Signals and System

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Lecture - 29
Properties of CTFT

So, now we have a unified formulation for both finite energy, and a large number of finite power signals also in the context of the Fourier transform. We find that if you deal with finite power signals in time, then these will be represented by Fourier transforms which have impulses in the frequency domain. Conversely if you have finite power signals in the frequency domain then their inverse transforms or their time domain counterparts will have impulses in the time domain, thus for example.

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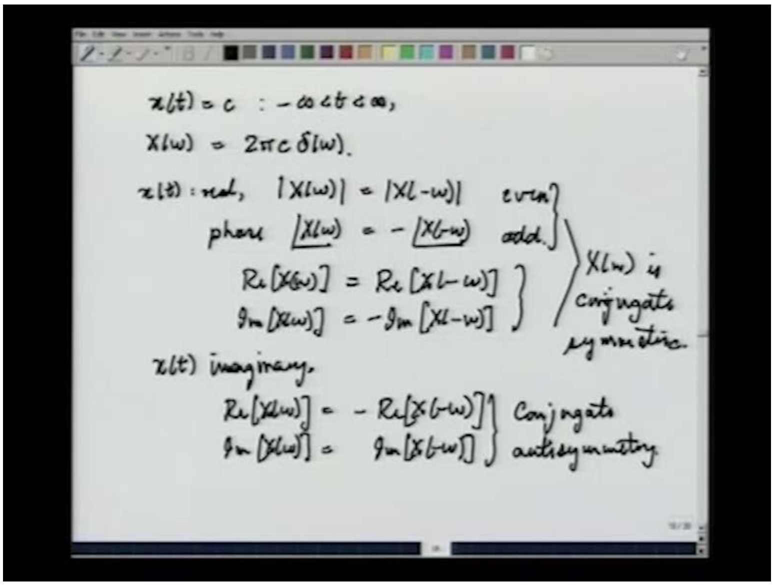

If you just took a DC signal for examples of value x t equal to c minus infinity less than t less than infinity, then x omega evaluates to 2 pi c delta omega which means there is an impulse of height of strength 2 pi c at omega equals to 0, alright. Now, how do we sketch the Fourier transforms, as I said there are always 2 parts to it, because the Fourier transform is generally complex the issue arises as to how you represent the Fourier transform just as with the Fourier series to spectra, you always have to have 2 plots either a real part and imaginary part plot which is sometimes used, but not very often or what is more common a magnitude and phase plot. So, if you had a magnitude and phase plot and if you consider the typical case of a real x t, so x t real than the magnitude plot would be even mod x omega would satisfy mod x of minus omega while the phase, which I denote by angle x omega would be equal to minus of angle of x minus omega. So, this is even, and this is odd. In rectangular representation where you deal with the real and imaginary parts x omega you would have real x omega equals real x of minus omega, and imaginary x omega equals minus imaginary x omega, sorry x minus omega. So, these are properties, when x t is real when x t is purely imaginary you would have a similar set of properties that you can easily evaluate using the knowledge that is already given. When x t is complex neither purely real nor purely imaginary, all these symmetric properties seized to hold, and the real and imaginary part can both have symmetric and non-symmetric components.

Now then as I have already mentioned in the context of the spectra of the Fourier series when a function has either this property or this property, either this or this property we say that x omega is conjugate symmetric. If it turned out as will be the case for a purely imaginary x omega you, you will get a purely imaginary x t, you will have conjugate anti-symmetry, that is x t imaginary, you would get real x omega equals minus real x minus omega, and imaginary x omega equals, imaginary x minus omega; this is said to be conjugate anti symmetry, conjugate anti-symmetry alright.

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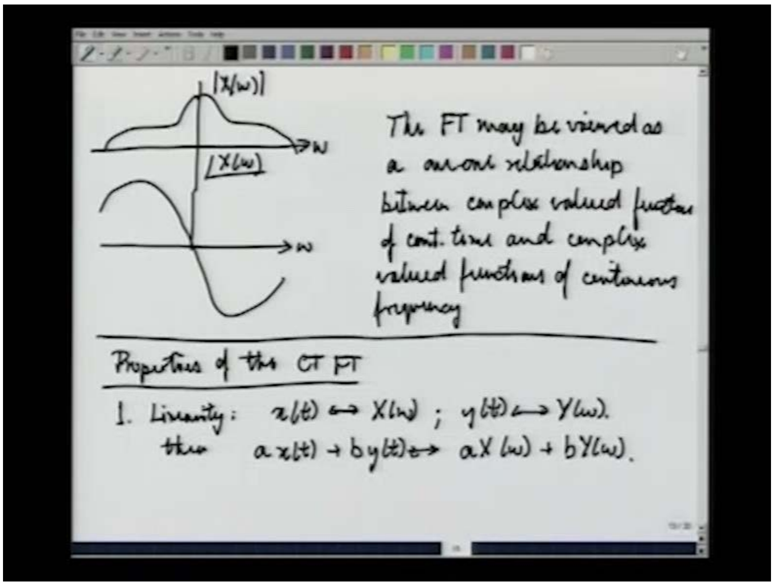

So, with these things in the background you would expect to see for real x t magnitude and phase plots that might look like this, magnitude plots would be symmetric about the origin. So, you would have this could be an example of mod x omega, whereas the phase would always be something like this, odd not even. So, this is what you would expect to look at. Now, with this much in the background, we can start examining different algebraic and other properties of the Fourier transform as a relationship between pairs of functions. Recognize that the Fourier transform can be seen as a one-one map between signals which are functions of continuous time, and signals which are functions of continuous frequency.

So, you have functions defined over similar independent variables time in the one case, continuous time in the one case and continuous frequency in the other case, and in both cases the functions both the continuous time functions as well as the continuous frequency functions can be complex valued. And the Fourier transform sets up a one-one relationship between each function here and the other function over there. So, we will say that the Fourier transform may be viewed as a one-one relationship between complex valued functions of continuous time of continuous time, and complex valued functions of complex valued functions of continuous frequency.

So, viewed in this manner, there are lots of very interesting properties, very symmetrical looking properties that you will soon find. We will now quickly run through these properties, we will not try to prove everyone of these properties only certain of them will be derived and proved, the others will be left as an exercise, alright. The first and most striking property of the continuous time Fourier transform is it linearity with respect to the input, it is both addictive and homogeneous as can be easily shown, if you use the defining equations for the synthesis or the analysis. You can easily see that if x t has a Fourier transform which is omega use an arrow at both ends of a small line to signify that x t and x omega are a Fourier transform there and y t similarly pairs with y omega, that is to say that y omega is the continuous time Fourier transform of y t.

Then a x t plus b y t would be a function that has a Fourier transform given by a x omega plus b y omega. So, much for linearity it is a very straightforward property it is a very, very fundamental property, and it is literally used all the time, it is invoked at sometimes or the other every few minutes, we discuss these things. The next properties deal with not its range, but with the domain of these two functions.

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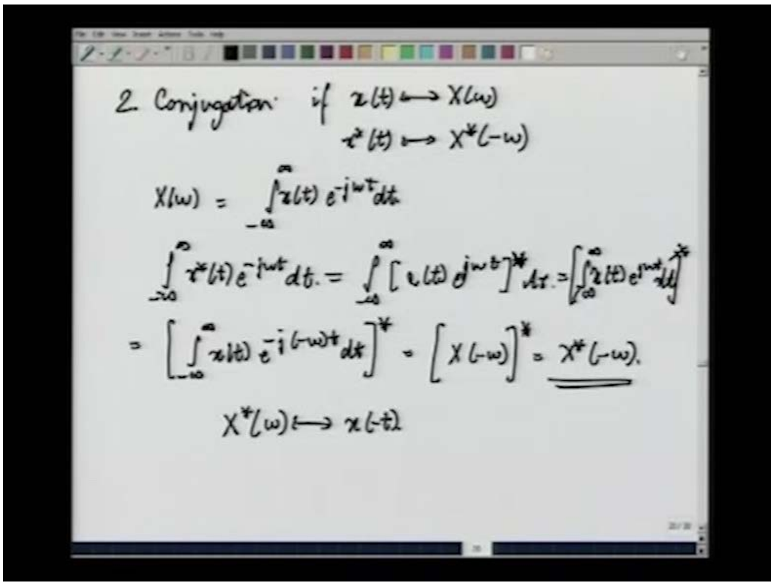

For example, I will ask the question well before I proceed there is just one more thing about the range, and that is this if this is the conjugation property, if x t goes to x omega then x star of t which is the complex conjugate of x t turns out to go to x star of minus omega. Now, since we are at the beginning of a set of very similar properties, I will actually go about and prove this 1 go about to prove this one, this something, we will do for every other property that follows, how do we do this? Suppose you have x t and the Fourier transform is x omega, then you have x omega equals integral x t integrated over all time e to the power minus j omega t d t, what we are however, interested in at this point is the integral over all time of x star t e to the minus j omega t d t. So, this is what we have?

Now, how do we find the Fourier transform of this we do this by trying to bring it to a familiar already known form, and that is carried out by rewriting this as integral minus infinity to infinity x t t to the j omega t the whole thing conjugated d t. Now, if this whole thing is conjugated we can look at this integral whose integrity is completely conjugated as a conjugate of the entire integral. In short we can rewrite this as integral minus infinity to infinity x t e to the j omega t d t whole thing conjugated, the whole thing conjugated. But now we have a problem inside this is not exactly the Fourier transform of x t, because it should be e to the minus j omega t that goes with it, and not e to the j omega t. So, we solve this by rewriting it as follows, x t e to the minus j minus omega t d t the entire integral conjugate.

Now, let us see what we have? We have the Fourier transform of x t with respect to the frequency variable minus omega, normally we have it with respect to the frequency variable omega, now we have written this with respect to the frequency variable minus omega. So, this then is nothing but x of minus omega the whole thing conjugated which is nothing but x star of minus omega which is what we predict. Now, it is not very hard to prove from this or using this, that if you have x star of omega, and wish to find its inverse transform that that will come to x of minus t, fine. So, this is about conjugation.

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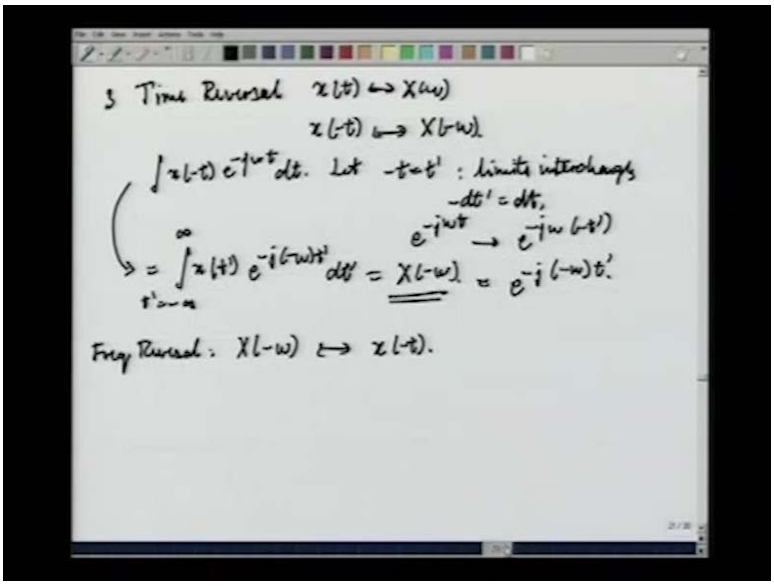

Next time reversal, if x t transforms to x omega, you can show that x of minus t transforms to x of minus omega easy to show again, this is nothing but integral x of minus t e to the minus j omega t d t. Let minus t be equal to t dash then the limits interchange, and minus d t dash equals minus equals d t d t alright. And e to the power minus j omega t becomes e to the minus j omega minus t dash, which is equal to e to the minus j minus omega t dash with, all these changes incorporated we can write that this is simply equal to integral x of t dash, t dash equals minus infinity to infinity e to the minus j minus omega t dash d t dash, where we have used the minus sign that went with d t dash to re-interchange the limits of the integration. And this expression we have write on hand is nothing but equal to x of minus omega.

So, that settles the matter x of minus t goes to x of minus omega, which also means that you have the opposite results right away, if x of x of t takes you to x omega and x of minus t takes you to x of minus omega, it also means that a frequency reversal; that is if x of omega takes you to x of t is this inverse transform, then x of minus omega will take you to x of minus t. So, x minus omega has an inverse transform of x of minus t, this is the time reversal property, and this actually is the frequency reversal property frequency reversal property, fine.

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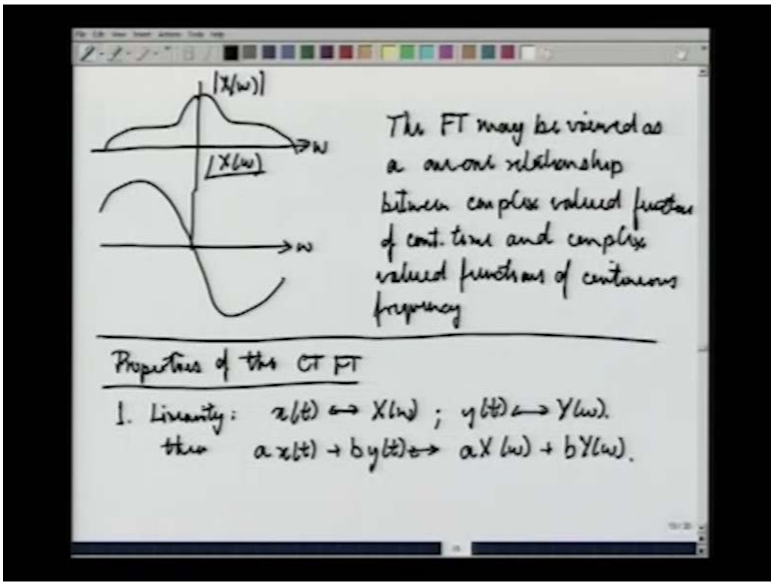

Now, after this we will go onto some more properties time and frequency scaling, time scaling. Suppose x t goes to x of omega then x of k times t, let us see what this does? It turns out that this comes to 1 by mod k x of omega by k, this is best shown by considering 2 separate cases; the first case k greater than 0 for which you will get that x k t transforms to 1 by k x of omega by k, and the second case is of k less than 0, where x of k t turns out to go to 1 by minus k x of omega by k. Now, each of these are not is not very is not at all difficult to evaluate, all you have to do is to substitute k t by t dash make the appropriate changes in the differential in the limits, and in the exponent of the Fourier colonel that is e to the j omega t and so on and simplify, and you will end up with this thing.

Now, if you combine these two together you will get this particular result, fine. Now, looking that this result itself, it is not very hard to see what frequency scaling would lead to. For example, this same expression can be looked upon the frequency scaling a factor 1 by k that is if you had x omega and that gave you x t, then x omega x of omega by k will just give you mod k x t. If x omega takes you to x t, then x of omega by k will take you to mod k x k t. So, clearly if you represent 1 by k by say l then you have that x l omega will have an inverse transform given by mod 1 by l x t by l. This is what you would have as the frequency the result of frequency scaling. Time scaling was numbered four. So, we will now go to property numbered five we will soon lose count. So, let us just keep numbering as long as we can remember to number them.

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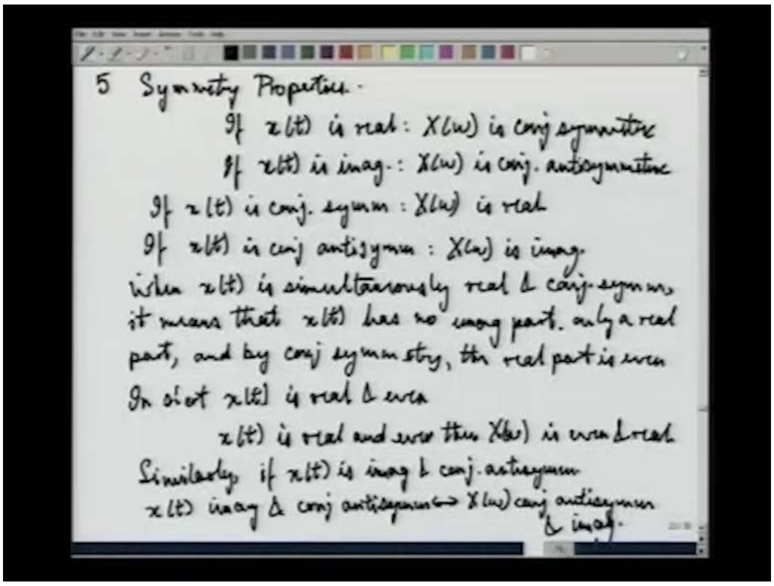

The next property is a, the symmetry property rather the symmetry properties, because there are so many symmetry properties over here. Now, this is also easy to show by just using the analysis and synthesis equations as a starting point, and what you will find is the following, that if x t is real x omega is conjugate is symmetric, if x t is imaginary. That means, not generally complex, but purely imaginary then x omega is conjugate anti symmetric alright, if x t is conjugate symmetric abbreviated by then x omega is real, if x t is conjugate anti-symmetric then x omega is imaginary. Now, each of these is in some sense a separate property, and show you can put two of these properties together and get the benefit of both the properties at once.

For example, I have said that if x t is real then x omega is conjugate symmetric; conjugate symmetric means that the real parts are even, and the imaginary parts are odd, suppose x t is both real and conjugate symmetric. Now, what does it mean? It means that x t only has a real part, it has no imaginary part, and that real part is an even function. So, let us say what does it mean, when x t is simultaneously real and conjugates symmetric it means that x t has no imaginary part only a real part, and by conjugate symmetry that real part is even. So, simply put x t is a real even function in short x t is real and even, now because x t is real by the symmetry properties that we have here, x omega is conjugate symmetric, and because x t is conjugate symmetric x omega will be real. So, x omega is also conjugate symmetric, and real at the same time which means by the same argument that we have used for x t over here, that x omega will be conjugates even and real. In short if x t is real and even then x omega is even and real.

Similarly, if x t were purely imaginary and conjugate anti-symmetric, if x t is imaginary and conjugate anti-symmetric, let us see what happens? Because x t is imaginary going by this properties x omega will be conjugate anti-symmetric. Now x t is also conjugate anti-symmetric, now conjugate anti-symmetry means that the real part is odd, and the imaginary part is even. Now, we are told that it is purely imaginary, so there is no real part anyway. So, x t simply has an imaginary part which is even. Now let us try to understand merely that, now it has an even shaped imaginary part.

Now, because it has an imaginary part x omega will be conjugate anti symmetric fine, and because it is conjugate anti-symmetric x omega will be purely imaginary. So, just like with the time domain function x c, which is conjugate anti-symmetric and imaginary, the frequency domain representation x omega will also be imaginary and conjugate anti-symmetric, so x t imaginary, and conjugate, anti-symmetric yields x omega conjugate, anti-symmetric, and imaginary. So, that was the summary of the symmetry properties of the Fourier transform. Now, let us go on and discuss some more startling properties of the Fourier transform. The next property we will deal with is the result that happens when one delays time signal time delay, will also deal with its conjugate with its counterpart frequency delay, time delay if x t maps to x omega under the Fourier transform.

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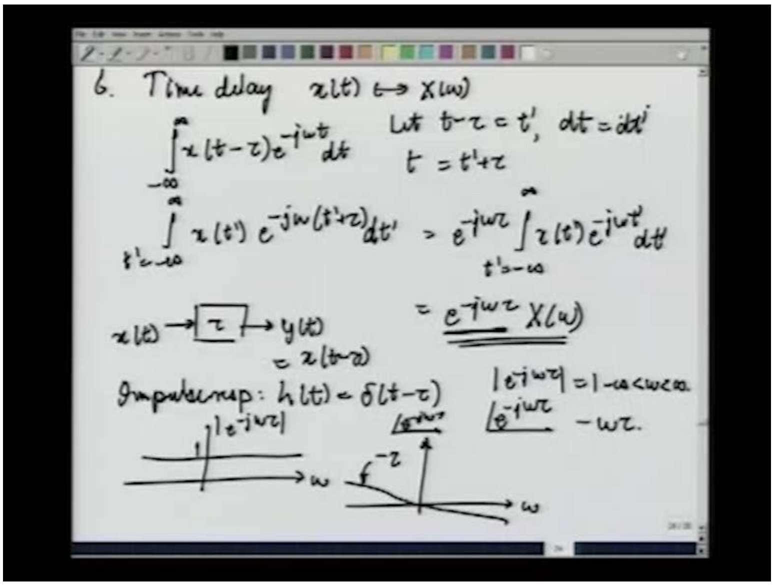

What happens to x t minus tau, let us see what happens to x t minus tau. So, let us just evaluate it e to the minus j omega t b t integrated over all time over all t. So, let t minus tau be equal to t dash then d t equals d t dash the limits do not change only thing t becomes t dash plus tau. So, that we get after making all these replacements integral t dash equal to minus infinity to infinity x t dash e to the minus j omega t dash plus tau d t dash, and this is nothing but e to the minus j omega tau integral t dash equals minus infinity to infinity x of t dash e to the minus j omega t dash d t dash which is nothing but e to the minus j omega it out x omega.

So, this is what happens? When you set a delay on the time function, it results in what is called a linear phase factor appearing along with the Fourier transform. I will discuss this briefly in terms of its impact. Now, this delay could well have been caused by a block, remember that we used delay blocks in the context of solving different equations differential equations and also, delay blocks are very common things they are appearing all kinds of places. So, now if you had a delay block and that delay block was a tau delay block, then you your input is x t and y t equal to x t minus tau, then the impulse response would be h t equals delta t minus t naught sorry, delta t minus tau, this would be the impulse response.

Now what I am interested to look at is the Fourier spectrum of the impulse response, the impulse response Fourier spectrum in this case can be easily evaluated as that appearing on account of this factor the factor e to the minus j omega tau, this must be viewed as a fraction of omega as a function of omega, because we are actually plotting a spectrum here from here. So, the magnitude of e to the minus j omega tau is always equal to 1 is always equals to 1 for minus infinity less than omega less than infinity, it is always equal to 1. What changes is the phase and how does this phase change e to the minus j omega tau has a phase component given by minus omega tau; that means, it is a straight line with the slope of minus tau. So, the magnitude plot is like this, a constant of magnitude equal to 1 and the face plot is a line that slopes downwards going to the origin like this with the slope of minus tau. So, the this is how the delay looks or the spectrum of a delay the Fourier spectrum of a delay looks it does nothing to the magnitude, but it gives a phase lag of an amount proportional to the frequency to different components of the signal that is been delayed.

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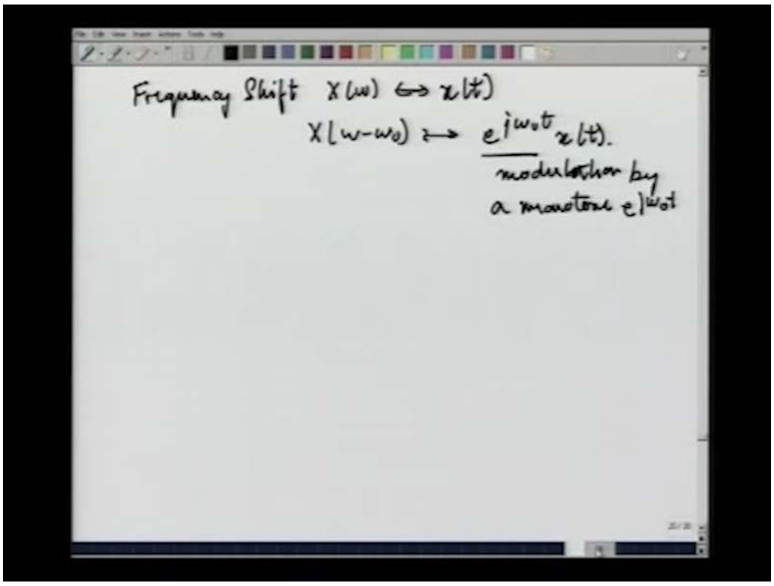

From what we have just discussed it is only a short step to see what happens, if you do a frequency shift. In short x omega maps to x t, what happens to x omega minus omega naught well, I will not derive this again since its pretty much derived by the same steps that we have just used it. It just turns out to have an inverse transform given by e to the j omega naught t x t, that is what happens? The only difference is there we had a minus sign the factor that came in, here there is no minus sign this is a function of time that is linear in time, it is in fact the same as multiplying by a complex exponential, this is just a periodic complex exponential and we are multiplying x t by a complex exponential.

So, delaying or shifting in the frequency domain results in multiplication by a complex exponential, this is often called modulation. So, now let us move on this is modulation by a monotone signal, modulation by a monotone signal e to the j omega naught, right.

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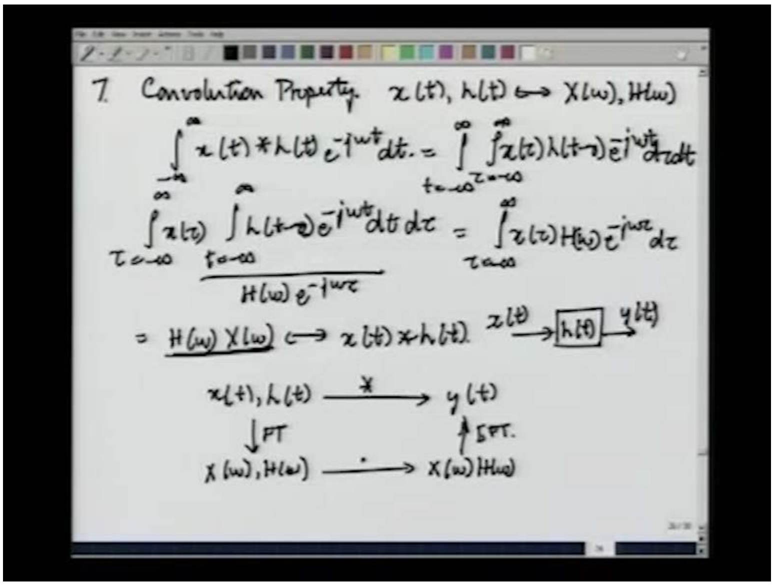

Now, let us more two are far more important property probably the most important property and the reason why we study Fourier transforms for so much it is called the convolution property. The convolution property seeks to find the result of the convolution of 2 signals on the spectrum of the convolution given the vector of the individual component signals. So, if you have x t and h t with respective transforms x omega and h omega, we want to see what happens? x t convolved with h t, we want to find the Fourier transform of this essentially we therefore want to find this, this is what we want to evaluate?

In order to evaluate this we first have to evaluate the convolution, and the convolution as we know is given by the following expression, now I am going to write the names of the variables that of the limits of the integration. So, first of all the integral we have already written is an integral with respective variable t. So, this is t t equals minus infinity to infinity, then we have the integral for the convolution to be put out. So, we will say tau equal to minus infinity by to infinity x tau h t minus tau e to the minus j omega t d tau e t that is the whole expression.

Now, let us see how we evaluate this expression, you see that x tau is an term over here is a factor over here which has nothing to do with t. So, we could exchange the integrals and keep x tau outside, we do that and write integral tau equals minus infinity to infinity x tau followed by integral t equals minus infinity to infinity h t minus tau h t minus tau e to the minus j omega t d t d tau, because we have exchanged the integrals now it was d tau d t now it is d t d tau. And then you can evaluate the inner integral over here in a pretty straightforward manner, we already know about the effect of delaying a signal on its Fourier transform, the inner integral will simply evaluates to h omega e to the minus j omega tau, just the previous result that we proved.

Hence now what we get is this integral tau equals minus infinity to infinity x tau h omega e to the minus j omega tau d tau, now h omega really has no business inside the integral, because it is no longer in terms of tau it is independent of tau. So, we can write this is equal to h omega, and then if you look at the rest of the integral it is simply the Fourier transform of x. So, we get x omega this then is the Fourier transform of x t convolved with h t, you see what has happened if you have to convolve two functions, and find the result of the convolution, you now have a different way of finding the result of the convolution. You can take the Fourier transforms of x as well as h multiplier the Fourier transforms point for point, and get the resulting spectrum h omega x omega and then find the inverse Fourier transform of this.

So, this would be another way to implement the convolution, it is often argued that the advantage of taking the Fourier transform the advantage gained by taking a Fourier transform is precisely the saving of computational effect, that results from avoiding the convolution and replacing it by a multiplication.

Now since this is such a widely held belief, I do not want to attack it too strongly, but I do want to point out that this so called advantage is often debatable. Let us make a small diagram of the steps involved in the calculation. Suppose you have x t h t and you want to find y t which is the typical situation when you have a system of impulse response h t an input of x t and a resulting output of y t. Suppose this is what the things comes to when we know that x t convolved with h t will give us y t.

So, let us explore the two routes of doing it; one is you have x t h t given alright, and you just convolve them, and get y t job done, but we know how horrendous convolution is to evaluate y t at each point of time. One has to evaluate a new integral a fresh integral, one has to therefore, compute integrals over and over again, and build y t point by point that is a lot of integration, because each integration itself is a point by point multiplication of x t and h t minus tau and then integration x tau and h t minus tau, and then the inner product essentially of these two real functions that much in computation just gives you one point of y t.

So, what you can do instead here is take the Fourier transforms of both of these to get x omega h omega, and then multiply the 2 to get well maybe I should not a put a cross it is too old-fashioned and just put a dot. So, that your you have here x omega h omega and then take an inverse Fourier transform this is the f t over here, and this is the i f t over here. So, if it is the f t and the i f t that, you also have the calculate, let us see what is the effort going one way and compare it with the effort going through the other route. Here we know what convolution is about. So, the direct path we know involves a lot of calculation, but is not really true that this involves less calculation this suppose depends upon the particular case in question.

But by and large we are really not replacing a convolution by a multiplication, we are replacing a convolution by a forward Fourier transform of two separate functions which is lot of work. Then of course, a very light job of multiplication followed by another lot of work in the form of the inverse Fourier transform to get to get y of t. So, it is this additional burden of to forward transforms of two different functions, and the inverse transform of one function, that has to be added in the burden of doing it in the alternate manner using the Fourier transform.

Before one can really pass judgment upon which is really more convenient and easy to accomplish, without doing this would really not by doing justice to to the seriousness of the task to evaluate which is easier or which is harder. So, this is the convolution property, we know how it works now? And we know that it has abruptly has a certain advantage in any case it does give us an alternate way of computing the output of the system other than convolution take the Fourier transform multiply them and inverse transform.