Signals and System
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Lecture - 27
Fourier Spectrum

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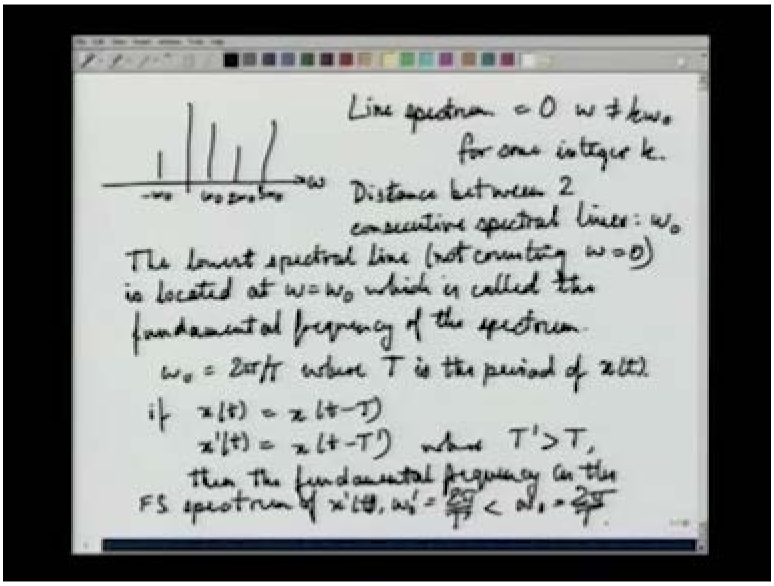

The Fourier series spectrum that we have learnt about is often called a line spectrum, and the reason is obvious this is because as we had made a few sketches. The last lecture, if omega is the variable against which against, which we make the plot rather than k, then you see that the lines occur at omega naught, 2 omega naught, 3 omega naught minus omega naught and so on. In short there are values at specific discrete points which are multiples of a certain quantity omega naught which is equal to 2 pi by t, where t is the period of the signal whose Fourier series expansion has been found at all other values of omega there is no value assigned to the Fourier spectrum to the Fourier series spectrum, that is why this is called a line spectrum.

Now, as we can see this line spectrum is equal to 0, at all omega not equal to k omega naught for some integer k, right. Now, what is the distance between 2 consecutive spectral lines? This is omega naught between 2 consecutive spectral lines this, place this is omega naught, and the lowest spectral line is at frequency omega naught not counting omega equal to 0 is located at omega equal to omega naught, which is called the fundamental frequency of the spectrum. The physical meaning that is attached, that is often attached to the Fourier series spectrum of a periodic signal is the following, we say that if non zero Fourier series components x k exists for larger values of k, then we will say that the signal the periodic signal x t has high frequency components. If it has significant values only for lower values of k or lower values of modulus k, then we will say that it is a low frequency signal or a low pass signal.

So, that is the kind of meaning that one attaches to the existence or non existence or largeness or smallness of the various spectral lines that constitute the Fourier series spectrum. Now, one thing that is clear at the end of this discussion is that the distance between the lines is dictated by omega naught, and omega naught is nothing but 2 pi by T, where T is the period of x t. So, if we take a different x dash t with a greater period, then it will give rise to a fundamental frequency omega naught dash which has a smaller period, which has smaller fundamental frequency. Thus for example, if x t equals x t minus T and x dash t equals x t minus T dash, where T dash is greater than T, then the fundamental frequency in the FS spectrum of x dash t namely omega naught dash equal to 2 pi by T dash is less than omega naught which is equal to 2 pi by T.

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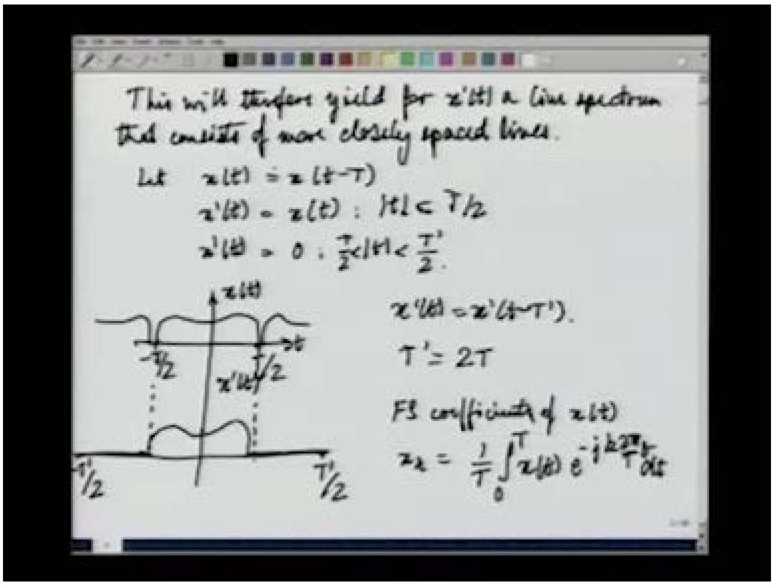

This will therefore yield for x dash t a line spectrum, that is or that consists of more closely spaced lines why simply, because for x dash of t not only is the fundamental frequency omega naught dash less than omega naught, which was the fundamental frequency for x t, but successive lines are found at 2 omega naught dash, 3 omega naught dash minus omega naught dash and so on. So, all the lines get closer to each other in the Fourier series spectrum of x dash of t.

So, now let us see where we can take this forward to, we now have an understanding of the way this whole this behaves, we have an understanding of how the spectrum changes if the period changes. So, the next thing we will do is take construct the story of two signals, which I will define as follows. Let x t be equal to x t minus T that is it has a period of t, fine. Now I will construct another signal now before that let me say that this is a a periodic signal which is of basic interest to me, I will now construct another periodic signal that is defined identically with x t over the period 0 to minus T by 2 to 2 T by 2, and elsewhere it is defined differently it is defined as zero.

So, what I will do is this? x dash t equals x t for mod t less than T by 2. Now, I will say that x dash of t equals 0 over the interval mod t T by 2 less than mod t less than T dash by 2, this means well, let us make as example sketch of this whole thing. So, things will become clear, let us say that this is t this is minus t and x t, sorry. Let me just make a change, let this be t let this be minus t, and let as therefore have x t given over this interval as some shape something like this not perfect. So, this is x t.

Now, what I am going to do is x t is itself a periodic signal. So, it would again start like this and go this way it would again start like this, and go this way, because it has a period T. Now, I will take x dash of t, now x dash of t which has a period greater than x t greater than that of x t, let us say that this is T dash, this then is minus T dash, and over this same interval x dash is defined the same as x t. So, it has, but outside this up to T dash x dash of t is 0 up to T dash, actually this is minus T by 2 to minus T by 2, this is minus T dash by 2 to T dash minus T dash 2 to 2 T dash by 2. So, that over one full period and then x dash t is also a periodic signal.

So, let me complete this definition over here by saying x dash of t equal to 0 over the interval T by 2 less than mod t less than T dash by 2, which is corresponding to this interval plus this interval. Now, x dash of t equals x dash of t minus T dash, this is what it follows this is the rule that it follows. So, you have a second signal x dash of t, which is defined identically with that of x of t over the interval minus T by 2 to T by 2 and is 0 outside it. So, these are the 2 signals that we have x t and x dash t, and we would like to understand, how their Fourier spectra relate the Fourier series lines spectral related to each other in order to do this lets for convenience make the assumption that T dash equals 2 t.

Now, let us first ex tract the Fourier series components of x t x k, which are the Fourier series components of x t, FS components another co-efficients. So, this is what we have? Now note that, since x t is a periodic signal, the integration to evaluate the Fourier series co-efficients, which we have been doing from 0 to t, could well be done over any interval any consecutive interval of t, it may not be from 0 to t, it could be from any point t naught to t plus t naught and we would still evaluate the same coefficients x k, let me write this down.

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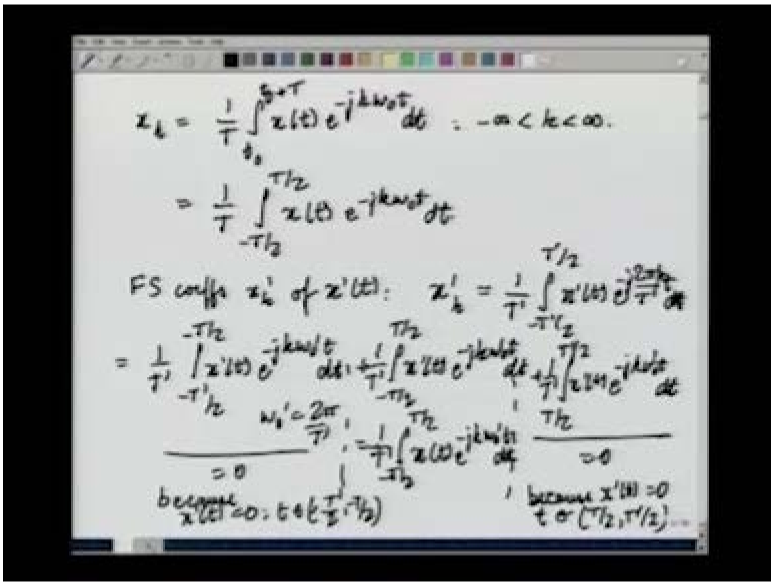

x k equals 1 by T integral t plus, let us say t 2 there is a t naught to t naught plus t, that is any contiguous into a left length t x t e to the minus j k omega naught t dt, and this e positive would hold for k. So, I could choose for example, this minus T by 2 to T by 2 which means I am choosing the t naught of minus T by 2 x t e to the minus j k omega naught t d t, suppose I choose this.

Now, I have all the Fourier series co-efficients x k corresponding to the signal x t, right. Now, let us look at the Fourier series co-efficients x k dash of x dash t, they would be given by using the same argument, and using the limits minus T dash by 2 to T dash by 2, we would have x dash k equals 1 by T dash integral minus T dash by 2 to T dash by 2 x dash t e to the minus j 2 pi k by dash T dash t d t, that is what you would get? But note that x dash t as defined over the T dash length contiguous in 12 minus T dash by 2 to T dash by 2 has the same value of x as x t over the interval minus T by 2 to plus T by 2, and is anyway 0 elsewhere. Hence this integral can be equally written as 1 by T dash integral minus T dash by 2 to minus T by 2 x dash t e to the minus j k omega naught dash t d t, where omega naught dash equals 2 pi by T dash plus integral 1 by T dash again of course, integral minus T by 2 to T by 2 x dash t e to the minus j k omega naught t d t plus the last term 1 by T dash integral T by 2 to T dash by 2 x dash t e to the minus j k omega naught t omega naught dash t here also omega naught dash t d t.

Now, this is 0, this is also equal to 0, because x dash t equal to 0 in this interval for t e minus T dash by 2 to minus T by 2, this is 0 because x dash, fine. So, that gets rid of this term, that gets rid of this term. Now finally, just the middle term, and this middle term equals 1 by T dash integral minus T by 2 to T by 2 of x t, because in this interval minus T by 2 to T by 2 x t, if you go back to the earlier equations over here. x t has been defined to be the same as that x dash t has been defined to be same as x t it is minus j k omega naught dash t d t, fine. So, this is what we have let us make sense out of this.

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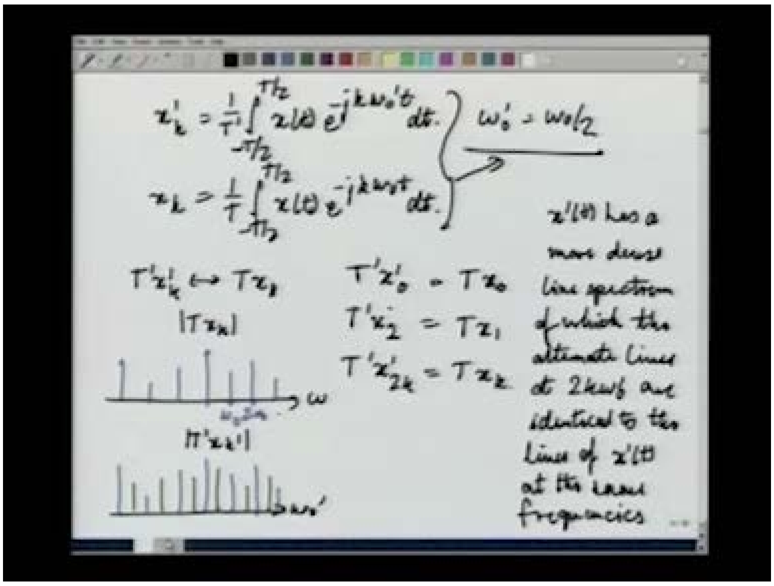

If you look at the equations we would finally get, it is the following we have x dash k equals integral one by T dash, sorry integral minus T by 2 to T by 2 x t not x dash t x t well x dash t also, but at anyway we can simplify to x t e to the minus j k omega naught dash t d t. Now, remember that omega naught dash equals omega naught by 2, this is something we have on the site compare this with the equation, x k equals 1 by T integral minus T by 2 to T by 2 of x t e to the minus j k omega naught t d t, compare this in light of this fact. And you get to see that if you take T dash x dash k and compare this with t x k, then you see the following T dash x dash 0 equals t x 0 T dash x dash 2 equals t x 1 T dash x dash 2 k in general will be equal to t x k. That means to say that alternate spectral lines in the Fourier series spectrum of x dash of t will be exactly the same as the alternate spectral lines, as as the spectral lines of x of t that is all spectral lines which occurs at multiples of omega naught which is 2 times omega naught dash will be the same for both the functions x dash t and x t. However, x dash t has additional lines that lie in between the original lines, the original lines contributed by x t and these additional lines occur at odd multiples of omega naught dash, such as omega naught dash, 3 omega naught dash, minus omega naught dash, and so on, fine.

So, let us put this down, let let us make a sample spectrum. And then we will know what we mean I am going to plot mod x x k; however, this has also happened please note when we have taken T dash over to the other side, we are not comparing the x k s anymore we are comparing actually T x k's. So, let us write that down over here, we are comparing T x k with T dash x k dash, and now when you compare the T x k's or different values of omega, you will get a a spectrum like this occurring that omega naught 2 omega naught and so on. Now, let us make the plot of the Fourier spectrum of T dash x k dash these will have the same lines at frequencies omega naught as x of t hat, but there will be additional lines.

So, let us draw the original lines first by copying these lines over from here, these are at the same frequencies multiples of omega naught, but then you have these additional lines in between the original lines say something like this, maybe something like this, this. Thus x dash T has a more denser spectrum dense line spectrum of which the alternate lines at 2 k omega naught dash 2 k omega naught dash are identical to the lines of x of t at the same frequencies the fact that it has more lines in addition makes it, what we call a more highly resolved or more finely resolved spectrum than that possessed of x t.

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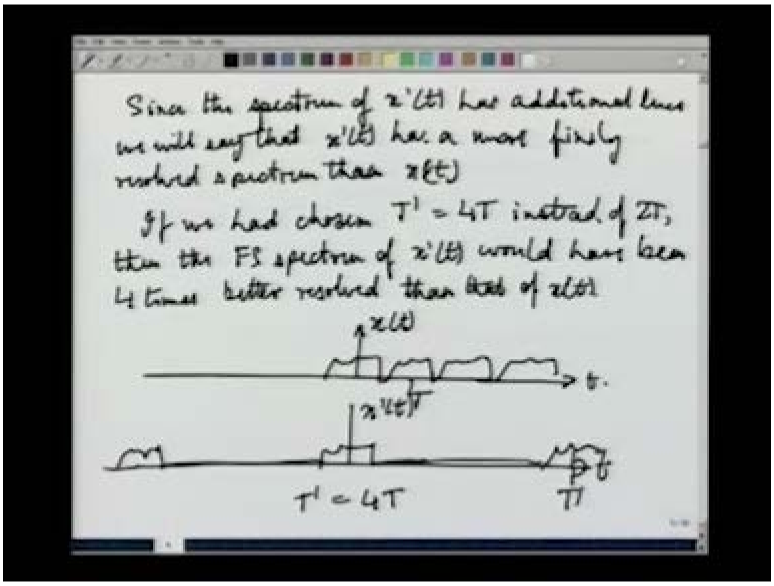

So, let us put this down since the spectrum of x dash t has additional lines, we will say that x dash t has a more finely resolved spectrum, then x of t and it became more finely resolved simply by changing the periodicity of the signal without changing the shape of the signal adding it with extra 0 space, and doubling the period from t from capital to capital T dash. If you wish you can go back to the definition and take a look this was the definition of x dash t with respect to x t.

Now, this same argument can be extended and it is easy to see that if we had chosen T dash equal to 4 T instead of 2 t, then the FS spectrum of x dash t would have been four times better resolved would have been four times better resolved than that of x t. Then we would have had a picture that goes like this x t would be a function that say. So, x t would have a shape like this followed by shape like this repeating itself well, I cannot write perfectly, but it it its quite clear that I want to repeat the same thing again, and again whereas x dash t would have a far more sparse repetition.

Then a gap for three of these, and then again making a copy here and again gap on this side for three periods and then having a copy here, this would be x dash of t, this is what would happen? If you set T dash equal to 4 T, because this would be T dash T dash whereas, this is t over here. So, T dash is four times t, and if you had this you would have a spectrum for T dash that has four lines in place of every single line of x of this spectrum of x t in short x k’s.

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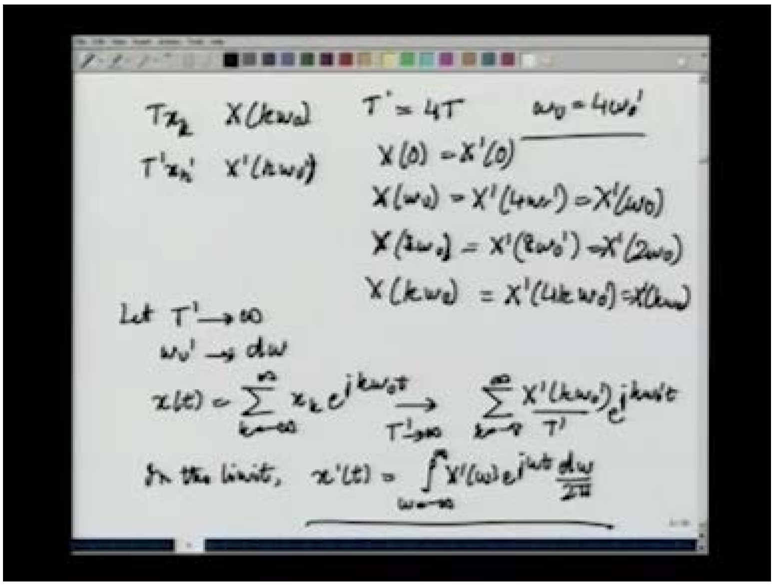

I am from, now on I will call the spectral lines T x k by the name x of k omega naught, because they are occurring at k omega naught and the spectral lines T dash x k dash, I will call this as x k dash k omega naught dash this is just a notation. I am going to use now using this notation it is evident that with T dash equal to four with T dash equal to 4 T, you would have x 0 equal to x dash 0, then x omega naught equal to x dash four omega naught dash equal to x dash omega naught. Because now remember that omega naught equals four omega naught dash, because of this. And next x 2 omega naught equals x dash 8 omega naught dash equals x dash 2 omega naught and so on.

So, in general in general you you would be able to write x of k omega naught equals x dash 4 k omega naught dash equals x dash k omega naught. So, this is the kind of spectrum you would get more, and more highly resolved spectrum results when you increase the period of x dash t with respect to x t. Now, we we just want to follow up on this lead, and find out what happens if we try to make T dash go to infinity. So, let T dash go to infinity, if T dash goes to infinity you will see that omega naught dash goes to an infinitesimally small component which we will call d omega, fine.

And then the summation of a countable number of discrete frequency components that existed to construct to synthesis x t becomes for the case of x dash t and integration. So, what you have for x t is x t equals summation k equals minus infinity to infinity x k e to the power j k omega naught t this transforms to as T dash tends to infinity an integral the x k's. Now will be replaced by x dash k omega naught dash by T dash, we will make it go towards within the limit due to the j k omega naught dash t, fine. Now, in the limit this becomes x dash t equals integral omega equals minus infinity to infinity, because essentially we were having discrete line frequencies k omega naught dash and omega naught dash is getting smaller, and smaller. And becoming d omega and x dash k omega naught essentially simply becomes x dash omega the continuous variable omega e to the j k omega naught t simply becomes e to j omega t, and this 1 by T dash where T dash is tending to infinity simply becomes d omega by 2 pi. So, this is the expression for x dash t the synthesis expression for x dash t when x dash t becomes non periodic.

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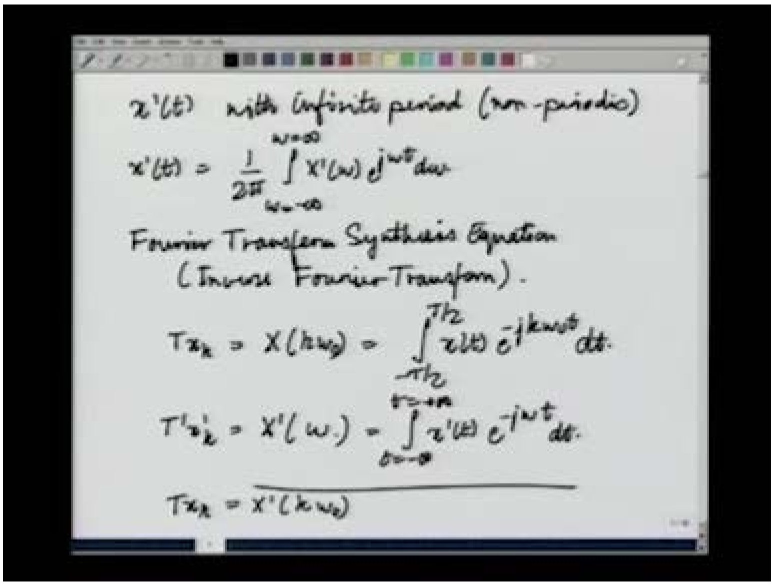

So, to summarize we have x dash t with infinite period that is to say non periodic alright, it has infinite period and is given by this is nothing but the synthesis equation or the inverse Fourier transform equation. What we have just discovered is the Fourier transform though we have actually landed up directly with the inverse Fourier transform.

So, this is called the Fourier transform synthesis equation, because it synthesizes the function Fourier transform synthesis equation or the inverse Fourier transform, what about the analysis as long as we were dealing with the Fourier series for a periodic signal. We had discrete Fourier series coefficients x k which when multiplied by the respective complex exponential e to the j k omega naught t gave us, what we call the

Fourier series components. So, the co-efficients over the x k s the components for x k multiplied by the respective complex exponential, this was the case for the Fourier series.

Now, from the equation of the synthesis we find that components are no longer at discrete positions alone, but they are everywhere for every value of omega, you have some components. And hence we have to ask the question what would be the form of the analysis equation here the analysis is nothing but evaluating x dash of omega for every value of omega, we want an expression for x dash of omega remember that t was taken over to the other side. So, we just had let us write the old equation first. So, that we remember what we had we had t x k which is nothing but x of k omega naught, and this was given by integral minus T by 2 to T by 2 x t e to the minus j k omega naught t d t this is what we had…

Now, T dash is standing to infinity and hence T dash x dash k, which is nothing but x dash of k omega naught will be equal to as T dash tends to infinity the limits of the integral also go to infinity. We are actually carrying out a co-relation as I said, but now a co-relation over infinite time of t going from minus infinity to t going to plus infinity x dash t e to the minus j k omega naught, now merges into the continuum omega t integrated with respect to t.

So, this is what we have this x dash k omega naught, if evaluated at frequencies k omega naught, you will get the same thing as this. And if evaluated for all values of omega, you will just get x dash of omega all we have done is to take the original x t take just one period of the original x t. And instead of making identical copies of it to construct a periodic signal on both sides of the time axis, we have replaced all the periodic copies by 0 to realize a non periodic signal. And therefore, there is still reason to ask if there is any relation between the T x k's of the original x t and the x omega x dash omega of the non periodic modification of x t. So, we have t x k on the one hand side and what is this equal to this is equal to x dash of k omega naught, where omega naught is the fundamental frequency of the periodic signal. So, let us make a plot of this to make an idea.

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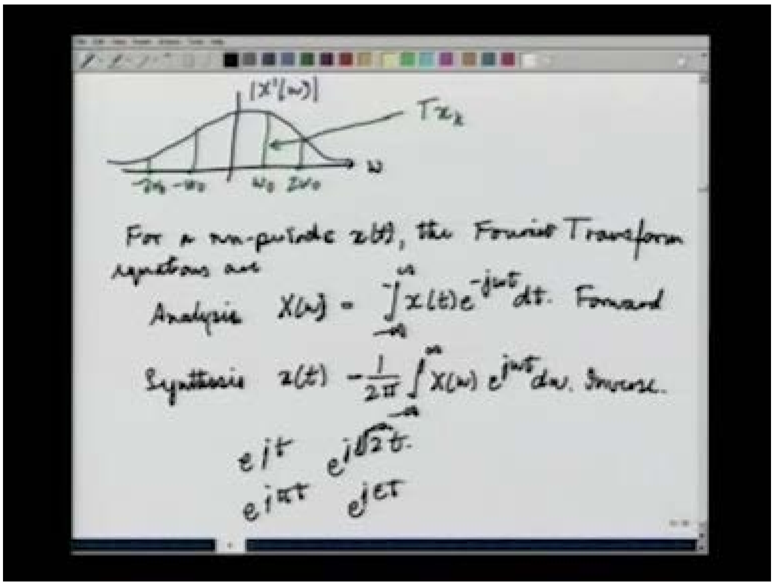

We now have a continuous spectrum, and what am I actually trying to plot is against omega I am going to plot mod x dash omega. So, let us plot it in color maybe something like this, now how does this relate to the Fourier Fourier series terms, Fourier series components of the original x t, it is like this if the period was some t. So, that the there was a fundamental frequency of omega naught, and let us say this was omega naught 2 omega naught minus omega naught, and minus 2 omega naught then the old spectrum which I am now plotting in green that is T x k's are in green then you will have I, then taking exactly these values at these points. So, this is the discrete lines spectrum in green, and how it relates to the continuous spectrum the infinitely resolved spectrum of x dash of t as shown in blue. So, let me let me just write that in blue, these are the lines spectral lines, and what we have I have here is mod x dash omega the periodic signal summarizing.

We now see how the periodic signal with its line spectrum has been transformed into the non periodic signal with the continuous spectrum, let us just put down the analysis, and synthesis equations for a non periodic x t the Fourier transform equations, equations are analysis x omega equals integral minus infinity to infinity x t. And infinite time co-relation minus integral minus j omega t d t, and the synthesis sorry x dash x of t given by one by 2 pi integral minus infinity to infinity, where omega is the variable of integration. Now, of x omega e to the j omega t d omega, this is what we have in the transform domain, the inverse transform this is called the forward transform, and this is called the inverse transform this is called the inverse transform.

So, together we have completed the story of the construction of the Fourier transform in the form of evolving it by by a means of evolving it from the Fourier series, but then there are some interesting troubling questions to ask we had said, now before we ask the question ask those questions. Let us look at what we have. Now, we have a linear combination, if you look at the synthesis equation over here a linear combination though, now the linear combination is been in the form of a integral rather than a summation, we have a linear combination of what are still periodic complex exponential functions e to the j omega t. And this linear combination of periodic complex exponentials is yielding at x t which is non periodic, but how is that possible had we not already said at one time the sum of periodic functions will always be periodic.

Then how come we are adding e to the j omega t with weights x omega, and generating a non periodic signal x t, how is it the addition of periodic signals is yielding a non periodic signal, this is something that we really should wonder about if that is we should wonder about, if we do not remember exactly what we said earlier we had not really said that the sum of 2 periodic functions is always periodic, we had only said that the sum of 2 periodic functions is periodic. If the periods of two functions being added are harmonically related here, that is clearly related not the case because e to the j omega t is a periodic function, but omega here can take all kinds of possible values; for example, you could be combining something like e to the j t with e to the j root 2 t.

Now, in one case omega is equal to one and the other case omega equals root 2, they are certainly not harmonically related. So, there is no problem such 2 signals, if they are added together will not yield a periodic signal or you could have e to the j pi t, and you could have e to the j e t where by e here, I mean the same base of the natural logarithm these two are again 2 irrational numbers, they are not harmonically related to each other their sum will not be a periodic signal. So, there is no inconsistency here that is all I wanted to point out.