Signals and Systems

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Lecture - 28
Fourier Transform

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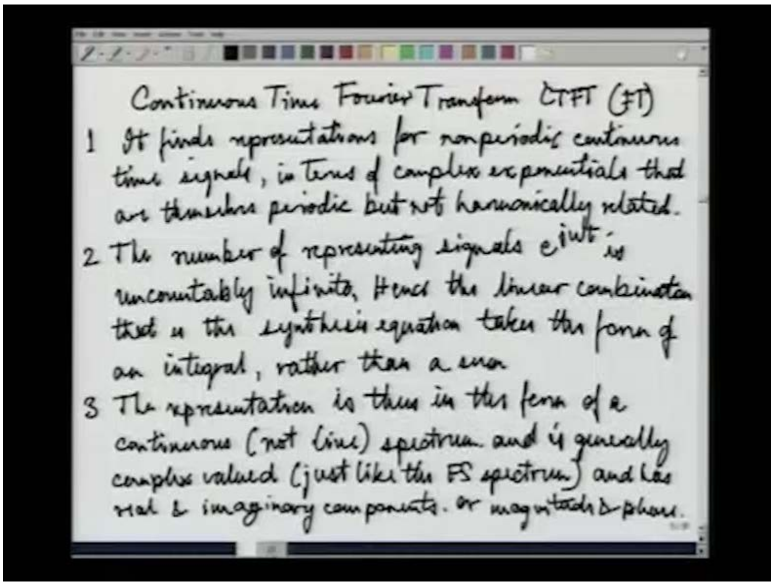

The Fourier transform that we have just discovered is what is more formally called the continuous time Fourier transform, which we abbreviate to CTFT or even to just FT where there is no room for confusion, but sometimes there is room for confusion and then we will call it the CTFT rather than just the FT. This is characterized by the following features it finds representations for non-periodic continuous time signals in terms of complex exponentials, that are themselves periodic, but not harmonically related, they are not harmonically related. This is what allows us to construct the non periodic signals using periodic representations, using periodic components. Secondly, the number of representing signals, that is the number of signals we use for representation namely the complex exponentials e to the j omega t is un-countably infinite.

Hence the linear combination that is the synthesis equation takes the form of an integral rather other than a sum rather than a sum, as was the case for the Fourier series, there it was a sum now it is an integral. So, this is relates number them this is point 1, point number 2, point number 3. The representation is thus in the form of a continuous spectrum not line continuous spectrum, and is generally complex valued, just like the

Fourier series spectrum, generally complex valued just like the Fourier series spectrum, and has a real and imaginary part, and has a real and imaginary components or magnitude and phase. So, let us plot this Fourier spectrum for some standard signals for some examples. So, that we get an idea of what the Fourier transform spectrum looks like, I will not trouble to go through the details of the derivation I will just give the Fourier transform pairs for the original, fine time function and for the transform of that time function.

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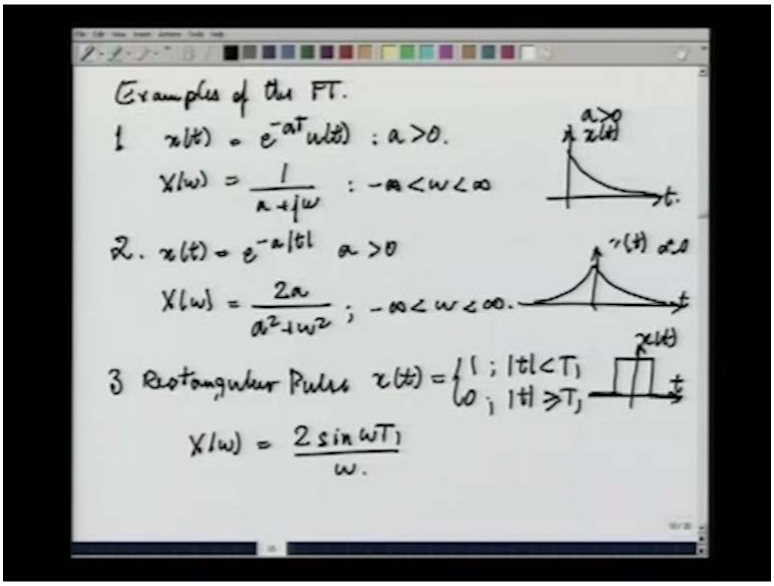

Examples of the Fourier transform as I said I will just give the pair’s, the transform pairs the original time function and its Fourier transform, it is very easy to derive these expressions, all you have to do is to substitute the expressions for the time function in the analysis transform in the analysis expression, and you will get the answer in few steps. So, one example is x t equal to e to the minus a t u t for a greater than 0, I cannot tell you right now why I am saying a greater than zero, but in a very little while in another few minutes that will become the topic of discussion, right now if we have x t equals e to the power minus a t u t then the Fourier transform of this turns out to be x omega equals 1 by a plus j omega for minus infinity less than omega less than infinity. So, this is one function, you can see that it is clearly complex valued you can find its magnitude part which will be 1 by and the root a square plus omega square.

It will have a face component which will be minus tan inverse omega by a and so on, but evidently at least it is the complex valued function of omega of the continuous variable omega.

Second example for the second example let me consider e to the power minus a mod t x t equals e to the minus a mod t. Now again I am assuming that a is greater than 0 for reasons which will be discussed in a few minutes. In fact, the reason is the same as the reason for the previous example, and x omega comes out to be under this assumption 1 by rather 2 a by a square plus omega square, so that is the second example. Now, a third example, the third example I will take is of what is called a rectangular function, if you have plotted all these functions over here on this side for example, if I plot x t the first 1 for a greater than 0 x t be a function that exponentially decays against time. The second function we have plotted is x t with a greater than 0 comes out to be symmetric on both sides of the time axis, and for a greater than 0 exponentials the decaying in both directions as I have drawn over here.

The third example is of what is called a rectangular pulse? x t equals 1 for mod t less than t 1 equals 0 for mod t greater than equal to t 1, the shape of the function would be like this. This is what you would have and it is Fourier transform x omega turns out to be equal to 2 sin omega t 1 by omega by omega, which is often called a form of the sync function will talk about the sync function in a moment.

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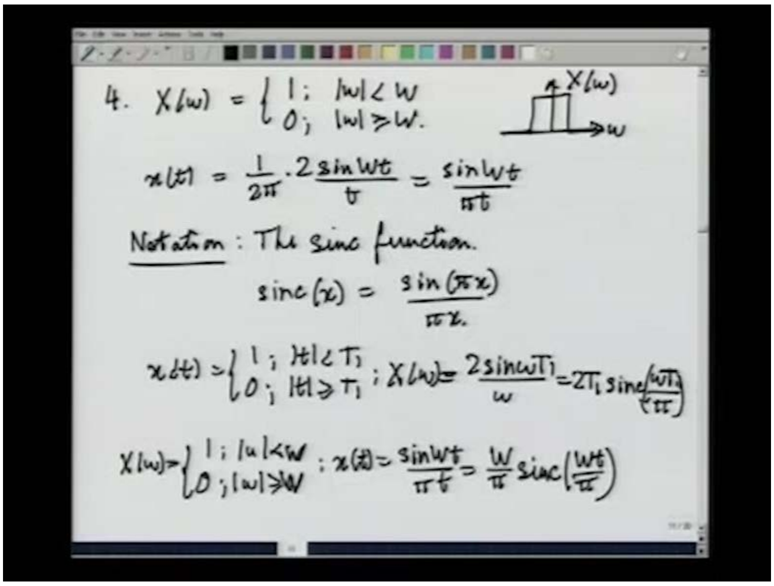

Now, let us take a forth example this will be of a rectangular pulse, but in the frequency domain, and what we will see is the inverse Fourier transform. So, let us specify x omega as given by 1 for mod omega less than some w and equal to 0 for mod omega greater than equal to w. Suppose this is what is given to us it is a rectangular pulse in the frequency domain is what has been given to us, and x t comes out to be equal to 1 by 2 phi to sign wt by t which simplifies to sign wt by phi t sign wt by phi t alright. So, this is what it comes to.

Now next point we will introduce the notation that we call the sync function, the sync function is defined as follows sync x is given by sign phi x divided by phi x, now with the definition of see this sort, you can see that the last 2 examples can be written in terms of the sync function, because they have sin of something divided by something else. Thus for example, with x t equal to 1 for mod t less than t t 1 equal to 0 for mod t greater than equal to t 1, we had x omega given by 2 sign omega t 1 by omega sign omega t 1 by omega which comes out to be equal to 2 t 1 sync omega t 1 by phi. Is not hard to derive it is just a matter of substituting number sign doing a little jaggery with the things.

Similarly when x omega was equal to 1 for mod omega less than w equal to 0 for mod omega greater than equal to the w, then we had x t given by sin w t by phi t which was equal to w by phi sync w t by phi, sorry. So, this is how you make use of the sync function to denote function of the form sin x by x or sin phi x by phi x alright. So, the next thing we were going to now talk about is this property of the continuous time Fourier transform to deal with continuous functions of continuous time, and yield spectrum yield a spectrum which is also a function of a continuous variable namely omega. What has to do with what we are going to talk about next is the following. So far we have been only concerned with finding or discovering the Fourier transform.

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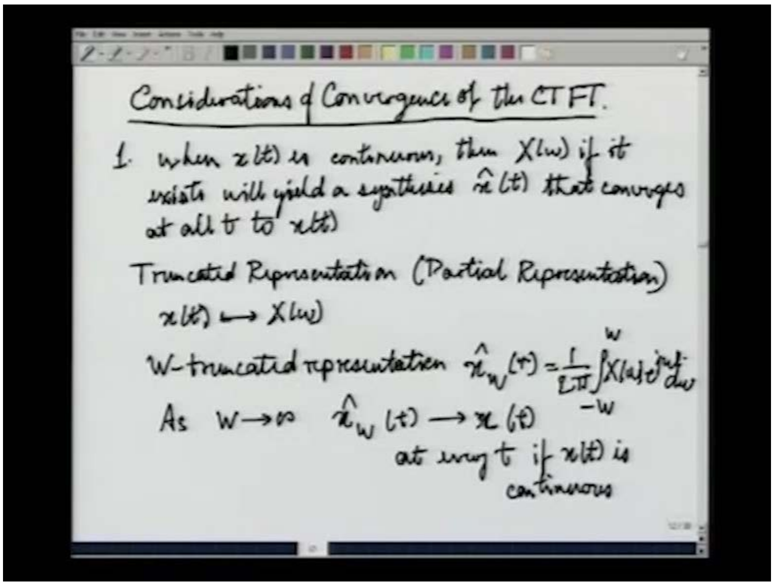

We still now have to ask lots of important questions, such as the convergence questions or considerations of convergence in the case of the Fourier transform, considerations of convergence of the CTFT; convergence considerations of the continuous time Fourier transforms, does the Fourier transform exist if so when does it converge what points does it converge? Similar questions were asked in the case of the Fourier series as well and we will find. In fact, that both these discussions are extraordinarily similar. So, let us briefly go through the issues of convergence.

So, first of all as was the case with Fourier series converging continuous functions, that is where x t is continuous finding a convergence is not difficult. So, when x t is continuous. So, let us start with the various discussions about convergence, the first is when x t is continuous then x omega if it exists, actually the existence issue is being dealt with separately we are saving that if x omega exists, we will yield a synthesis which following our earlier notation, we will denote by x hat of t that converges at all t to x t. That means, there is no point problem of convergence when x omega exists for the continuous function x t a function without discontinuities then the synthesized function x hat of t will converge to x t at all points t.

Now, we are also we can speak of what we can call a truncated representation. What do we mean by a truncated representation here, suppose x t has a Fourier transform x omega, truncated representation also called partial representation, fine. What do we mean by a truncated representation is let x t have a transform of x omega, then we will denote the truncated representation we will say the t truncated representation x hat t subscript t of t given by 1 by 2 phi integral minus t to t x omega e to the j omega t d omega instead of integrating for all omega from omega equal to minus, sorry. This is not called the t truncated this should be called the w truncated representation for consistency of notation.

And therefore, we will call this x w of t x hat w of t is the syntheses carried out from minus w to w, since w is finite and not equal to minus to infinity and minus w is not equal to minus infinity, some of the information in the transform is been lost and only partial information is being used to reconstruct the original signal or to attempt to reconstruct the original signal, that is why we call this x hat w of t. Now what we are saying is as w goes to infinity x hat w of t tends to x of t at every t, if x t is continuous, this is what I meant when I said converges at all points t.

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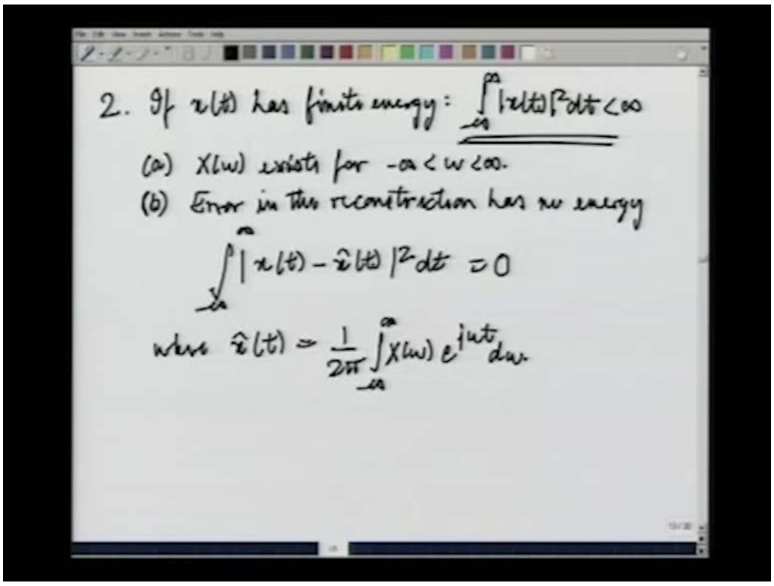

Now, let us look at other criteria for convergence, the second criterion for convergence consider functions which are not which are possibly not continuous for all t which could have discontinuities and other diseases, but which satisfy the property of square integralablity over t equal to minus infinity to infinity. These signals are called finite energy signals. So, if x t has finite energy that is to say integral minus infinity to infinity mod x t squared d t is finite, then… So, many things happen, but before we talk about those. So, many things, let us compare this particular statement which the counterpart of it that appeared in the context of the Fourier series. There we said that it should have finite energy over 1 period or equivalently if it has finite energy over 1 period it would have finite power. That means, it would have finite energy over any finite interval that was the case for periodic signals, now we are concerned with non periodic signals and we derived the whole theory of representation of non periodic signals using Fourier transform by gradually making the period go to infinity. In that sense this new criterion which speaks of finite energy relates very easily to the old criterion for the Fourier series which spoke of finite power, there finite power meant finite energy over 1 period. Since it will have the same same energy over all successive periods, because they are identical to this.

So, here finite energy over all time is certainly equivalent to finite energy over 1 period because now 1 period itself is all-time, we are still saying finite energy overall time, but now over over 1 period, but now 1 period extends from minus infinity to infinity. In this sense this criterion is not fundamentally different from what we spoke of over there. Now let us see what this criterion ensures, and what it guarantees to us. Now if x t has finite energy and satisfies this expression, then a - x omega exists for minus infinity less than omega less than infinity that is point number 1 the transform existence of the transform is ensured. Now, if you go back to the previous criterion for convergence where we said that x the reconstructed signal x hat will converge to x t at all points of time, we said that that is to if x omega exists we did not say when x omega will exist. Now we have a criterion with this we have a condition that ensures that x omega exists that is make sure that, it is square integral it has finite energy, fine.

Now, x omega will exist for all minus infinity for all omega from minus infinity to infinity. Second the error between the reconstruction and the original signal x t has no energy, that is to say lets write this down. The error in the reconstruction has no… What is the reconstruction error? Reconstruction error is the difference between the original and the reconstruction. So, this is essentially the reconstruction error x t minus x hat t where x hat t is 1 by 2 phi integral minus infinity to infinity x omega e to the j omega t d omega, this is x hat t, and this is the reconstruction error x t minus x hat x t minus x hat t. Now if you have x t minus x hat t that is the reconstruction error.

The square of that reconstruction error is this and if integral over all time minus infinity to infinity is the total energy in the error signal, and this is what we are saying is equal to 0. So, the reconstructed signal has error probably has error with respect to the original signal, but the error has no energy, this much is guaranteed if we confine ourselves to square integrable signals; signals which satisfies this criterion. It however, does not tell us it tells us that the error signal might be non-zero in places, that is to say that x hat of t may not be equal to x of t for all t. There might be places where they are not equal, but unfortunately it does not tell us where they are not going to be equal.

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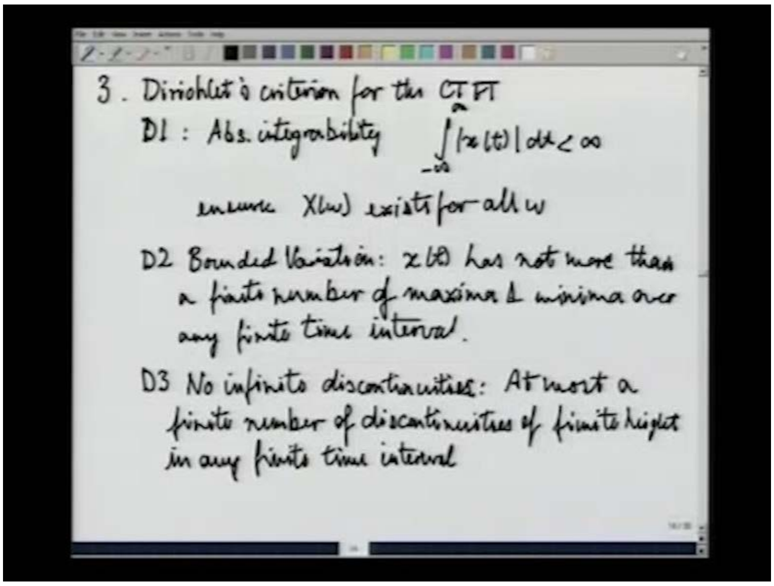

If you want that kind of information we go to the next member of the criteria set of criteria, which is the criterion applied for the Fourier transform; the Dirioklets for the Fourier transform is the following, it sets 3 Diroklets constraints on the signal that is to be transformed. First I will call this d 1, d 2, and d 3; d 1 says absolute integral ability of x t absolute integral ability simply means in this case integral over 1 period of the absolute value 1 period is now minus infinity to infinity absolute value of the signal that is x t d t must be finite. This absolute integral ability ensures x omega exists for all omega, but there are other criteria absolute integral ability will ensure that x omega exists, but it goes further than that you have bounded variation which requires the criterion of the constraint of bounded variation requires that x t has lot more than a finite number of maxima and minima over any finite time interval.

The third and last Dirioklets criterion d 3 says that there should be no discontinuities, we are allowed to have discontinuities, but there should be no infinite discontinuities infinite discontinuities are those where the jump of the discontinuity is infinite. For example, what you will find in 1 by x at x equal to 0, 1 by x at x equal to 0 is an infinite discontinuity, no infinite discontinuities and utmost a finite number of discontinuities of finite height in any finite time interval in any finite time interval; these are the 3 constraints of the Dirioklets criterion. Now, what you get if you follow these constraints that is what do you get if certain x t needs all these constraints you get the following.

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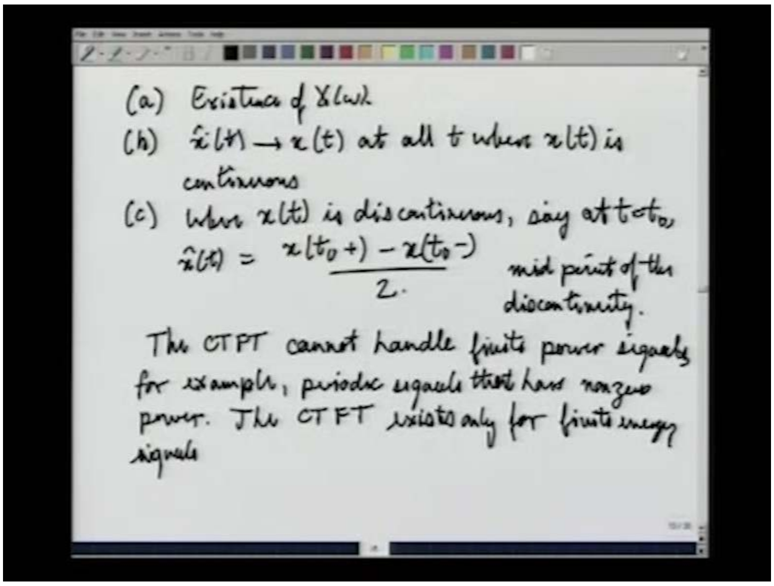

Existence of x omega b convergence at all points where x t is finite x hat of t converges to x t at all t, where x t is continuous, the third point is where x t is discontinuous say at t equal to t naught x hat of t equals x t naught plus minus x t naught minus by 2, where x t naught plus is the left side limit, and x t naught minus is the right side limit of x t at the discontinuous point t naught. So, it takes you to the midpoint of the discontinuity.

So, at all points, where x t is continuous the reconstruction works perfectly, and only those points where the where x t is discontinuous, the reconstruction lands up at the midpoint of the 2 limits; the left and right side limits, this is what the Dirioklets condition tells us with this we now have an idea of what to expect from the reconstructions and the presentations that are provided by the Fourier transform. Now one important constraint on the Fourier transform you see is that the Fourier transform will exists only for signals which are finite energy, signal should have finite energy otherwise the Fourier transform will not exist.

Now given this fact what happens to for example, an x t which is periodic, if an x t has non-zero energy and is periodic, then clearly it will non-zero power and is periodic then overall it will certainly have infinite energy, this is common sense, because it has a certain amount of energy per period which is what we call the power. And that gets added over and over for successive periods there are an infinite number of periods from t equals to minus infinity to infinity. And so over that period of time over the entire period of the time axis the energy will become infinite. So, there is a problem with handling periodic signals or finite our signals, the c t f t cannot handle finite power signals.

For example periodic signals, that have periodic signals that have non-zero power non-zero power this is an inherent limitation of the continuous time Fourier transform. Now we will try to see if we can address this limitation in some way, and ameliorated for certain kinds of cases at least in order to do this. We will recall an acquaintance that we have not dealt with for a long time the Dirac delta function. So, this is all motivated by trying to extend the Fourier transform to deal with finite power signals it only deals with finite energy signals. So, let us just write that here, the CTFT exists only for finite energy signals.

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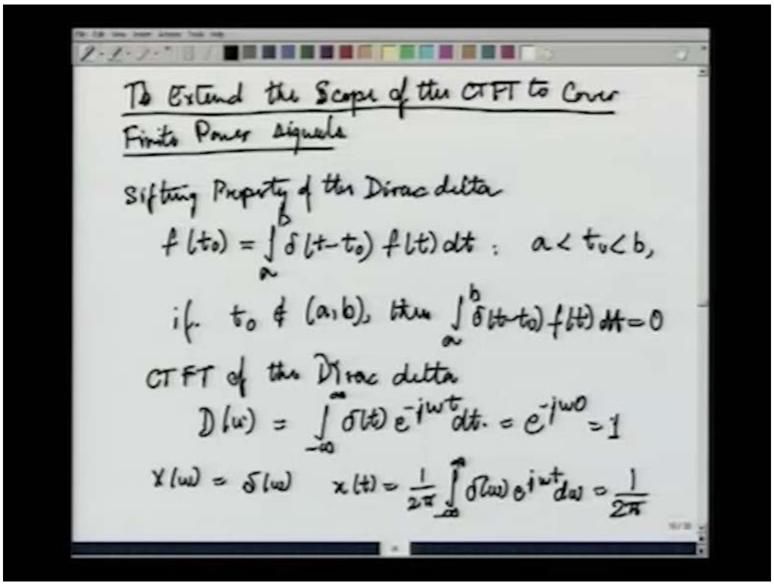

So, the next thing we will talk about is to extend, the scope of the CTFT to cover finite power signals it is for this that the needed to recall the Dirac delta. Remember that the Dirac delta was given was defined in terms of a property, that it had called the swifting property; swifting - swifting property of the Dirac delta said that if you integrated delta t minus t naught along with some f of t over an interval, which includes t naught over an interval a, b such that a less than t naught less than t then this integral was equal to f of t naught that is what we had.

So, if t naught lies in the interval where the Dirac delta occurs, then this kinds of an integration over, and interval that includes the point of occurrence of the Dirac delta will field an integration of value equal to f of t naught. It will evaluate the companion function f of t at the point where the impulse occurs provided the impulse occurs within the range of integration, if it does not occur within the range of integration then of course the integral will evaluate to 0, that is to say if t naught is not in a b then integral a to b delta t minus t naught f f t d t equals 0, this was the swifting property of the Dirac delta.

Let us see how this becomes useful to us now suppose we ask the question, what is the Fourier transform of the Dirac delta itself? Does the Dirac delta have a Fourier transform the swifting property of the Dirac delta, and how it can be used to find its own Fourier transform. Suppose we had Dirac delta then d f omega d f omega would be equal to integral minus infinity to infinity dealt of t which is the Dirac delta multiplied by e to the power minus j omega t d t. This just looks like an application of the swifting property, the Dirac impulse occurs at t equals to 0. So, this entire integral will simply evaluate to the value of e to the minus j omega t at t equal to 0, that is to say it is equal to e to the minus j omega 0 which is equal to 1. That means, the Fourier transform of the Dirac delta is the constant 1, now that is one side of the story suppose we look at the same story from the other side, suppose we had a dirac impulse in the frequency domain, that is to say let x omega be equal to delta omega.

Then what will be x t x t will be equal to 1 by 2 phi integral minus infinity to infinity delta omega e to the j omega t d omega, this again is an application of the swifting property where the impulse in the frequency domain occurs omega equals to 0. So, if the impulse occurs at omega equal to 0, you just evaluate the companion functions in that place and that gives you the answer. The companion function is e to the j omega t set omega equal to 0, you get unity and therefore this evaluates to 1 by 2 phi.

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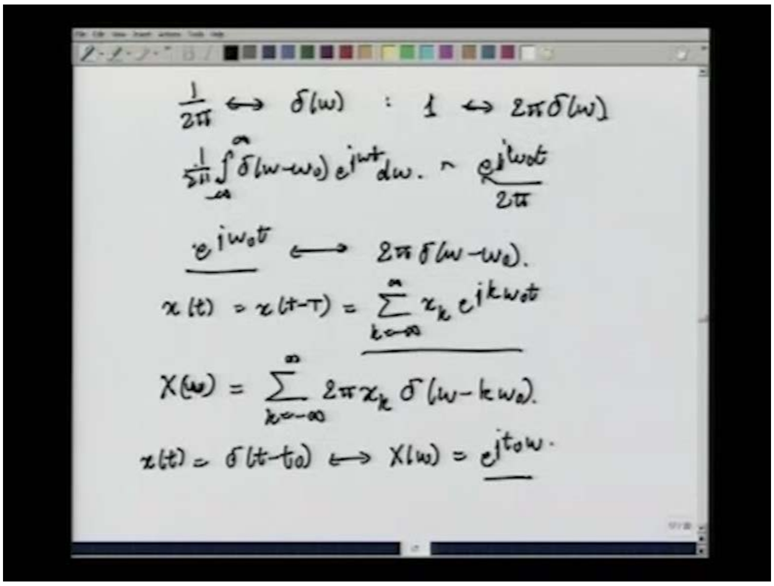

But this means we have discovered something interesting, we are saying now that 1 by 2 phi which is a constant function a value 1 by 2 phi for all time is equal to 1 by 2 phi from t equals minus infinity to infinity has a Fourier transform equal to delta of omega, alternately you can say 1 in the time domain has a Fourier transform equal to 2 phi delta omega right, but look at this constant functions like 1 by 2 phi and 1, they are not finite energy signals. They are in fact finite power signals certainly not finite energy signals; that means, we have found the Fourier transform of 1 at least of at least 1 a finite power signal which is not a finite energy signal.

Now, let us see if we can generalize this further, suppose I had a delta in the frequency domain occurring not at omega equal to 0, but at omega equal to some other omega naught. What would we get if we had integral 1 by 2 phi minus infinity to infinity is the limit, delta omega minus omega naught e to the j omega t d omega sorry, d omega, then what happens. Then you have to evaluate the companion function at omega equal to omega naught e to the j omega naught t by 2 phi, that is to say that x t is e to the j omega naught t. The Fourier transform pair you now have is that the complex exponential periodic function e to the j omega naught t has a Fourier transform given by 2 phi delta omega minus omega naught.

Now, clearly what you have on the left side is the function of time which is complex valued, but that is not relevant us right, now that it is complex value it is periodic and has finite power but not finite energy, it is one of those functions for which we were afraid we did not have a Fourier transform representation. Now, we find that there is a Fourier transform representation provided you allow the transform domain expression to have impulses what we originally derived was a Fourier transform where neither the time, domain nor the frequency domain representations were allowed to have impulses, but now when you allow impulses in the frequency domain you are able to construct time domain functions which are its transform counterparts; transform counterparts which are actually finite power. But not finite energy this means we have been able to do some extension of the Fourier transform to signals which are not a finite energy, but have finite power at least 2 periodic signals, how would we extend this to an arbitrary periodic signal very easy where practically, they are already suppose x t equals x of t minus t.

Then we know that it has a Fourier a Fourier series representation, I am writing the synthesis equation over here, k equals minus infinity to infinity x k e to the j k omega naught t this is this Fourier series. So, we have an x t which we suppose meets all the criteria for convergence that is, it does have a Fourier series representation, if it has a Fourier series representation, then this is the synthesis for x t for this periodic x t which is finite power, but not finite energy. Now, if you find the Fourier transform of this expression here of this summation then you can clearly see that this will have a Fourier transform which I will call omega given by, now for each term in the summation you will have 2 phi delta omega minus k omega naught times x k, and it is a summation of all such terms.

So, you get a summation k equals minus infinity to infinity 2 phi x k delta omega minus k omega naught, this is what you have fine. Now we have a means of representing periodic non-finite energy, but finite power signals in time domain in terms of a Fourier transform representation, that is now possessed of impulses inverse transforms will works. Similarly in short if you had impulses in the time domain these yield periodic functions of omega thus.

For example, suppose you had delta t minus t naught suppose x t equals delta t minus t naught, then it is Fourier transform is easily found using the analysis equation, and applying the swifting property, you would get this has a Fourier transform of x omega equal to e to the j t naught omega. That means, it is periodic in omega and so it has no finite power it has finite power, but not finite energy the Fourier transform has finite power, but not finite energy, and it is time domain counterpart x t is an impulse.