Signals and Systems
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Lecture - 30
Properties of CTFT

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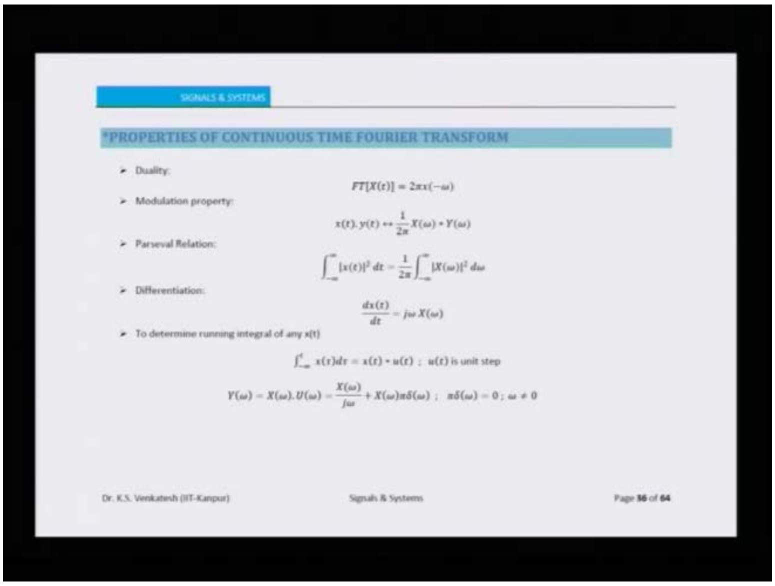

We now come to what is often referred to as a very aesthetically significant property of the Fourier transform is called the duality.

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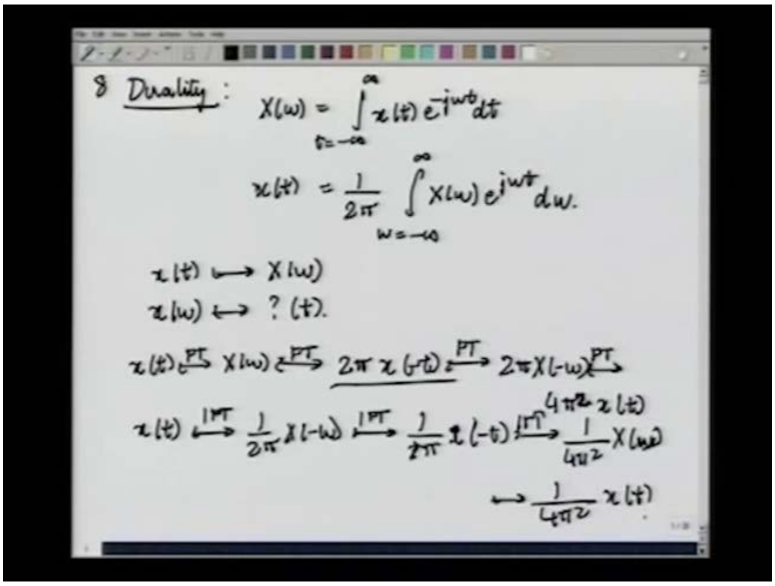

Duality refers to the fact that both the forward Fourier transform or the analysis and the inverse Fourier transform namely the synthesis or mathematically very similar though not identical operations. So, let us just write down these two expressions very familiar to us by now already, but let us just write them down X omega equals integral minus infinity to infinity here. Of course, the variable of integration is t, t equals minus infinity to infinity x t e to the minus j omega t d t whereas, we have x t equals 1 by 2 pi omega equals minus infinity to infinity X omega e to the j omega t d omega. In the first case the integral with respect to t in the second case the integral with respect to omega.

In the first case, the integral contains x t in the second case it contains X omega, in the first case there is e to the minus j omega t and here we have e to the plus j omega t, so there is a change in sign apart from all these. Of course, there is the multiplying factor outside 1 by 2 pi this all this is of course, a very minor difference between the way you calculate the inverse transform and the forward transform. Thus often if for a particular kind of function x t you have already taken the efforts of evaluating the integral to find X omega.

Then if in a different situation you encounter X omega which has the same form as the x t for which you already had the Fourier transform, then you could calculate the inverse transform of X omega from your knowledge of how you compute at the forward transform of x t? Let me put it another way suppose for some x t you have evaluated X omega. Now, suppose in the omega domain you had the same function small x that is to say you have X omega then the similarity between these two expressions will be able to give you the expression for the inverse transform, some function of t can be evaluated using your experience of this in order to substantiate this further.

Let us carry out a simple exercise let us for example, take x t find its Fourier transform X omega and then let us find the Fourier transform once again of X omega in order to not confuse variables. We will now be clear that we are trying to find the forward Fourier transform of x omega using the new frequency variable which we will call say X omega dash. So, then what happens if you carry out a forward Fourier transform of X omega once again it turns out that you get 2 pi x of minus t, if this is the Fourier transform of this and this is the Fourier transform.

Then, this is what you get in short we have come from x t back to x t back to x t by carrying out two forward Fourier transforms or two Fourier analysis. First step yielded as X omega decision and a further Fourier analysis of X omega has yielded 2 pi times x of minus t. This is interesting because we can continue this chain of events and take another Fourier transform and this will give us 2 pi times, now recalling the properties we have already observed x minus t being Fourier transform will give us X of minus omega this is the third in the sequence of Fourier transforms we have carried out on x t itself.

Now, let us carry out just one more last Fourier transform another F T and what you expect to get is 2 pi x minus omega here when we did it two steps previously we got 2 pi x omega gave us 2 pi x of minus t, now this 2 pi is already there. So, what you will get is 4 pi squared x of t that means the four sequential applications of the Fourier transform on x t and the results will yield 4 pi squared x t once again after a cycle length of four steps alternately you could easily see that if you carried out inverse Fourier transforms of x t.

At each step it would be just going backwards to this same sequence of results that we alreadyhave 4 pi squared x t subjected to four a sequence of four inverse Fourier transforms would yield you x t. In short if you started with x t with x t you normally apply and a Fourier transform because it is already a time domain function, but if you applied an inverse Fourier transform on x t and did this over and over again what you would finally end up with at the first step you would end up with one by 2 pi x of minus omega. If you carried out another F T you would end up with 1 by 2 pi x of minus t, one more time.

You would get 1 by 4 pi squared x of omega which subjected to one more inverse Fourier transform give you 1 by 4 pi squared x t. So, you started with x t and ended up after four steps of inverse Fourier transformation with 1 by 4 pi squared x t. All this essentially uses the duality of the Fourier transform, the similarity, high mathematical similarity between the forward and the inverse transforms and this comes in useful in a lot of situations.

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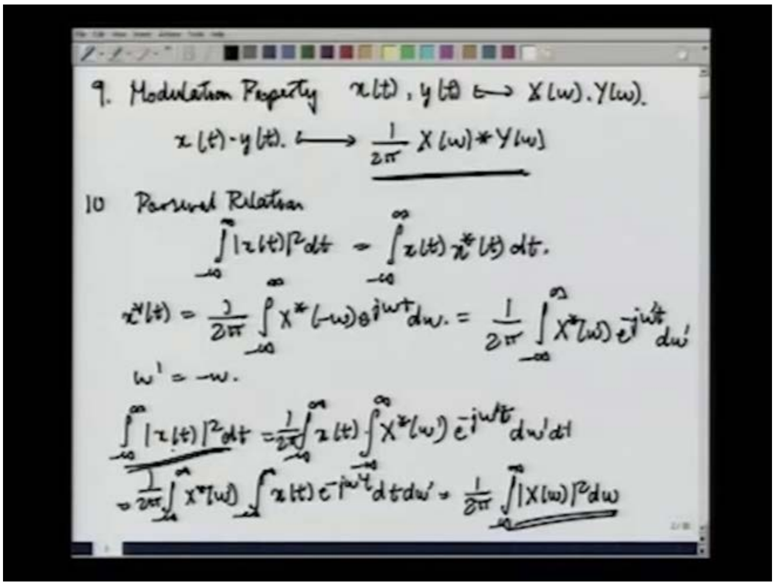

For example, now we are going to study a new property of the Fourier transform called the modulation property were very addressed the question of what is the Fourier transform of the product of two time functions. Suppose, we had x t and y t, x t y t with respective transforms X omega Y omega, then we ask the question what is the Fourier transform of x t times y t. Now, recall that we already know what happens if you have the convolution of two time functions in the time domain then if Fourier transform would has already turned out to be the product of their Fourier transforms.

Now, we have a product of two functions, but this time in the time domain and instead of carrying out a forward Fourier transform as we did I mean vocalized instead of having, instead of starting with the time domain as we did. We now have a product in the time domain instead of the frequency domain and we want to find the Fourier transform of this product by applying the duality property suitably you can easily see.

I will not try to spend time on evaluating it here that this come to 1 by 2 pi X omega convolved with Y omega alright more properties there is something called Parseval's relation. That establishes a relationship between the energy x t and the energy in X omega they turn out to be proportional to each other with the scale factor of 1 by 2 pi that relates them. Now, the energy in x t is given by x t mod squared d t integrated over all time this is the energy in x t. Now, let us find the Fourier, let us find what this comes out to be this is nothing but minus infinity to infinity x t x star t, because it was mod squared x t d t.

Now, x star t equals X star minus omega it does not equal it, it is the inverse transform of X star of minus omega. Let me just rewrite that part d omega, now replacing omega by minus omega dash this comes out to be equal to 1 by 2 pi integral minus infinity to infinity X star of omega e to the minus j omega t another X star omega dash e to the minus j omega dash t d omega dash this has happened by setting the omega dash equal to minus omega. Now, we will use this over here and write integral minus infinity to infinity mod x t squared d t equals integral minus infinity to infinity x t into X star omega dash e to the minus j omega dash t d omega dash d t.

Now, we can reverse the order of these integrals sorry there is a 1 by 2 pi; this is all 1 by 2 pi another integral minus infinity to infinity over here. What we have, we can exchange integrals and get integral minus infinity to infinity 1 by 2 pi of course, outside X star of omega times minus infinity to infinity x t e to the minus j omega dash t there is omega dash here also d t d omega dash which yields 1 by 2 pi integral. Now, just call them omega dash omega you can as well leave it as omega dash if you like you get minus infinity to infinity you have X star of omega over here and the second integral inside is nothing but X omega.

So, you get X star omega X omega which is nothing but mod X omega whole squared d omega. So, we have that this equals this which is what is called the Parseval’s theorem, now this is the energy in any function x t. If you have a function X omega the energy in that is given by integral X omega mod X omega squared integrated over all omega, it turns out that the energy in X omega is related to the energy in x t through this factor 1 by 2 pi that is essentially the statement of the Parseval’s theorem. Next, what we will now deal with is a slight generalization of the Parseval’s theorem where of course, the significance of energy no longer holds.

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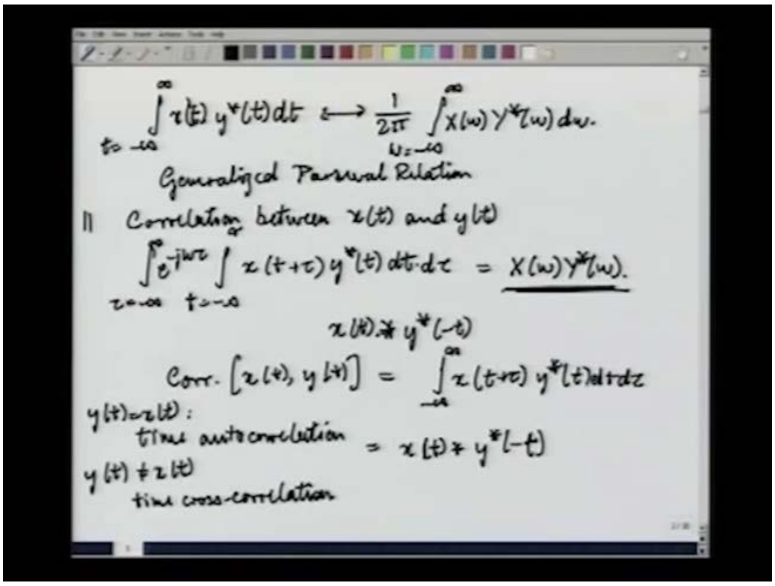

Suppose, you had integral x t y star t d t integrated over all time which is the analog the more generalized analog of x t x star t d t that we had. Then it is turns out that it is Fourier transform results in 1 by 2 pi integral minus infinity to infinity this time of course, this was with respect to time, this is with respect to omega X omega Y star omega d omega. This is just the generalization of the existing Parseval's relation, which reduces to the existing Parseval's relation if you replace y t by x t. So, this is just the generalized Parseval's relation; in the same way, you can go around evaluating the Fourier transforms of the interactions between various functions.

This is one example of an interaction between two functions x t and y star t. You want to find out the Fourier transform of x t y star t, earlier we found out the Fourier transform of the product x t y t even before that we found the interaction the convolutional interaction of two functions x t convolved with y t or x t convolved with h t. So, these are all interactions of various sorts one very commonly used interaction is the correlation between two functions correlation is called the autocorrelation. If you correlate the same function with itself its different shifts it is called graft correlation, if they are two different functions which are correlated with different shifts.

So, the correlation in general between two functions between say x t and y t is a plot of the inner product of x t and y t minus tau or x t plus tau and y t evaluated against tau that means the inner product is related like this. The correlation is given by integral t equals minus infinity to infinity x t plus tau y star t d t this might look somewhat similar to convolution, it is not because here we are integrating against t and what we therefore get is and also we are conjugating the second function. Now, if you evaluated this the correlations of these two functions and you wanted the Fourier transform of the correlation of these two functions that is to say you wanted over all tau that is e to the minus j omega tau, tau equals minus infinity to infinity.

This whole is a function of tau the correlation itself is a function of tau of the relative shift between x and y. Therefore, of this correlation function if we evaluated the Fourier transform by putting d tau over here what would you get it turns out to be equal to X of omega Y star of omega. Now, look at this as the Fourier transform of the correlation of two functions that is one way of doing it one way of looking at it. Suppose, you looked at this just as the result of a convolution remember if you convolved two functions you get a product of Fourier transforms. So, suppose I ask you now what are the two functions of time that ought to have been convolved to get this expression that is not hard to see this is the product of X omega.

Y star omega produce inverse transform of X omega the inverse transform of X omega is x t and the inverse transform of Y star omega what is it? Remember that y star t would give you Y star of minus omega, so Y star of omega will give you y star of minus t. So, it is the convolution of y star minus t with x of t that is equivalent to correlating x t and y t. So, the correlation is simply equal to x t convolved with y star of minus t. Now, if you look at different textbooks on signals and systems you will find that there are variations in the way in which correlation is defined.

For example, I have said that it is the integral of x t plus tau y star t some others would put it as x t y star t plus tau some people would put it as x star t y t minus tau and so on and so forth. There are various variations you get on this thing depending upon exactly the choice of the author, but the fundamental fact remains the same that they are all correlations and the measure the similarity between two signals and their different amounts of shifts. If y t is replaced by x t, then you have, what is called autocorrelation or the time autocorrelation. If y t is not equal to x t this is when y t equals x t when y t is not equal to x t you get what is called the time cross correlation.

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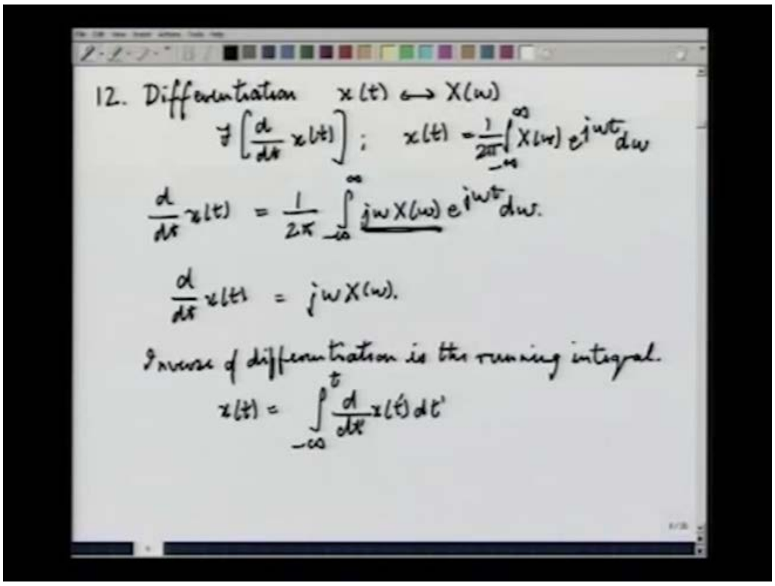

One of the final properties of the Fourier transform that we will study is about what happens to the Fourier transform of a signal that is differentiated, so differentiation. Suppose, x t has a transform of X omega, and we want to know what happens to the Fourier transform of something we denote by this d by d t of x t in order to find this out. We shall use the synthesis equation x t equals integral 1 by 2 pi of course, minus infinity to infinity X omega e to the j omega t d omega and differentiate this equation with respect to time. On the left side, you will get d by d t of x t and the right side; we will find has only one factor that depends upon t and e to the j omega t, when that is differentiated against time, we will get j omega times e to the j omega t, so you get in form this expression.

Simply, the Fourier transform of this quantity that is to say we can now claim that is just the inverse Fourier transform of this quantity and so we can say that d by d t of x t equals j omega x omega essentially. This means that when you differentiate a signal, you simply multiply its Fourier transform by j omega that means by imaginary quantity proportional to the frequency of signal to the respective frequency component of the signal. Now, having said it is differentiation, let us now look at integration the inverse of differentiation as we know is the running integral, so this is the running integral and we want to find the Fourier transform of the running integral of a signal, if we know the Fourier transform of the original signal. Now, this requires first that we prepare the ground by making a one of the final properties of the Fourier transform that we will study is about what happens to the Fourier transform of a signal that is differentiated.

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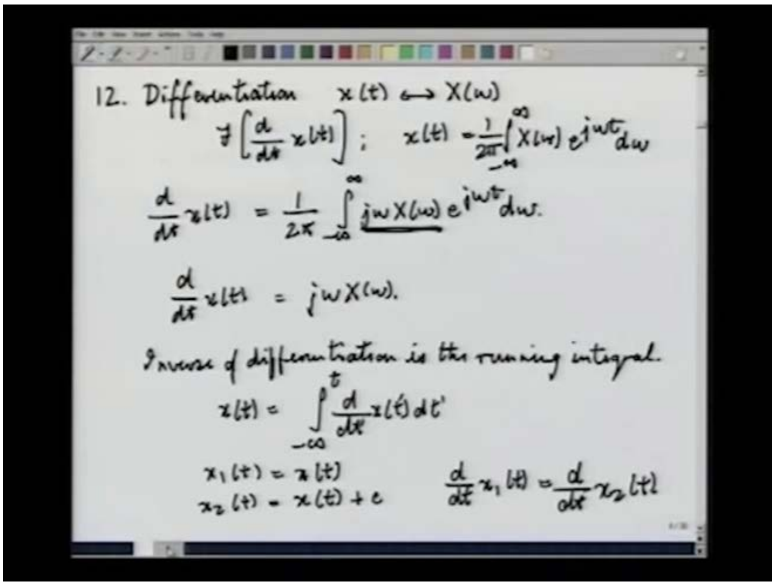

So, differentiation suppose, x t has a transform of X omega and we want to know what happens to the Fourier transform something we denote by this d by d t of x t in order to find this out. We shall use the synthesis equation x t equals integral 1 by 2 pi of course, minus infinity to infinity X omega e to the j omega t d omega and differentiate this equation with respect to time. On the left side you will get d by d t of x t and the right side we will find that has only one factor that depends upon t and that is e to the j omega t when that is differentiated against time we will get j omega times e to the j omega t.

So, you get in form this expression is simply the Fourier transform of this quantity that is to say we can now claim that is just the inverse Fourier transform of this quantity. So, we can say that d by d t of x t equals j omega X omega essentially this means that when you differentiate a signal you simply multiply its Fourier transform by j omega. That means by imaginary quantity proportional to the frequency of the signal to the respective frequency components of the signal. Now, having said it is differentiation, let us now look at integration the inverse of differentiation as we know is the running integral.

So, this is the running integral and we want to find the Fourier transform of the running integral of a signal. If we know the Fourier transform of the original signal, now this requires first that we prepare the ground by making the point is this when you differentiate a signal and get its derivatives. As we all know very well two different signals two quite different signals can have the same derivative though they are themselves different in order to illustrate this point. We will take a certain signal x t and then set x 1 t equal to x t set x 2 t equal to x t plus c where c is some constant.

Now, it is clear that d by d t of x 1 t equals d by d t or of x 2 t that actually x 1 t and x 2 t are not equal to each other. So, this is a fact that we must keep in mind when we try to integrate a function find the running integral of a function and find its Fourier transform. Now, let me point out as a next step in preparing the ground a certain property of the Fourier transform.

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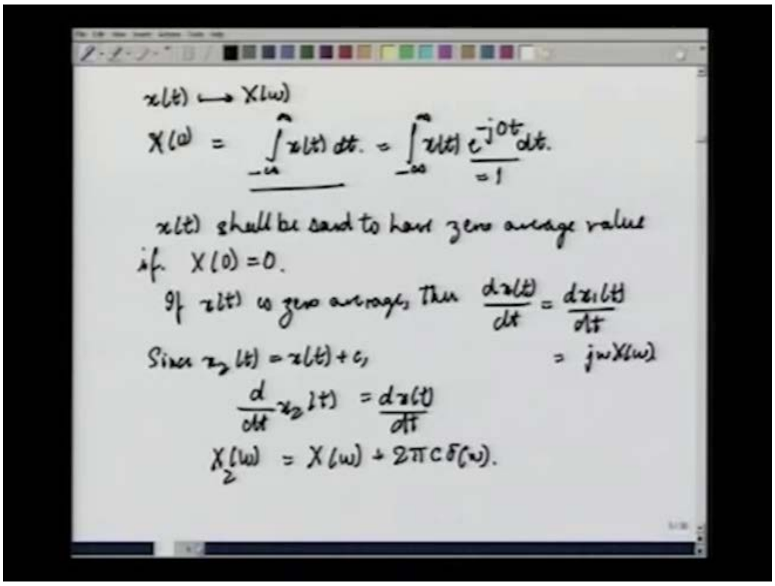

We have x t transforming to a very interesting and very nice convenient feature we have is that X 0 the value of the Fourier transform at omega equals to 0 is simply equal to integral minus infinity to infinity x t d t. That simply because if you wanted to be explained x t e to the minus j 0 t d t and this after all equals 1 and that is how you get this. So, we have this property, now what we are going to do is this given any signal it can be divided into two parts, the part which has zero average value and the rest the average value, so let me put it like this.

Suppose, you have x t I will say that x t has zero average value if X 0 equal 0, so on this basis we see that if you differentiate such a signal and comparing it this is just another way of saying that there is no d c component in the in x t. So, going back to our two newly defined signals x 1 t and x 2 t, it turns out that if x t is 0 average, then d x t by d t equals d x 1 t by d t equals j omega X omega since x 2 t equals x t plus c d x t by d t d x 2 t by d t sorry equals d x t by d t. These two are the same thing, but then x 2 t X 2 omega is equals X omega plus 2 pi c delta omega 2 pi c delta omega, now this is the only important thing that we need to keep in mind to go the next.

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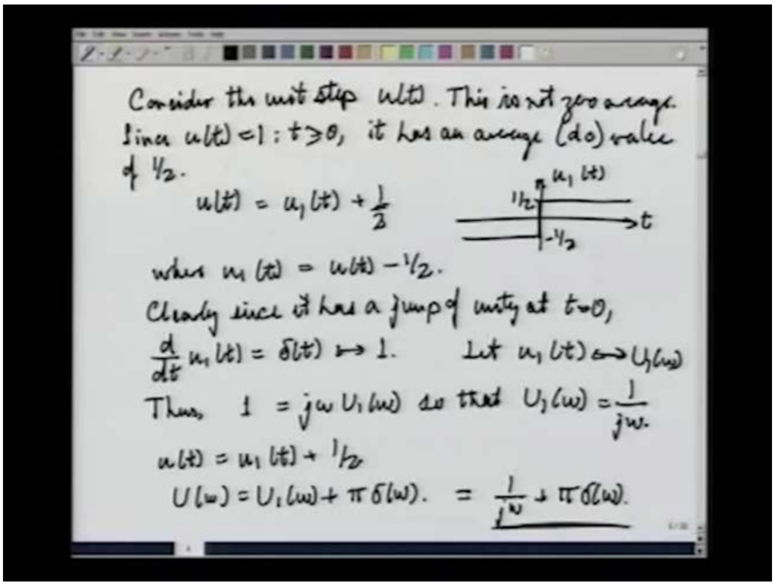

Now, consider the unit step u of t now u of t is not zero average since u of t equals 1 for t greater than equal to 0 it has an average value or d c value of half. So, I will expect I will express u of t as a sum of a zero average component and a component which is the d c part u 1 of t plus half, where is simply equal to u of t minus half. So, this is what I have for u 1 of t, so u 1 of t looks like this, this is u 1 of t, now clearly since it has a jump of unity at t equal to 0 d by d t of u 1 of t equals delta t which has a Fourier transform. As we know of unity data t is Fourier transform of 1, so what is U 1 of omega by which of course, I mean the Fourier transform of u 1 of t have a transform U 1 of omega.

Then U 1 of omega equals j omega times 1, sorry I am just going back by as second then one which is the Fourier transform of delta omega equals j 1 j of omega U 1 of omega, so that U 1 of omega equals to 1 by j omega, but then there is also the other component. So, we are just a little away from finding the Fourier transform of u of t, now u of t equals u 1 of t plus half. So, U of omega equals U 1 of omega plus the Fourier transform of which is 2 pi into half into delta omega which is just pi delta omega that is to say U of omega equals to 1 by j omega plus pi times delta omega. So, this is what we have, this is the Fourier transform of the nth step, now how does this help us to integrate functions or to find the running integral of a function very easy.

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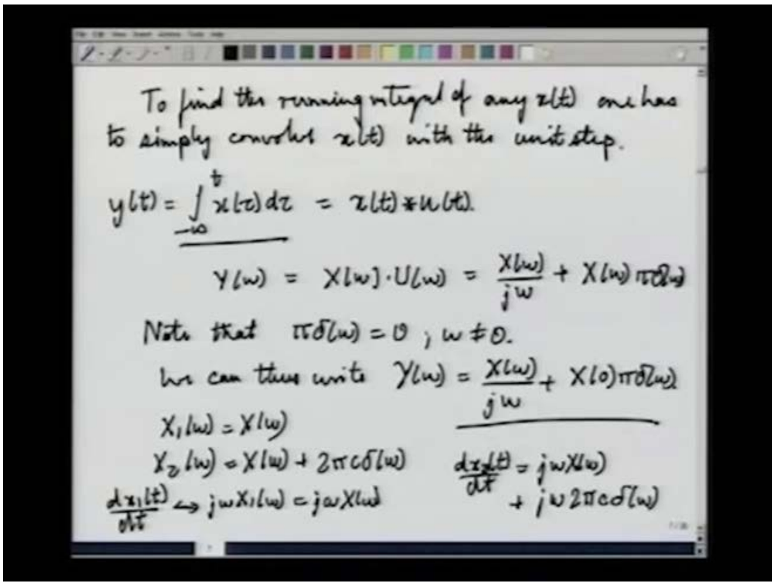

You will probably recall that to find the running integral of any x of t, one has to simply convolve x t with the unit step, that is to say that integral minus infinity to t x tau d tau for any x is nothing but x t convolved with u t. Where u is the unit step given this fact and given our favorite convolution theorem. We can immediately say that the Fourier transform of this side just be the product of the respective Fourier transforms here, so this is equal to just a second let me call this as say y t. Then you have Y of omega equals X omega multiplied by U of omega, which is equal to X omega by j omega plus pi X omega pi delta omega.

This is what you get, but note that pi delta omega is always 0 except at omega equal to zero, so we can simplify this all. Now, one final question does this all tie up properly, we had x 1 t equal to x t where x t was taken to be a zero average signal. Therefore, X 0 was 0 and then we had x 2 t which had an additional d c component of c, now will it turn out under the analysis. We have done that both of them when differentiated since both of them will differentiate to the same function will the Fourier transforms be the same what is X 1 omega. X 1 omega equals X omega X 2 omega will contain X omega, but it also contains a d c component.

Now, if I differentiate these two functions I really ought to get the same answer, if I differentiate the two-time functions and using the results we have obtained find the Fourier transforms of the derivatives, do I get the same answer? Let us see for the first case, I will get d x 1 t by d t transforms to j omega X 1 omega equals j omega X omega. For the second case I will get d x 2 t by d t j omega X omega plus there is a second term j omega 2 pi c delta omega, but then both these should have been the same.

Are they the same? What I mean to say is j the derivative the Fourier transform of the derivative of x 2 t must be equal to the Fourier transform of the derivative of x 1 t. But, then here we just got j omega X omega whereas, here we got j omega x omega plus this function, this additional term over here, so we really have to check if j omega 2 pi c tau omega is 0.

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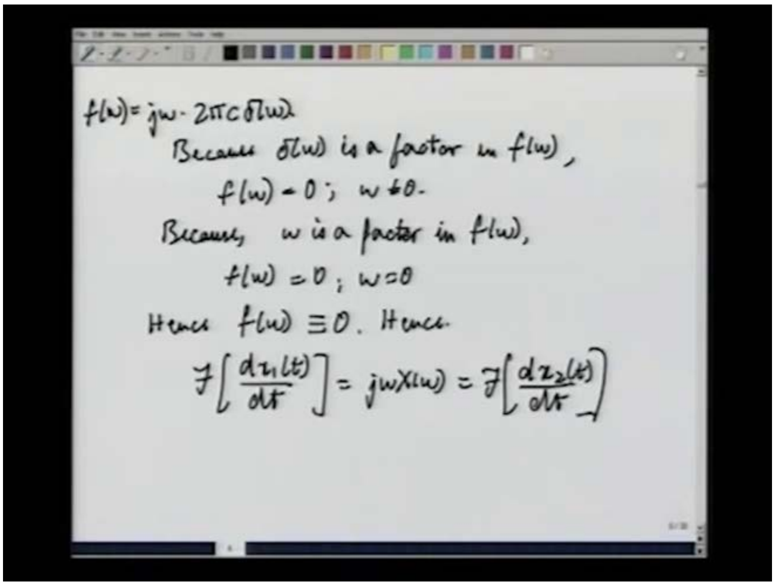

Consider this function because delta omega is the factor, we will just call this some f of omega, will call this f of omega because delta omega is a factor in f omega. f omega equals 0 for omega not equal to 0, because j omega is a factor or more precisely, because omega is a factor in f omega, f omega equals 0 at omega equal to 0. Hence, f omega is identically 0, hence the Fourier transform of d x 1 t by d t equal to j omega X omega equals the Fourier transform of the x 2 t by d t.