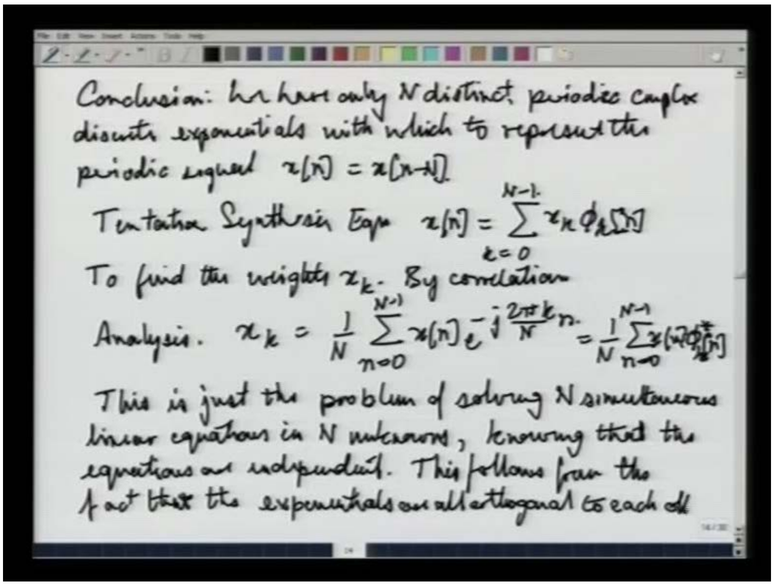

So, n distinct ones after this we find that phi n equals phi 0 phi n plus 1 equals phi 1 and so on towards the right and towards the left we find phi minus 1 equals phi n minus 1 and so on towards the left. So, in neither direction do you find any new periodic complex exponential you therefore, find for period n exactly n distinct complex exponentials only conclusion we have only n distinct periodic complex discrete exponentials no more no less with which to represent the periodic signal x n equals x n minus n this is all we have no more than this ; that means, that when we write a synthesis equation similar to what we wrote for x T we can be very compact about it it will be a sum only of a finite number of terms.

So, we will write the synthesis equation the tentative synthesis equation x n equals summation x k which are the weights with which we should add the complex exponential components phi k n sorry phi k n as k goes from 0 to n minus 1 this seems. So, much more comfortable than dealing with summing an infinite sequence of components we just have a finite number of components that have to be added to present x n that is all we have. So, with this finite set of components there are some nice things which happen?

There are times when we do not know if the transform representation or the signal representation we have for x n will actually converge to x n here there is no such issue because we are just adding a finite number of terms and the sum will also therefore, be finite because each term is finite. So, this is the synthesis equation the synthesis equation has dealt in terms of these phi k's of n each phi k of n being e to the j 2 pi k by n into n now how do we find the weights x k to find the weights x k we hardly need to repeat that we do it exactly in the same way as we handled the problem in the continuous time fourier series case we do it by correlating x n with the corresponding complex exponential. So, x k turns out to be equal to 1 by n times summation this again is a finite summation n equals 0 to n minus 1 x n e to the minus j 2 pi k by n into n.

Fine, now this gives the analysis we have a minus sign in the exponent of the complex exponential here simply because we have to take the conjugate of the function when one is carrying out a correlation. Now taking the synthesis and the analysis equations side by side as we have found and rewriting them in this form the synthesis equation can be rewritten as for example, one by n summation n equals 0 to n minus 1 x n phi star k of n this would be the analysis equation since both the analysis and the synthesis consist of finite sums and since they both carry out sums over the same set of basis terms phi k of n we can. Now write this in the form of a matrix operation the whole business of finding the x k's unlike in the case of the continuous time analysis turns out to be simply a matter of solving n simultaneous linear equations in n unknowns n simultaneous linear equations in n unknowns fine with the further assurance knowing that the equations are independent why they are independent I will come to in a moment it is because these exponentials that we have are all orthogonal to 1 another they are all orthogonal to 1 another which is very easy to demonstrate you just take say for example, a summation over one period n equal to 0 to n minus 1 of phi k n and phi star L n.

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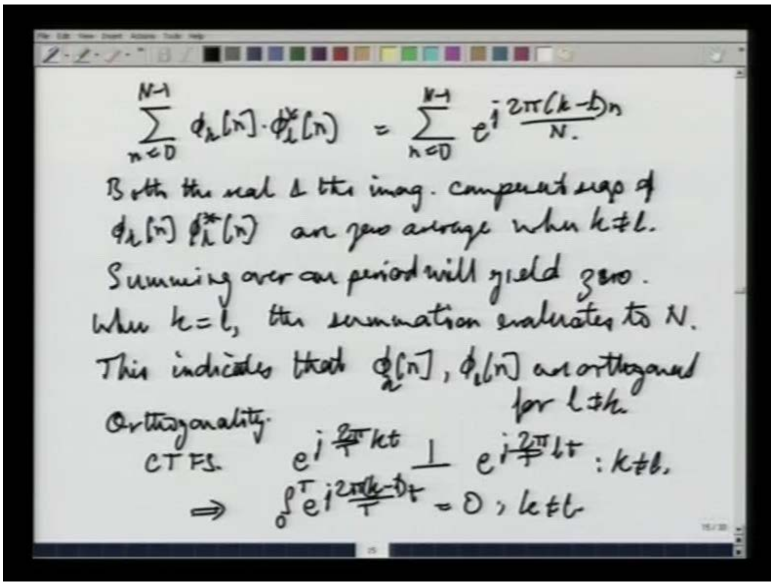

Where k is not equal to L this will be equal to n equal to 0 to n minus 1 of e to the j 2 pi k minus L n by n now this is a 0 average signal both its real and imaginary parts are 0 are 0 average real and the imaginary component sequences of phi k n phi star L n both the real and imaginary parts are 0 average of course, when k and L are different and hence summing over one period will yield 0 summing over one period will yield 0 this; however, will not happen when k equals L k equals L you will get the summation evaluates to capital L that is what happens this is why we say or will I say this is this indicates that phi k n and phi L n are orthogonal for L not equal to k orthogonality it turns out is a repeated thing in our study of various kinds of representations for the fourier series we have already shown that the Fourier the complex exponentials e to the j omega naught T and e to the j 2 omega naught T. For example, are in general e to the j k omega naught T and e to the j L omega naught T were always orthogonal.

So, let us just have a small about the orthogonality of these representing functions in the continuous time Fourier series we have e to the j two pi by T k T is orthogonal to e to the j 2 pi by T L T for k not equal to L orthogonal is orthogonal to; that means, that over one period 0 to T e to the j 2 pi k minus L by T of T equals 0 for k not equal to L this is what we have this is the proof of orthogonality for the c T f s for the c T f T.

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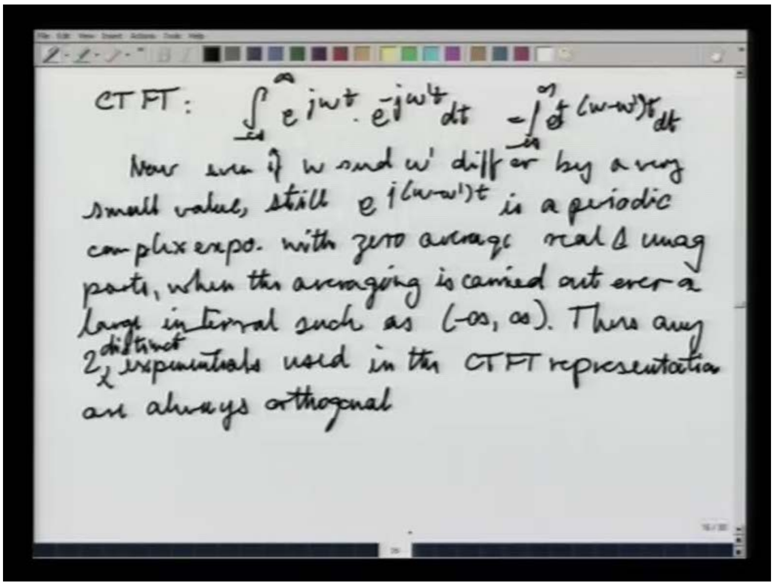

We have similarly again that the integral the inner product this time evaluated again over one period, but now the period is infinity from T equals minus infinity to infinity still yields 0 that is if you have an integral from minus infinity to infinity of e to the j omega T multiplied by e to the minus j omega dash T d T which is the same as saying integral minus infinity to infinity e to the minus. Sorry e to the j omega minus omega dash T ready to go now even if omega and omega dash differ by a very small value differ by an extremely small value still e to the j omega minus omega dash T is a periodic complex exponential with 0 average real and imaginary parts when the averaging is carried out over a large interval such as minus infinity to infinity thus any two exponentials used any two distinct exponentials used in the CTFT representation are always orthogonal.

So, that is true as it turns out as much for the CTFT as it was for the continuous time fourier series and again we see a manifestation of the same thing when we dealt with the discrete time fourier series we have only n different complex exponentials there which have all the period of capital n and each of them is orthogonal to every of the every one of the other members of the set and it is, because these different exponentials are orthogonal with respect to each other that we get a set of independent linear equations and because they are independent what we get is a matrix equation to solve where the matrix has full rank we are now going to write the equation in matrix form the…

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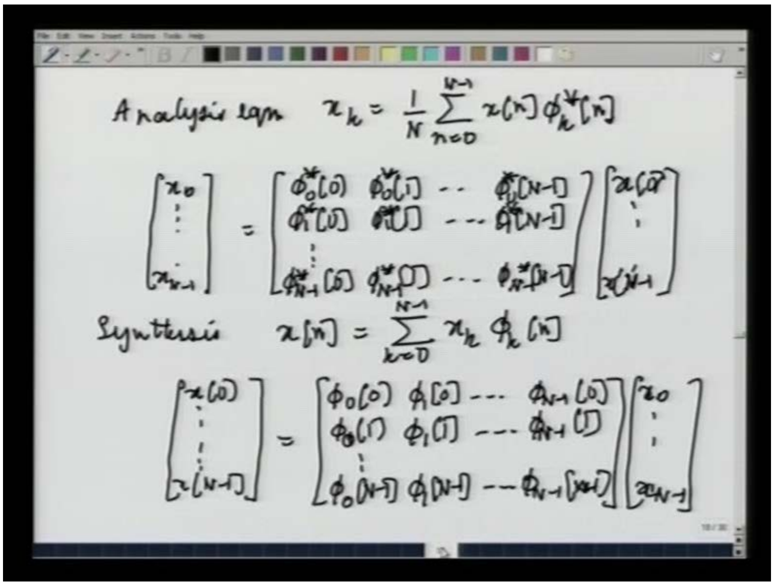

So, now let us make a matrix of all the equations that go into determining the analysis. So, for the analysis we have the analysis equation x k equals one by n summation n equals 0 to n minus 1 x n phi k n star this is what we have this written out in plain language will give you a matrix of this form a matrix equation of this form you have x k which is x 0 going on to x n minus 1 equals the matrix phi 0 zero phi 0 on to phi 0 n minus 1 next row is phi 1 0 phi one one going on to phi 1 n minus 1 and finally, phi n minus 1 0 phi n minus 1 one phi n minus 1 n minus 1 all of them of course, conjugated something I forgot.

So, you can put star star star star this thing times x 0 to x n minus 1 this is the analysis equation now for synthesis, we have the equation x n equals summation k equals 0 to n minus 1 x k e to the I will just write in terms of our abbreviated notation I will write x k phi k n. So, now, we do a summation over k's instead of n's and thus we have the synthesis equation in matrix form as x 0 to x n minus 1 equals phi 0 0 phi1 0, so on to phi n minus 1 0 phi 1 0 phi sorry phi 0 phi 0 of 1 phi 0 of 1 phi 0 of one was the first term in the second row then phi 1 of one going onto phi n minus 1 of one and so on until you finally, have phi 0 of n minus 1 phi 1 of n minus 1 going onto phi n minus 1 of n minus 1 times x 0 to x n minus 1.

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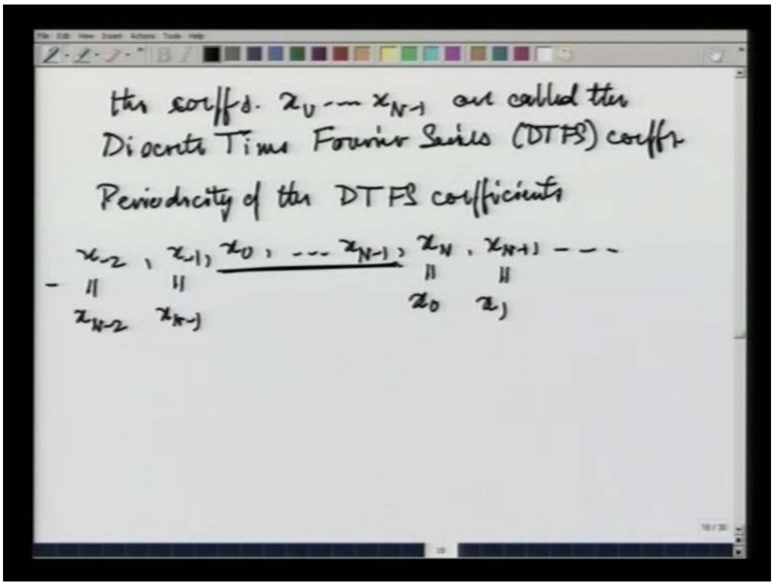

The coefficients x 0 x n minus 1 are called the discrete time Fourier series coefficients coefficients. Now it is important to ask whether we have only eight only whether we have only n different discrete time Fourier series coefficients or there are more the answer is that there are only n because these Fourier series coefficients are found as correlations of the corresponding complex exponential discrete complex exponential with the original function since there are only n different discrete complex exponentials periodic complex exponentials which can be correlated with x n we will get only n different discrete d T f s coefficients.

So, we have the d T f s coefficients satisfying a periodicity property you have x 0 going on to x n minus 1 then if you write x n it turns out to be equal to 0 x n plus 1 will be equal to x 1 and so on this side and if you go towards the left and write x minus 1 that will turn out to be equal to x n minus 1 if you write x minus 2 that will turn out to be equal to x n minus 2 and so on towards the left side. So, in short you only have these many really distinct coefficients no more than these and the periodicity that you observe is a very, very important property of the discrete time Fourier series coefficients you do not have an unlimited number of coefficients like was the case with the continuous time Fourier series, but only a finite n number of different coefficients.