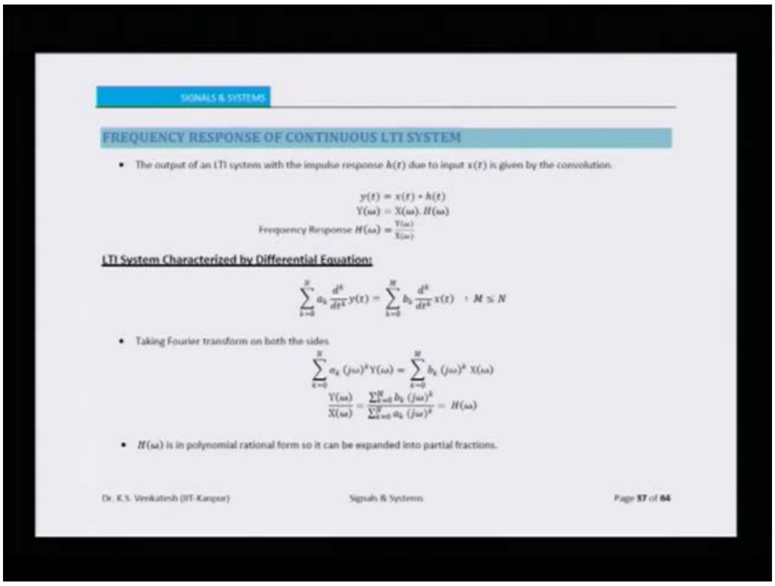
Signals and System
Prof. K. S. Venkatesh
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

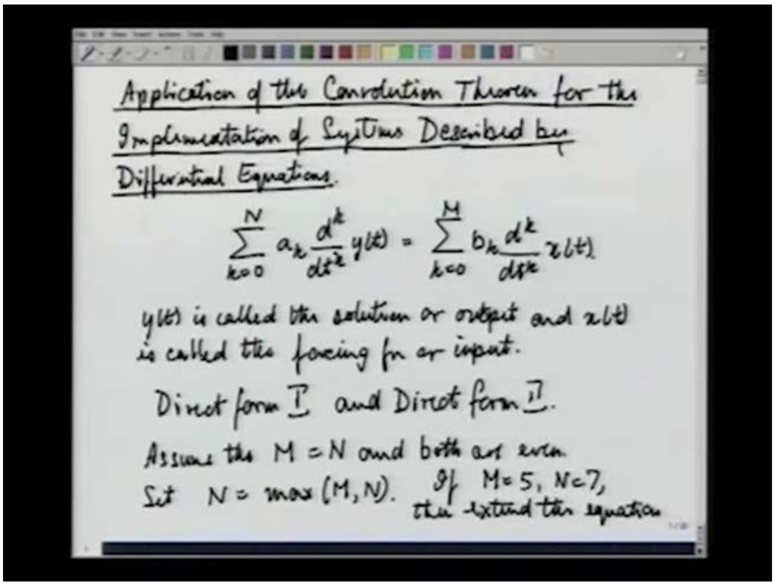
Lecture - 31
Frequency Response of Continuous System

(Refer Slide Time: 00:16)



We now consider the use of the convolution theorem for the solution of differential equations for implementation of systems that are described by differential equations. About the implementation of systems described by differential equations, we have already had some exposure. If you recall, we are only concerned with linear constant coefficient, ordinary differential equations, which were given the general form.

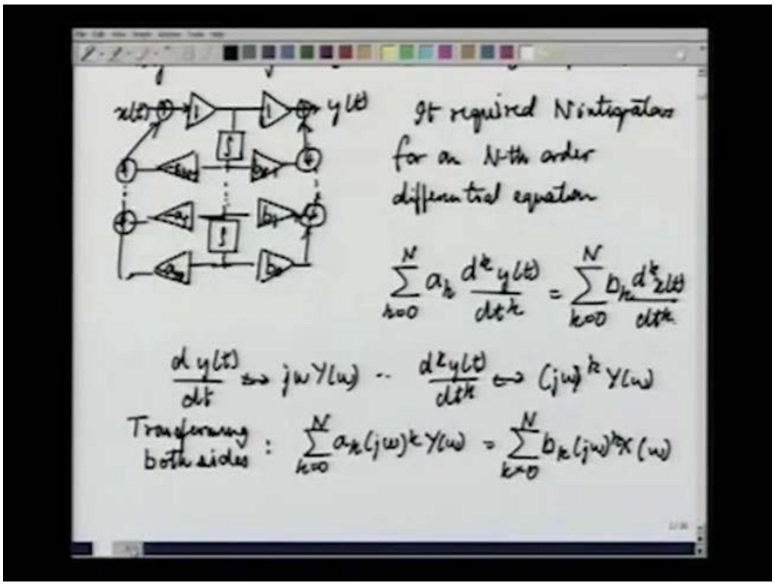
(Refer Slide Time: 00:54)



Just to recall the terminology, let us remember that y t is called variously the solution or output and x t is called the forcing function or input. We had already studied systems described by such equations and we had even studied certain implementations. If you recall their names, they were called a direct form 1 and direct form two implementations. Before we proceed, we will make some harmless assumptions, some assumptions that can be made without any loss of generality. This was something we did even earlier and this is simply that we will assume that M equals N and both are even if these assumption is not actually valid in a particular example of a differential equation. We can just set N equal to the max of M and N and add a fewer 0's. For example, if M equals 5 N equals 7, then extend the equation.

By including a 8 equal to 0 that would make N even and b 6 equals b 7 equals b 8 equals to 0, then you would have eight terms of the higher derivatives of both x t on the right side and y t on the left side. So, you could write this equation as summation k equal 0 to 8 a k d k y t by d t to the k where a 8 is been designated as 0. Similarly, on the right side write such a similar expression where b 6 b 7 and b 8 are designated as 0. So, we will proceed with this assumption just because of the convenience it gives us. As we can see from what we have shown over here, it does not take anything away from generality of the formulation.

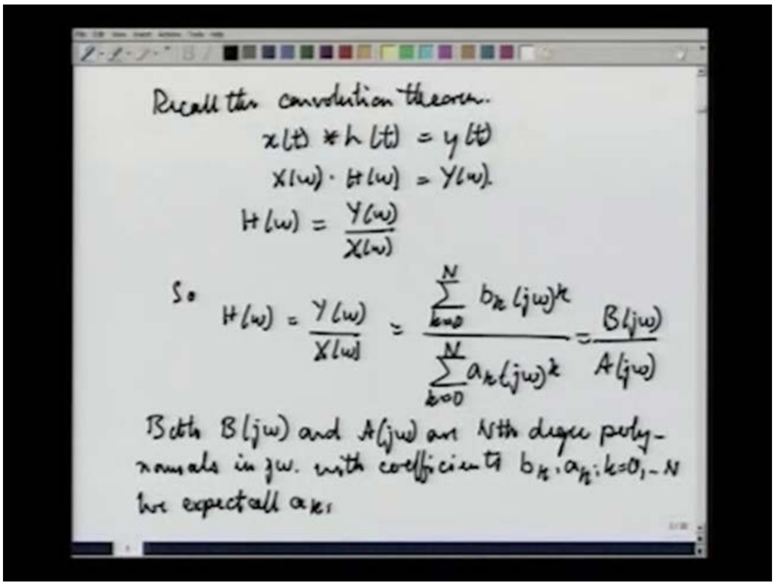
(Refer Slide Time: 04:20)



Now, if you recall direct form 1, then it was implemented using a set of integrators and the overall form looked like this. This was the direct form 2 implementation, it required N integrators for an N th order differential equation. So, this is the only way we could implement of course, apart from direct form 1 a system that was described by this differential equation. Now, with the strength of the Fourier transform and it is convolution theorem, the powerful convolution theorem, there are other ways in which one could implement a differential equation of this sort a system described by differential equation of this kind.

This is what we are now going to examine, let us return to the differential equation. Now, suppose we use the differentiation property, the differentiation property says that d y t by d t will transform to j omega Y omega and more generally d k y t by d t to the k th derivative of y t will transform to j omega to the k Y omega. So, using this property both for y as well as x we can write down the differential equation by transforming both sides by taking the Fourier transform of every term of both the signs this is what we have.

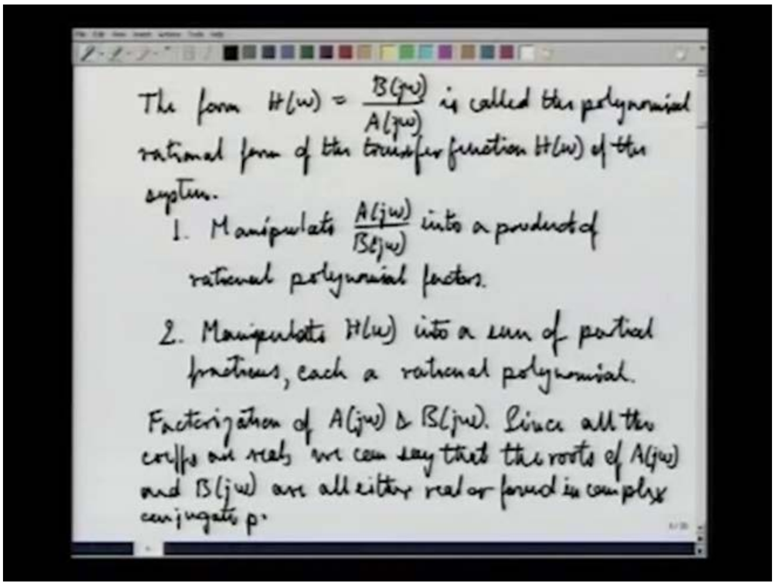
(Refer Slide Time: 11:44)



Now, let us go on recall the convolution theorem, it said that if you had an input x t and an output y t of linear time invariant system with impulse response h t. Then, the complete transformation of the following expression x t convolved with h t equal to y t was that X omega times H omega equals Y omega. This is the transformation of that equation, but from this it follows that H omega can be written as Y omega by X omega. Now, if we go back to the equation, we have just obtained by transforming both sides of the differential equation, we find that Y omega is a common factor in all the terms on the left side and X omega is a common factor in all the terms on the right side.

Hence, we could factorize this expression, divide Y omega by X omega to get an expression for H omega. Now, both are N th degree polynomials, the numerator and the denominator, I will call this B of j omega over a of j omega. Then, both B j omega and A j omega are N th degree polynomials in j omega with coefficients given by b k's and a k's respectively with these coefficients. Now, for a real-world system, a system that actually exist in this world and describes some kind of a practical problem, we expect all the a k's and b k's to be real.

(Refer Slide Time: 15:45)

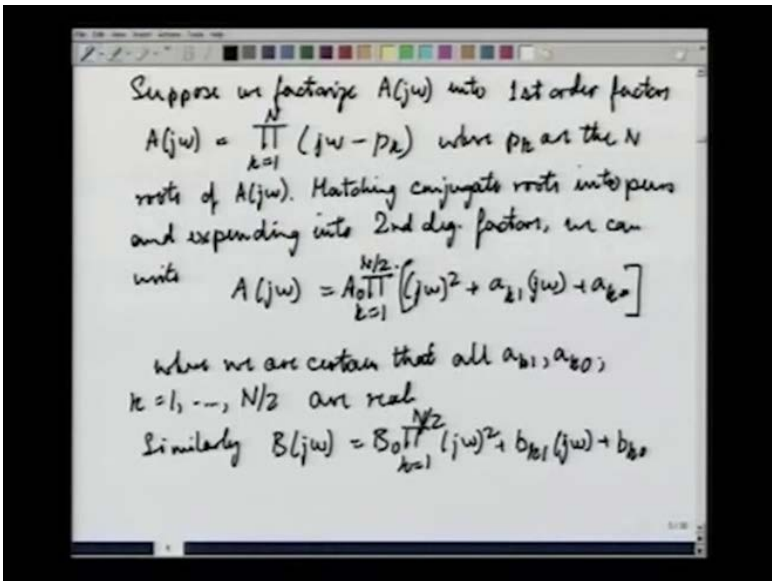


Now, let us see, this is called the polynomial rational form for the system of the transfer function H omega of the system. Now, where is all this leading to? We will know in a very short while where it is leading us to is this, you see, earlier as I said, we had only direct form 1 and direct form 2 ways of implementing the system, but now other possibilities open up. These possibilities open up by manipulating the polynomial rational form in two different but very, very interesting directions.

The first approach is to manipulate A j omega by B j omega into a product of factors, rational polynomial factors. The second approach is to manipulate H omega into a sum of partial fractions, each a rational polynomial is in itself a rational polynomial. So, this is the second thing that we can do, so each of these is going to give us an interesting result. In order to be able to do either of this the first thing, we have to figure out is how to factorize A omega A j omega and B j omega. So, factorization of A j omega and B j omega since all the a k's and b k's are real this much can be said.

We can say that the roots of A j omega and B j omega are all either real or found in complex conjugate pairs. They are all either real or found in complex conjugate pairs, so what we will try to do is to factorize A j omega to begin with into second-order factors. If we factorize them into first order factors, we may or may not get real coefficients in those factors, but let us just in any case do it to understand what happens suppose we factorize into first order factors.

(Refer Slide Time: 21:06)

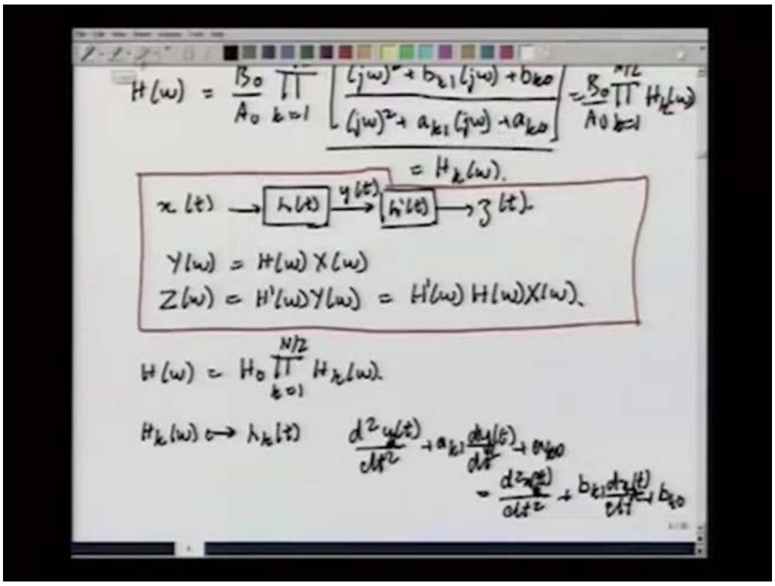


So, we will write A j omega equals pi k equals 1 to N j omega minus p k, where p k are the roots, the n roots of A j omega. As long as the p k's are real, there is no problem, if it happens that they are not real, then we will pair them up, so that the complex ones are matched along are matched against their conjugates. If we match each complex root with its conjugate and develop it into a second order factor by expansion. Then, finally we can say that we will be able to find an expression for A j omega in the following form matching gate, making matching conjugate routes into pairs and expanding into second order factor, second degree.

We can write A j omega equals pi k equals 1to N by 2, remember that we have taken N to be even. Now, we know why we have taken into be even, so that we get a nice complete set of second order factors, second degree factors. Then we will write j omega square plus a k 1 j omega plus a k 0, this will be A j omega will be equal to a product of second order factors of this kind, where we are certain that all a k 1 a k 0 k equal to 0, k equal to 1 to N by 2 are real. When you match two factors, two first order factors with matched conjugate routes and expanded since the roots are conjugate, all these coefficients a k 1 and a k 0 will come out to be real.

In addition to this, there could be some constant outside which I will call say is 0 that of course, will be the scale factor that lies outside this entire expression. Similarly, we could do the same exercise with the b polynomial and following the same arguments make the same claim about the second-order factors of the B polynomial. So, we will write similarly, B j omega can be written as some b naught capital b naught times the product k equal to 1 to N by 2 of j omega whole squared plus b k 0, sorry b k 1 j omega plus b k 0. We could write this then; taking this whole thing together, we can now express our H omega has a product of N by 2 different second order rational polynomial factors as follows.

(Refer Slide Time: 26:49)



I will it write as B 0 by A 0 product k equals 1 to N by 2 of H k omega. So, it is a product of each of these factors and a factor like this is what I call H k omega. Now, let us understand one implication of the convolution theorem. The convolution theorem said that if you apply X omega or x t to a system with an impulse response h t and got output y t, then Y omega was X omega times H omega. From this, it followed that if you applied Y omega further to another system say h impulse response h dash of t.

Therefore, a Fourier transform of H dash of omega to get say z of t then z of t would have a transform Z omega, which was related to X omega and Y omega as follows. So, let us consider what is called a cascading arrangement of multiple systems, x t goes to a system with impulse response h t to yield y t. This again goes to a second system h dash of t which goes, which yields h t. Let us have this, then we have Y omega equals H omega X omega Z omega equals H dash omega Y omega equals H dash omega H omega X omega.

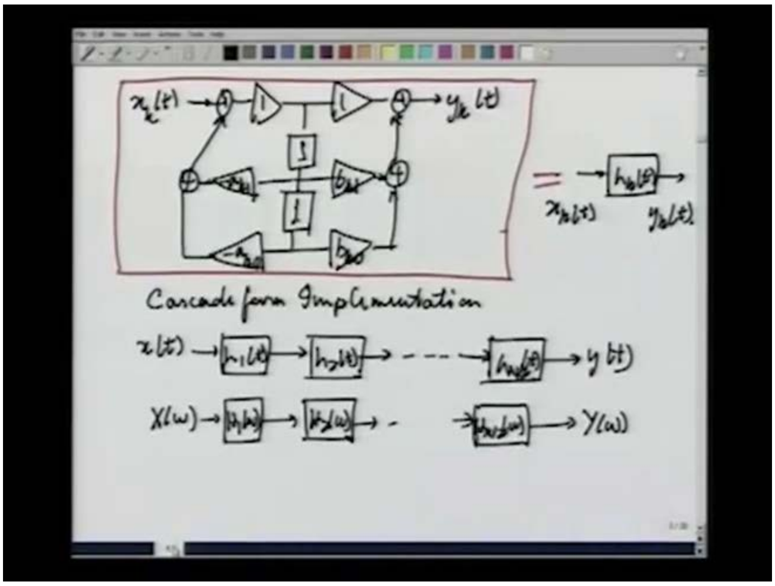
So, this is what we have, thus it follows that means if you have a cascading arrangements of several systems, then you simply multiply the transfer functions of the respective systems in the cascade. So, this follows from this the fact that convolution and multiplication are commutative. It means that you need not even arrange them in any particular order, I could write H dash omega times H omega times X omega equally as H omega times H dash omega times X omega or X omega times H dash omega times H omega in any way I like.

The order does not really matter; the order in which we arrange these systems does not really matter as far as the theory is concerned. So, that means if we now look at H omega of our earlier system, let me just put a box on all these. This was a digression not so much a digression as a clarification of some of the implications of the convolution theorem. We will now return to the original discussion and see the following here, we have H omega equals some B 0 by A 0. So, I will just call, say H 0 which is just a multiplicative constant time the product k equals 1 to N by 2 of H k of omega.

Each H k of omega is the typical second-order system with both moving average and autoregressive components. How would you implement H k of omega? H k of omega has an impulse response h of t, h k of t and is equivalent to a system described by a differential equation. What is this differential equation that will describe this h k of omega? The differential equation is simple, it has as you can see terms j omega squared b k 1 j omega plus b k 0 and on the other side has j omega squared a k 1 j omega and a k zero. So, this would correspond to the differential equation d squared y t by d t squared plus a k 1 d y by d t plus a k 0 equals d squared x t by d t square plus b k 1 d x t by d t plus b k 0.

If you are not convinced of this, you can again take the Fourier transform of the left side and the right side here. Of course, y t and x t are not the same thing as before, in fact maybe we should call them as y k y k x k and x k to say that there are the input x k of t and the output y k of t of the k th system in the cascade alright. So, you can just take the Fourier transform on and do all the same things that we did a few minutes ago and you will find that h k omega given by this expression over here given by this expression over here will indeed be given by this differential equation, this difference equation. Now, how do we implement this that differential equation in our conventional direct form two kind of format very, very simple, you can recall what we used to do it was simply that.

(Refer Slide Time: 34:37)

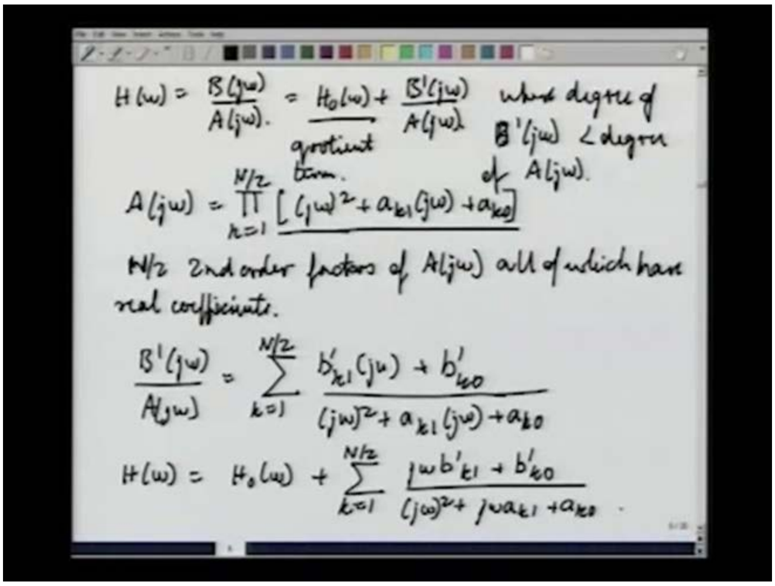


If I had x k of t going into the system, then there were a few summing blocks, scaling blocks and you got y k of t over here. Then, you had a pair of integrators just to and then here we had b k 0 here we had b k 1 and here we had this sum of all those on this side. We have minus a k 0 minus a k 1 and then these were getting summed up and added to this. This is how you would implement it, this whole thing is nothing but an implementation of h k of t with x k of t as the input and y k as the output.

Now, we have a cascade of such systems, we have a cascade of N by 2, such systems, because the Fourier transform of h omega is a product of all these plus there is this gain term that was there. Just go and recall what the gain term was, it was B 0 by A 0. So, let us write now a new implementation of a system described by the same differential equation and this implementation will be called the cascade form implementation, x t goes in to the first block which is h 1 of t. Next block h 2 of t and so on until you come to h n by 2 of t and you get y t in transform domain terms.

You would write x omega entering h 1 omega leading to h 2 omega and eventually leading you to h n by 2 omega. This gives you Y omega, so this is what you would have this is the cascade form implementation this is an entirely new way of implementing. The differential equation which you really could not have done, because we had no means to factorize these things, the whole thing was in one piece. Now, you are able to factorize them. Now, factorization was not the only approach, if we go back a couple of slides we will see that there was another approach is that is to manipulate H omega this one, to manipulate H omega into a sum of partial fractions each a rational polynomial in its own.

(Refer Slide Time: 39:16)



So, this is the next thing we will look at, so we already have h omega in the form of B j omega by A j omega. Now, depending upon the actual values of M and N, which are the orders of the equation of the derivatives on the right side and the left side respectively of the original differential equation; this polynomial rational form over here for H omega might or might not be a proper fraction. In the sense that the degree of the numerator might or might not be less than the degree of the denominator.

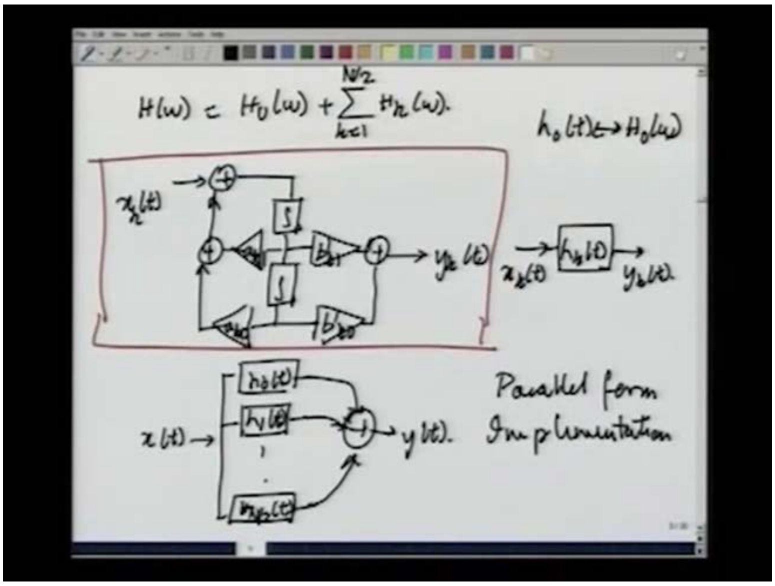
If we take the general case that it may not be proper fraction then we can always rewrite this as H 0 of omega plus we will write B dash j omega by A j omega, where now B dash, degree of b dash j omega is less than degree of A j omega. This is what I mean it is just done by long division, nothing very special about it. We have this term, which this x 0 of j omega we will call is the quotient term, it will just be a polynomial we have already broken up H omega into a sum of two parts. This is just the beginning; we are going to break up B dash j omega by A j omega into a series of second order terms into a sum of second order terms.

Now, A j omega is the same as we had earlier in the cascade form as well. Therefore, its factorization in this case would be really no difference from its factorization into second-order factors in the case of the cascade form implementation. So, we will again write A j omega as a product pi k equal to 1 to N by 2 product of the terms of the form j omega whole square plus a k 1 j omega plus a k 0. This is what we will have for the denominator, so essentially we are looking at a situation where there are N by 2 second-order factors of A j omega, all of which have real coefficients as far as b dash is concerned.

Since, it is degree is less than that of the numerator after carrying out a partial fractions expansion. What you will get is a sum of factor of some of terms where each denominator is one of these factors of A j omega and whose numerator is a first order term that is a factor of B j omega that is depended upon B dash j omega. In short you will get H omega or rather we will just focus on B dash j omega by A j omega can be written as the sum of B dash k 1 j omega plus B dash k 0 divided by k equals 1 to N by 2.

Here, you have j omega squared plus a k 1 times j omega plus a k 0, this is what you will have and therefore, finally, you can write that H omega equals H 0 omega plus this sum. Now, I will rewrite this as H omega equals H 0 omega plus summation k equals 1 to N by 2 H k of omega. Of course, these h k's are not the same as h k's we got in the cascade form. They are quite different alright because these are essentially partial fraction terms where there we had second-order rational polynomial factors.

(Refer Slide Time: 46:21)



So, now we have this how do we implement this? Implementing any h k for k equal to going from 1 to N by 2 is very straightforward. It would have an implementation like this it would have an input x k of t that would go into the system B dash k 0 B dash k 1. This would be y k of t and on this side you would have a dash no minus a k 1 and here you would have minus a k 0 that would give you the implementation of the autoregressive part. So, you have this would be the implementation of one h k of omega, so this would give me one block. Now, how do I put the all the blocks together? In the earlier form we cascaded all of them, but here we have to add all of them.

So, we actually do the following: we apply x t to a kind of a distributor which passes it successively through H zero of t which is the Fourier transform. I am just pointing out that H zero of t is the inverse transform of H 0 of omega and then you will have H 1 of t, the inverse form of H 1 of omega and so on up to H N by 2 of t. All this would have to contribute to a common sum and the final result would be y of t in both the cascade form and the parallel form implementations.

We have succeeded in decomposing effectively the larger more complex N th order system into a combination either a cascade combination or is called a parallel combination . We have succeeded in decomposing larger and more complex system into component systems either in parallel form or in cascade form where none of the component systems has a degree or has an order, which is greater than 2. This completes our discussion of the study of continuous time signals and their representation through discrete time Fourier series.

For periodic functions, sorry, not discrete time continues time Fourier series for periodic continuous functions continuous time functions and continuous time Fourier transform for non periodic continuous time functions. We also extended continuous time Fourier transform to obtain a unified formulation that admitted of both finite energy and finite power signals, so that we really only need to carry the Fourier transform along with us in our minds. We have seen how the system, this theory of signal representation has allowed us to do many things in a new way it has given us physical interpretations of notions like frequency which are certainly easily palpably meaningful.

In the context of say the analysis of music or in the analysis of detail in an image, so since we cannot go deep into these into the study of these applications we will merely point out that one of the truly original things that has a reason out of our study of the Fourier transform. As a means of representing systems is signals and systems is these two new completely new ways of implementing systems described by differential equations. What we have seen is truly a new way of implementing the differential equation, the cascade form as well as the parallel form. These forms of implementation would not have been possible if we did not know how to decompose be higher order differential equation into a combination of lower order differential equations.

That could happen only because we used the theory of signal representation provided by the Fourier transform. So, that completes our discussion of the Fourier transform and the Fourier series for continuous time signals. The next thing we will touch upon is the study of discrete time signals and systems, which process them, and how we represent discrete time signals, how we find representations for discrete time signals.