Signals and Systems
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Lecture - 26
Fourier series

This gives us now both a formula for constructing the representation or doing the synthesis as well as for finding the weights of the representation, which is called doing the analysis.

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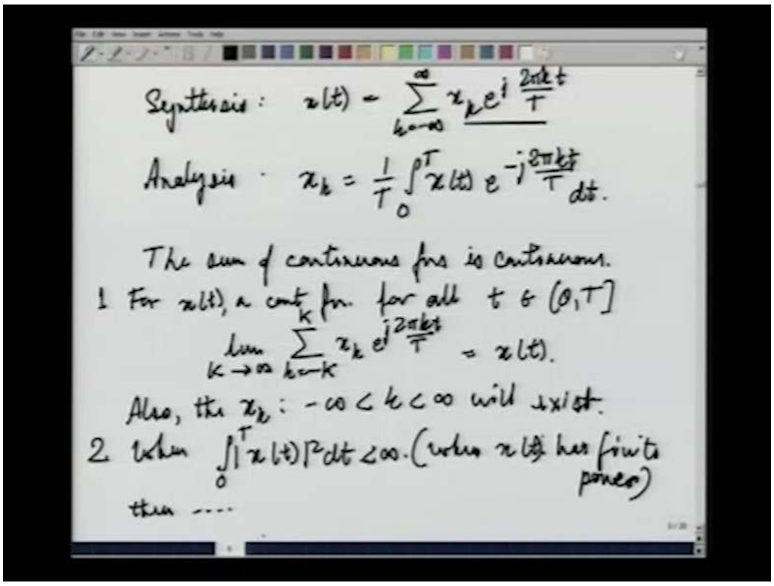

So, synthesis as we said x t is given by this kind of a sum k equals minus infinity to infinity x k e to the j 2 pi k t by T it is a sum of exponential functions. Each exponential function is periodic and harmonically related to the x t in question. Then we have the analysis equation x k is given by 1 by T integral 0 to T x t e to the minus j, the minus is there because the complex conjugate the second member of the co relation 2 pi k t by T d t.

So, now we have defined the Fourier series with these two equations, the definition of the Fourier series is complete. But as I said there are still a lot of questions to answer lots of things to explore. For example, we would like to know what sorts of signals this thing is definitely this approach is going to work for what sorts of signals, this approach will work. Let us take a look at certain obvious properties. Now one of the facts common sense facts is that when you add continuous functions of time, the result is always a continuous function of time.

So, if this is true. Then since each component in the synthesis is a continuous function because periodic complex exponentials are imminently continuous functions, extremely continuous functions they do not show discontinuities anywhere at all. Then their sum must also be a continuous function. And hence it is not possible at this level of analysis; at this level of reasoning to ever believe that x t such as a square wave can be represented using the Fourier series. Because if we could do that then clearly we are constructing a non continuous function, a discontinuous function using continuous components. That is just not possible, but there is something else to look at something that should show for us down. And that is the fact that while the sum of finite number of continuous functions is continuous.

Here we are constructing a sum of an infinite number of continuous functions. And maybe something funny is happening over there that allows us to make a discontinuous function by adding up continuous functions. If you just add large numbers of them in sufficient numbers. Now, this is not a discussion that would be very convenient to do at this level in any kind of detail, because it goes into all kinds of intricacies in the mathematics of Fourier analysis. This is really not expected in the course at this level. So, all I will do is to say that continuous functions if x t is a continuous function then there is absolutely no problem in representing it. Using the Fourier series you can even say several more things about it. For example, you can say that if x t is continuous then for every t in the interval 0 to T the partial representation. That is the truncated representation of the Fourier series will come closer and closer to x t.

Let us put this down for x t a continuous function for all t in 0 T that is in one period. It is the case that limit as k tends to infinity of k equal to minus capital k to plus capital k this is also limit as capital k tends to infinity x k e to the j 2 pi k t by k equals x t; that means, for every point of time over which x t is periodic that is from 0 to T this representation will converge to x t. So, this is the simplest and the best case.

Now, there are other things that can be said about the existence of the x k s so this is point number one. Point number two existence of the x k s by the way for the case one, when x t is a continuous function. We will also say before we start on point number two that will exist. Second we have in fact, we will take up the discussion of when these x k s will exist what is the necessary condition to be satisfied for the x k s to exist. The necessary condition is the following. When integral 0 to T x t when this is finite that is when x t has finite power then we can say the following. This representation minus the original x t will have no energy. Also all x k s will exist.

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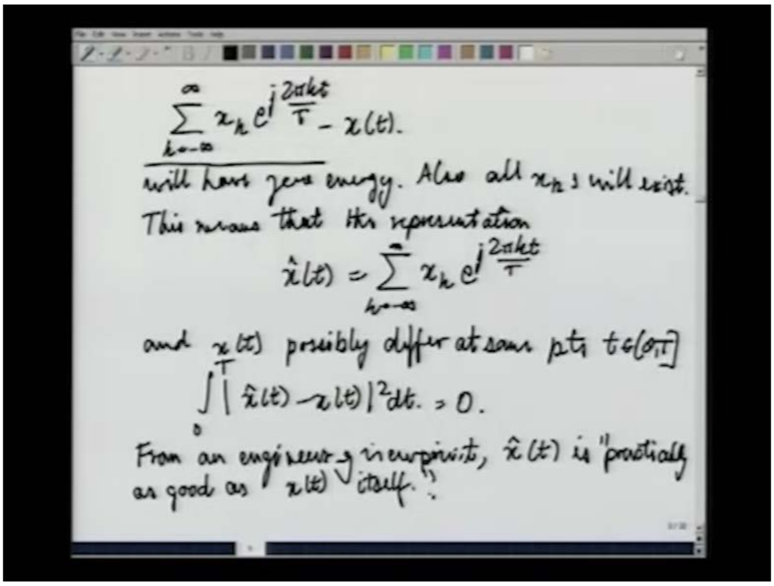

Now, there are two different things being said over here. First that the coefficients of the representation will exist when x t has finite power. The second thing is when x t has finite power. Then the difference between x t and its representation will have zero energy. Now that is a very ominous thing that is almost like saying. In fact, it is saying that this representation need not necessarily be equal to x t at all points T in the interval 0 to T. This means that the representation and the original signal possibly differ at some points.

This is nothing to get alarmed about. Now, remember that for the case to that we are presently discussing we have not imposed upon x t the constraint that it should be continuous. That was case one when x t was continuous so you had all the nice things in life. For case two, we are considering any x t which has finite power without imposing any further constraint on x t itself. For example, x t could be a square wave, which is certainly not a continuous signal. But it still can have finite power. In fact, normal square waves have finite power and therefore, it is designable to Fourier series representation.

However, it is said that the representation will not match the signal probably at all points but its power will be 0. In short we will say that if we evaluate not less than if I write over here we are just saying that x hat t minus x t this whole thing squared d t integrated from 0 to T will be zero.

Now, it is an important fact of engineering life. That when two signals do not differ in energy or do not differ in power. Then for all engineering purposes they may indeed be treated as identical. If they differ in power then alone will the engineering system be able to distinguish the signal from the representation. If they do not differ in energy then the error has zero energy. And therefore, the two signals the original and the representation cannot really be distinguished. That is from an engineering view point x hat t is practically as good as x t itself.

So, while the purest or the mathematician will still find this very exciting. We say well well that good enough for us. If they do not differ in energy then that is fine enough for us. A more precise treatment of I mean you see this point second statement that we have made is not telling us exactly its only telling us that the representation and x t will not be identical to each other.

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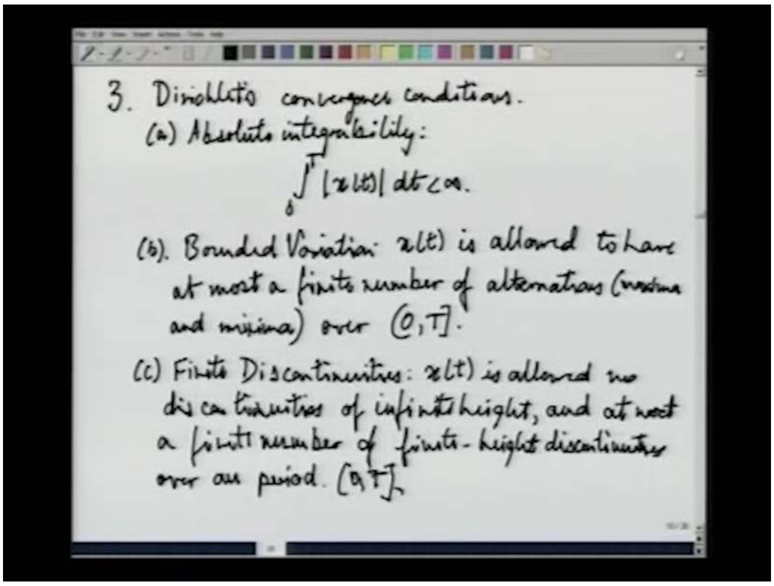

It does not tell us at which point will it indeed be identical it is not telling us any of this information. However, as we said it really does not matter to us as engineers we do not have to be worried beyond this point. However, point three which is discusses the dirichlet conditions is a very nice treatment that uses more precise requirements to meet a certain kind of resemblance. Now, dirichlet conversion conditions essentially impose three requirements on x t. The first requirement is absolute integrability; that means the integral of the absolute value of x t over one period must be finite write.

Now, the second constraint is bounded variation. That is to say that x t is allowed to have at most if finite number of alternations maxima and minima in other words over one period. Third, x t is allowed to have own at most finitely many discontinuities of finite height. x t is allowed to have no discontinuities of infinite height, which is a constraint, but nevertheless the happy constraint because the positive way of looking at this constraint is to say; that means, you are allowed to have discontinuities of finite height. However, is allowed no discontinuities of infinite height and at most if finite number of finite height discontinuities over 1 period.

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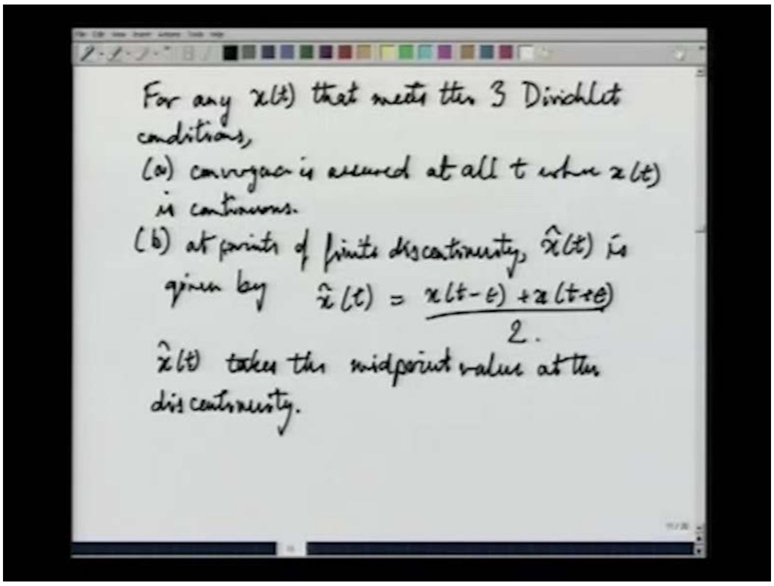

So, three conditions in all… So, what do we get for obeying all these three conditions after all if you obey the rules you must get something for it this is what we get. For any x t that meets the three Dirichlet conditions convergence is assured at all points where x t is continuous at all points t, where x t is continuous. At points of finites discontinuities at points of finite discontinuity x hat t is given by x hat t equals x of t minus epsilon plus x of t plus epsilon divided by 2. So, it just the x hat t takes the midpoint value at the discontinuity. So, the Dirichlet conditions of probably the most transparent and frank discussion of the convergence properties of Fourier series.

If we look at convergence issues in terms of the Dirichlet conditions we not only get to know whether it will converge. We also get to know where it converge and where it will not converge and also beyond all this. How much are what value x hat t will have at those places where it does not converge. There is almost nothing else left to be said except of course, if curious to ask what happens if it does not satisfy all the three Dirichlet condition.

Now, suppose you do ask that question then the answer is very, very long. It goes on and on for the different cases of different kinds of complications. All we can say is that the guarantees a and b given in this page regarding dirichlet conditions will not be met. So, it is difficult to discuss such signals, now so much about the properties of the existence and conversions of the Fourier series. Now, we will go on to one more thing. Question b the only thing that was left or rather I do not know if it was question c, let me just go back couple of pages and take a look. Now, the question of what we can say about the truncated representation as we go on adding more and more terms to the truncated representation to bring it to the complete representation.

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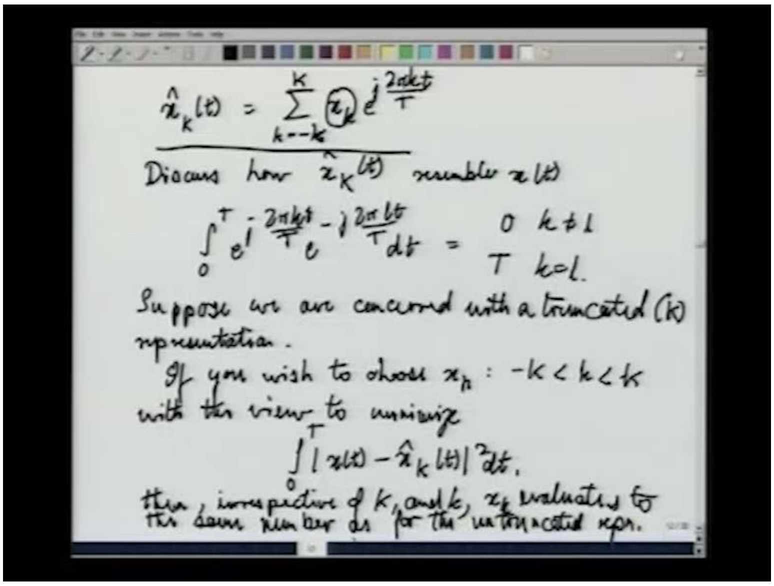

What can we say in short of the value of what I will call x hat capital k of t given by the sum small k equals to minus capital k to plus capital k of x k e to the j 2 pi k t by T. Now what I want do, therefore ask is I want to discuss how x capital k hat of t which is the k th approximation. You can say the first approximation is the corset, second is a little finer and so on the k th approximation. Discuss how x k of t resembles x t. In order to discuss this there are various straight forward nice properties. That the Fourier series has the most nice properties is that signals taken from different preserved classes absolutely do not correlate it with one another.

In short they are what you call a linear algebra as orthogonal; that means, if you take integral e to the j 2 pi k t by T and correlate it with e to the j 2 pi l t by T that is e to the minus j 2 pi l t by T. Remember I am putting minus sign here because the second member of an inner product has to be conjugated. d t for minus infinity or oh sorry this is from 0 to T. What can we say about this as I was this is something, we have already discussed it is a pretty obvious fact. It is simply that this evaluates to zero 1 k is not equal to l, equals zero for k not equal to l, equals T when k equals l we have this fact.

Now, what this can be shown to lead to is the following fact that if x t has a certain representation in terms of a Fourier series. Then this representation as seen from the truncated series x hat k of t get closer and closer in energy terms or other empowered terms. If you like energy per period terms to x t as we add more and more coefficient as we add more and more terms in the truncated representation. Furthermore the choice of coefficients small x k over here is not determent the choice of best coefficient is not determent by the choice of the truncation limit. Let us put these both these statements down properly.

Now, suppose we are concerned with the truncated representation. Let us say with truncated to k point representation. Then one does need to ask a question since now we have a truncated representation as shown over here. One does need to ask the questions are the x k s that act as weight for combining the exponentials. The same values as you would use for the complete representation. In short the choice of the coefficients the weights of the combination will they change, if you have a truncated representation instead of a complete representation.

Now, the answer to this is very, very nice the answer is no. In short if you only wish to minimize the energy, that is to say if you calculate you wish to choose x k minus capital k less than small k less than capital k, if you wish to choose x k with the view to minimize integral 0 to T mod x t minus x hat k of t square d t that is to say. If you wish to minimize the energy of the error and choose such x k s as to minimize would be an error. Then irrespective of k of both capital k and small k, x k is the same evaluates to the same number as for the untruncated representation. The coefficients do not have to be recalculated. Thus out of say practical considerations we wish to represent an x t with the first hundred Fourier coefficients may be you do not have enough money to compute more coefficients. But later on tomorrow you are richer and you want to have a more accurate representation for x t.

Then you do not have to go back and recalculate the old x k s. They will continue to hold they are still the best x k s you will have. You can now calculate another hundred coefficient terms and have a two hundred term representation for x t. The new hundred coefficients will certainly have to be obtained, but the existing hundred coefficients which we use to construct the hundred point truncated representation will not be affected by extension of the representation to 200 coefficients. That is the crux of the fact that this is an orthogonal representation.

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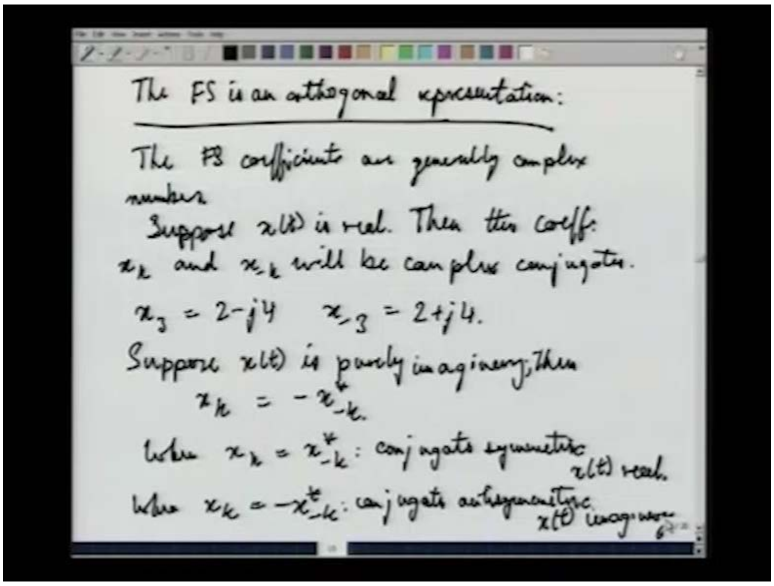

And I said, it simply means, that if you take two of the representing functions of two separate preserved classes. Then their inner product over one period evaluates to zero. If they are only from the same preserved class you get nonzero coefficients of co relation. So, their coefficient of co relation is zero when they come from different preserved classes. So, it is because we have an orthogonal representation. That the choice of coefficients that you will use for a truncated representation is the same that you use for the un truncated representation.

So, that is an important milestone in our study of the Fourier series. Now let us see, whether we can say something more about the Fourier series how to take it further, for example as to solving examples to obtain in the Fourier series representation of various functions. We are not going to do that here there are enough text books this video course is meant to support information that you can already get from books. So, we now going to spend time solving simple examples over here.

Now an important property of the Fourier series coefficients is that they are generally complex numbers. And because they are complex numbers we will have to see how when these complex weights are used to generate a combination of the preserved bases functions. How it is possible construct a real function time in x t. Suppose an x t is real that means, the Fourier series evaluated over all the values of k should adapt to a real number equal to the value of x t for all T between zero and capital T, that is for all T for 1 period. The Fourier series should not make silly mistakes like going wrong and giving complex values for a function which is itself inherently real.

Now how does this really happen, it happens by the fact that the coefficients x k for negative k and the coefficients x k for positive k are complex conjugates of each other when x t is real. The coefficients x k and x of minus k will be complex conjugates. Thus for example, if you have x 3 equal to 2 minus j 4 then x minus 3 will be equal to 2 plus j 4. It is because of this pattern followed by the coefficients that the complex valued functions bases functions add up. So, as to cancel their imaginary parts and leave behind a purely real function x t real cause we started with a real function.

Now, suppose we started with a purely imaginary function, then what happens? Suppose x t is purely imaginary, then pairs of coefficients x k and x minus k would be conjugates and have a minus sign. x k equals minus x star of minus k. This is the only way you can make sure that the real parts cancel leaving behind only the imaginary parts. Of course, this ensures the second part ensures that x t is purely imaginary, the first the earlier case we discussed ensures that x t is purely real. So, that is how this is ensured.

Now, there is a nice name we have for this relationship between the corresponding pairs of Fourier series coefficients for k positive and k negative, when x k equals x star of minus k which is the k s for x t real we say that they are conjugate symmetric and that is when you have x t real. On the other hand, when x k equals minus x star of minus k we say that the conjugate anti symmetric. And x t this corresponds to the case of x t imaginary fine. Since, we have an array of discrete co efficient index for different values of k, k running from minus infinity to the infinity.

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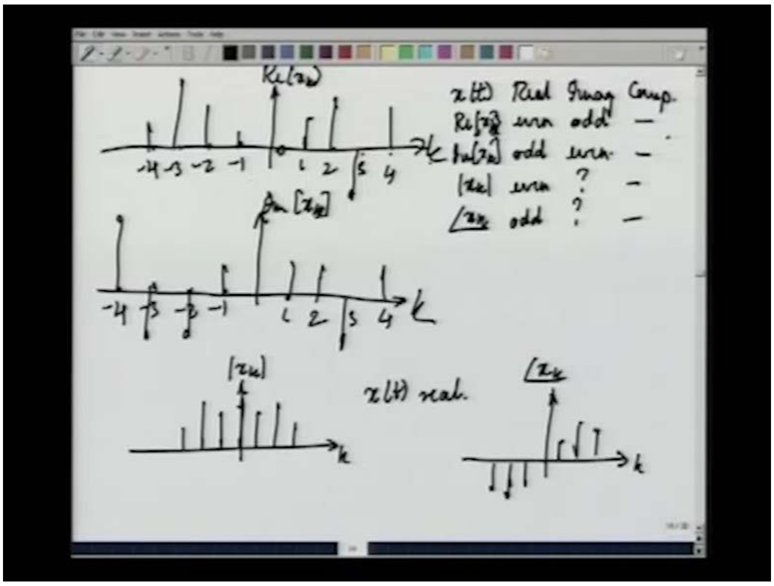

It becomes worth looking at making a plot of these x k s for different values of k. If you made a plot like that, it would ideally have 0 in the middle and for different values of k you will have k equal to 1, 2, 3, 4 on this side, minus 1, minus 2, minus 3, minus 4, on this side. Now, what can we plot on this. We really cannot plot x k itself because x k is a complex number. And you cannot just put down only the real part or only the imaginary part of complex number. You really have to make two plots therefore, this you can say is real x k.

And then you can say its value here is this much, this value here is this much, this value here is this much, it can be both positive and negative value here is so much. Here you can say its value here is this much, this much, this much, this much whatever. We have made similarly you will have a different simultaneous plot for the imaginary part. Here again you will say it is this much over here, this much over here, this much over here, this much over here for 1, 2, 3, 4 and may be for minus 1, minus 2, minus 3, minus 4 you can have some values like again this, this, this, this whatever.

So, you need 2 plots to completely describe these. And these are the essentially discrete plots. The sorts of plot you would make for a discrete time signal you are making then for the Fourier series representation. It is a discrete sequence indexed by k going from minus infinity to infinity. Alternately you can have what is called a polar plot. Well, it is not really a polar plot so much as a plot where you express the x k s not in terms of the real and imaginary parts, but instead in terms of the magnitude and phase.

So, you would have one plot, where you plot mod x k and another plot, where you would have angle x k fine. This is what you would have; now you see essential feature is this also is discrete, mod the magnitude plot is also discrete, the phase plot is also discrete, because it is available only for a discrete values of k. So, you will get some plot which look like the real and imaginary plot which we have made, but then certain patterns can be discovered. When x t is real conjugate symmetry it holds and in such a situation you will find the mod x k is even let us list all these properties over here. Let us say x t real and here we will say imaginary then real x k.

So, this is for x t real or x t imaginary. Real x k will be even imaginary x k will be odd; that means, to say x k will be equal to x of minus k, real x k will be equal to real x of minus k and let us list all these properties over here. x t can be purely real, purely imaginary or neither, that is generally complex. So, let say x t is real, imaginary or complex, fine. Now when x t is this we will look at the symmetry properties of the real part of x k, the imaginary parts of x k, the magnitude parts of x k and phase parts of x k. So, let us consider real x k, the real part of x k will be even when x t is real. The imaginary part of x k will be odd when x t is real. The magnitude part of x k will be even when x t is real. And the phase part of x k will be odd when x t is real. This is the set of properties you can expect when x t is real.

When x t is imaginary precisely the opposite happens for example, the real part will be odd, the imaginary part will be even, as to the magnitude and phase I will let you find out by yourself. What it is going to be please remember that the magnitude function can never be odd because odd functions have to be negative in some places at least. And magnitude functions are never odd. So, try to figure out what you will have to write here and what you will have to over here, what will be the magnitude and what will be the phase. What properties will they have what symmetry properties they will have.

Now, coming to the third column, where x t is complex this is the most general case. And so generally no symmetry properties may be expected all we can say is that dash, dash, dash, dash you cannot say anything about this symmetry properties for the general complex case. Now, if for the real valued x t which is what we often like to deal with real life. We are to make a magnitude plot it would like this symmetrical as you can see conjugate symmetry or simply even. Now, the phase on the other hand would be like this. This is also phase would be. So, this is just an example for x t real. So, now we know how to make magnitude and phase plots. Now, let us just start looking at what happens to the Fourier spectrum if time undergoes certain changes. For example, let us first try to relate what these positions are along the k axis.

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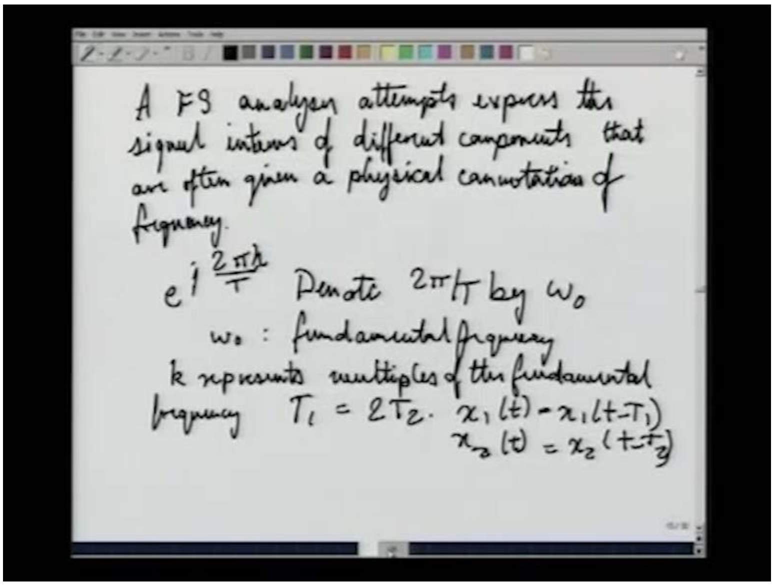

What do the cases really stand for the k s the different values of k s stand for different so called frequency. A Fourier series analysis attempts to express the signal in terms of different components that are often given a physical connotation of frequency. That means, to say that we say that these different Fourier series coefficients corresponds to the contribution of different physical frequencies to the signal x t. Now there can be a lot of debate about taking this meaning of frequency, but let us not enter into the debate. Because it is very very widely held that have certain physical meanings they do indeed have physical meanings, but this notion of relating frequency or rate if change to different Fourier series coefficient is problematic at certain level. I will merely say this and leave the debate at that.

Now, let us instead look at what happens to these different frequency coefficients. We see that essentially in the frequencies or in the parameters of the exponential that are used in the representation. We can indeed give a connotation of frequency to the parameter. when you have e to the j 2 pi k T as a certain periodic complex exponential. We can denote 2 pi by T by some frequency omega naught if you give this notation of frequency. If we write this as omega naught then omega naught will be called the fundamental frequency.

Clearly, the larger the period of x t the smaller the fundamental frequency, because is 2 pi by T. where, T is the period of x t. So, if T is large omega naught gets small. This allows us to compare these spectra of two signals with different periods. And I am just going to make some sample spectra of two signal with different periods, so that we can look at them. I will just make the magnitude plot not the real or imaginary or the phase, just the magnitude plot to show what happens. The second thing is what this k is doing over here.

k represents multiples of the fundamental frequency. So, you have constraint multiples increasing integral multiples of the fundamental frequency that contribute to the Fourier spectrum of any x t periodic x t. So, let us make let us say T 1 equals 2 T 2. And we are concerned with x 1 of t equal to x 1 of t minus T 1, x 2 of t equal to x 2 of t minus T 2. Suppose we have T 1 and T 2 given in this format.

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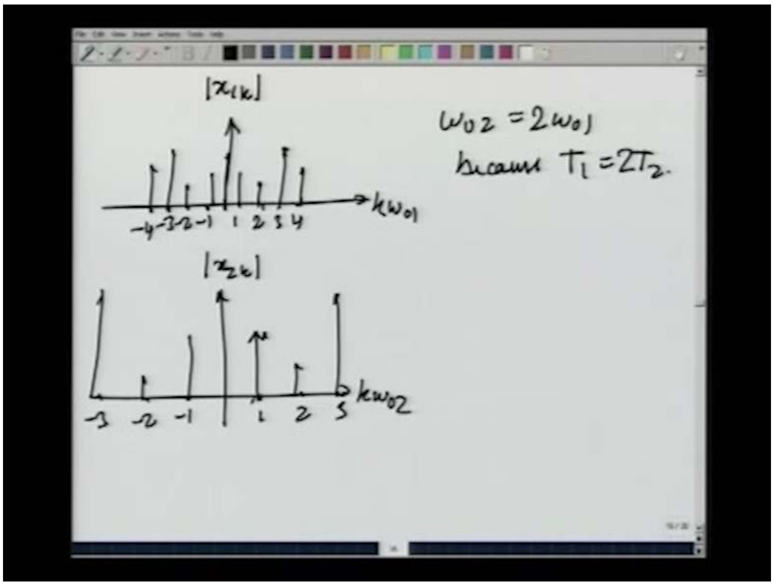

Then let us try to understand, what the Fourier spectra are likely to look like. Suppose I plot mod x 1 k. Remember that T 1 equals two times of T 2; that means, omega naught 1 which is the fundamental frequency for x 1 of t will be half of the fundamental frequency of x 2 of t. So, here you have k or we can write this as k omega naught and write this as 1, 2, 3, 4, minus 1, minus 2, minus 3, minus 4 and so on. So, then you will find that you have pillars in each of these places I am trying to represent a real valued signal. So, it will be conjugate symmetric. No negative values because it is the magnitude plot magnitude is never negative fine.

Now, mod x 2 sorry x 2 k here we had k omega naught 1. Now we have k omega naught two, but omega naught 2 equals 2 times omega naught 1 because T 1 equals 2 T 2. Keeping this in mind these points at which we make the Fourier spectrum plot must be chosen twice further apart from each other then was the case for x 1. So, you will have 1 over here, 2 over here, 3 over here far apart from each other minus 1, minus 2, minus 3. Then you would have some plots like this. The fundamental difference between this plot and the previous plot is that the magnitude spectrum of x 2 T, x 2 k looks extremely sparse compared to that of x 1 k.

Now, this is the important significant point about the effect of the choice of the period effect of the period of the periodic signal upon its Fourier spectrum. From now on we will not plot Fourier spectra for Fourier series using k as the index, but using k omega naught as the index. Because the moment we do this we have a handle on the actual physical frequencies that are involved. We wouldn't have that if we just dealt with k and suppressed omega naught.