Signals and System
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Lecture - 25
Representation of Periodic Signal

So until now we have spent a lot of time and effort on discussing the fundamental principle of signal representation. In fact that is what I will call it for rest of the future the fundamental principles of signal representation. We have for example to summarize said that instead of dealing with the understanding of a processor in terms of what it does to each possible signal in the signal space. In terms of how it maps it to the output signal, we will find it much more convenient to decompose the signal space into small classes, if not decompose it at least to identify small preserved classes of signals inside the signal space.

So, that given the system on hand any signal in a preserved class is preserved in the same class under operation by the processor, under the operation by the system. Next we said that arbitrary signals not found in any of the preserved classes should be representable as a combination probably a linear combination, but not necessarily a linear combination; a combination of members of the preserved classes.

If this could be done, and if the system was invariant to the combining process that is to say if you combined and processed a signal, a collection of signals it was the same it would yield the same result as processing first and combining later. Then we have the advantage of finding a representation for an arbitrary signal in terms of representatives one each from each of the preserved classes.

So, when you have one representative each from the preserved classes and their combination is the signal of interest the input signal of interest, and equivalent to applying the input signal of interest to the system would be to apply each member of the representing classes, that represent this particular signal to a copy of the system at hand. And processing each such component separately. Since each of these members is found in a particular preserved class, the corresponding outputs out of these several copies of the system at hand would also lie in same preserved classes, and these outputs would have to be combined in the same manner that the inputs were combined the input components were combined to generate the input. In other to constitute the complete output signal, that is the whole idea behind signal representation, but this level of discourse is extremely general. And we will now proceed to make this more specific and more related to the normal working context of people who work with signals and systems. Well there are two essential points that I would like summarize as the lessons of the preceding discussion.

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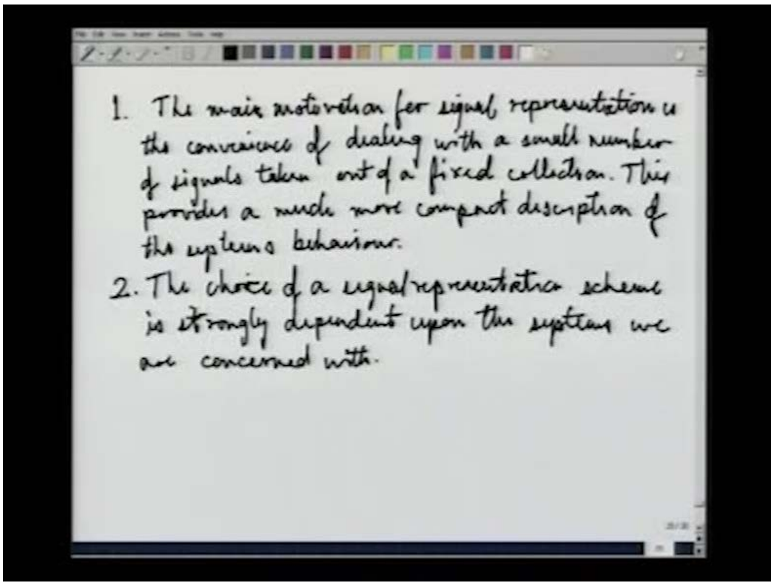

The first point is the following we have discovered that the main motivation for signal representation is the advantages the convenience of dealing with a small number of signals taken out of a fixed collection. The essential convenience is that we have a much more compact description of the behavior of the processor. Next and this is probably very important particularly, so because most text books do not lay in my opinion sufficient emphasize on this very, very important point.

The fact that the choice of a signal representation scheme is strongly dependent upon the systems, we are concerned with. This is not a choice you make irrespective of or independent of the system you work with, for a certain family of systems that you want to work with you choose a signal representation that is appropriate. However, it is understandable why most people do not emphasize this point, it is because most treatments of signal and system theory are practically confined if not entirely confined to the study of signals processed by linear time in-variant systems.

Systems of other kinds and there can be a large verity of systems of other kinds. In fact, one can even say that the family of linear time in-variant systems is a very smaller sub set of the family of all possible systems. Because of this fact that we are pre occupied with linear time in-variance systems practically only this family of systems most people do not even worry about whether signal representations can be proposed for systems which are not linear and time in-variant. So, we just go along with the particular case pretending that it was the only possible case, it is not the only the possible case and there is much larger context a much larger wider theory of signal representation that is possible, though this is generally not discussed ok.

So, these are the two points now moving on as I said to more specific terrains, the kinds of signal processing that we do most of the time that we confine ourselves to most of the time as I said we confine our self to linear time in-variance systems. So, if we have systems with are linear and time invariant, then we would now like to ask the question what are the preserved classes under linear time in-variant processing. Remember that we not only want preserved classes of signals, we want not only want to identify preserved classes of signals, we also want to have preserved classes that are as small as possible. Indeed if we did not impose this secondary requirement that the preserved classes should be as small as possible, though they may be large in number we want each preserved class to be small, because that makes the representation more powerful, more compact, and more convenient.

If we do not impose this requirement, then obviously any kind of processing has at least 1 preserved class, any kind of system associated with every kind system we have the preserved class that is the whole of the signal space, because no system takes an input signal from the signal space, and produce the something, which is not of the signal space. So, clearly there is no big deal it just having preserved classes, because we can always have one single big preserved class which is always available for every kind of system. What we want is small preserved classes? In any case let us start our exploration, we have linear time in-variant systems.

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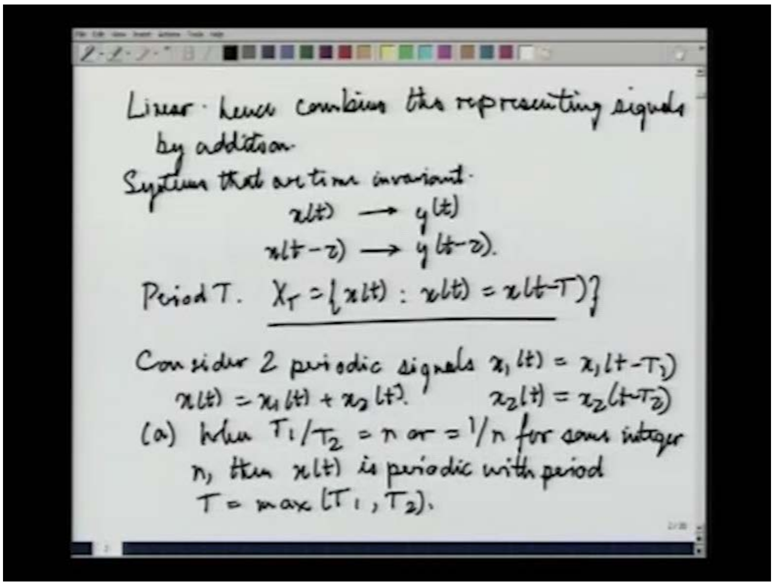

Now, because the systems are linear, evidently the form of combination that we would like to use for signals taken out of different preserved classes representing signals taken out of different preserved classes would be addition, that is we will be combine by addition, that is good we took care of one problem at least. We have interpreted the property linearity appropriately and taken a decision upon how it is going to influence our construction our framework, that is combination will be by addition, because linear system obeys supper position. Next the question of what signals will be preserved by linear time in-variant systems.

Now, let us consider time in-variance by itself for the time being let us consider system switcher time in-variant. That means they follow the principle that if x(t) yields y(t), then x(t) minus tau must yield y(t) minus tau, we have already discussed systems of this sort. And we have even found that they do have preserved classes, which are classes of periodic signals in short, if you take the set of all signals, which have a certain period let us say a period t.

That means consider the set X T of signals x(t) that satisfy x(t) equals x(t) minus T, this is what I will call class capital X T. Now, this x(t) is clearly preserved by time in-variant operation this is something we have already demonstrated, so there is no need to show again periodic signals with period t will be preserved by time in-variant processing. So, periodic signals with all of which have a common period t constitute a preserved class for time in-variant systems this is something we already know, now since linear time in-variant systems are also time in-variant, this same preserved this same class will also be preserved under linear time in-variant processing.

Now, to understand this further and to also give a further shape to the act of representation, let us see what happens? If we add 2 signals - 2 periodic signals of different periods T 1 and T 2; x 1 (t) equal to x 1 of t minus T 1, x 2 t equal x 2 t minus T 2. suppose we have these 2 signals now I want to see what happens to x(t) equal to x 1 (t) plus x 2 t, is it periodic if it is periodic what is it is period these are the questions, we wish to ask. So, there are clearly three cases that are possible the first case is the simplest one, when T 1 by T 2 either equals n or equals 1 by n for some integer n then x(t) is periodic with period T equal to max T 1 T 2, this is the first case.

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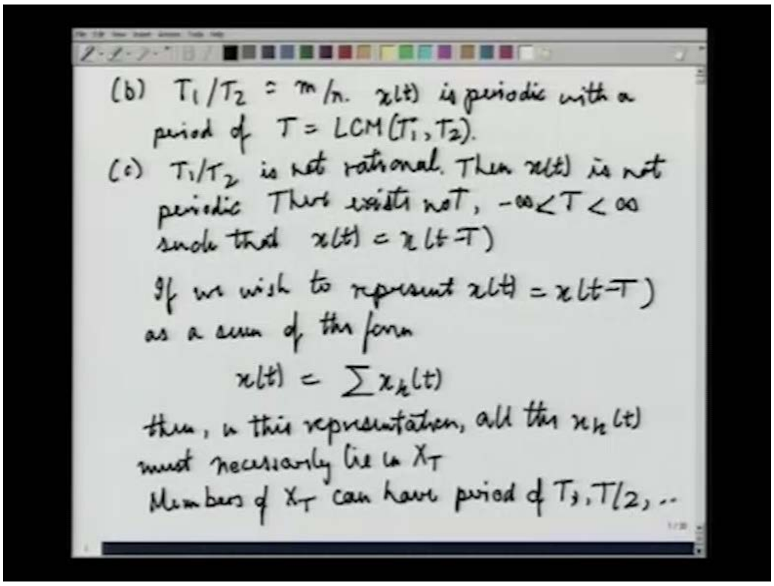

Now, the second one, the second case is when T 1 by T 2 is a rational number m by n, when this is the case, it is not hard to show that x(t) is periodic with a period of t equal to LCM of T 1 and T 2, the third case is when T 1 by T 2 is not rational. In this case it turns out that x(t) is not periodic, that is to say there is no finite T, such that x(t) equals x(t) minus t, this tells us something it tells us that if we are concerned with representing periodic signals with period t, then both case b and case c are not convenient; case c is definitely not convenient, case b is also not convenient, because it yields signals with a larger period.

In short if we wish to represent x(t) equal to x t minus T as a sum need not be a sum, but let us say a sum a combination of the form x k of t, then in this representation; all the x k of t must necessarily lie in X T. The set X T that we designated a little while ago, we have learnt something. But the set X T is still a very large set, it contents of all periodic signals of whatever shape, but possessing a periodicity of T, please note that a signal which lies in x(t) necessarily it necessarily not something, which has a period of capital T, it might have a period T by 2, T by 3, T by 4, and so on. That is to say that members of X T of T, T by 2.

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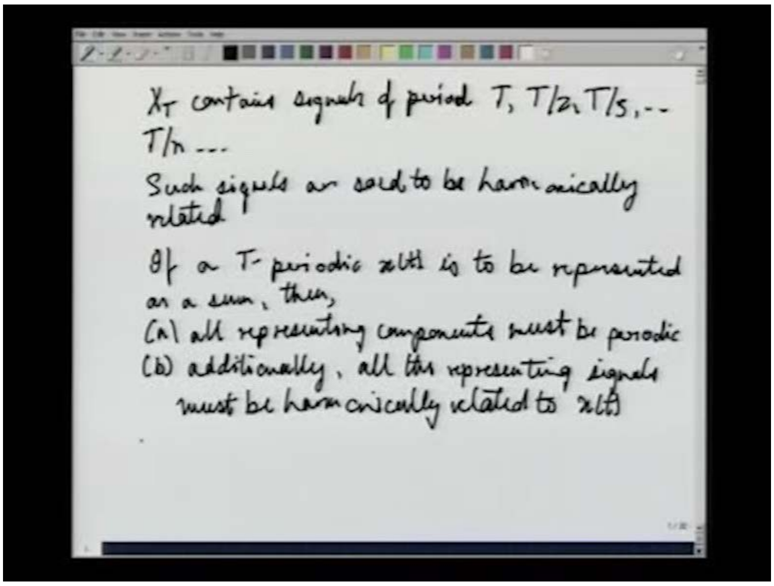

So, we have signals in x(t) of period t itself T by 2, T by 3, and so on, T by n for various values of n, such signals are set to be harmonically related. Now, seems only signals, which are harmonically related constitute the members of x t. It is evident that such signals being closed under addition harmonically related signals remain harmonically related meaning may continue to have a period of T, when they are added to each other this much is clear that, if a signal is not harmonically related to a particular x(t), that we wished to represent using components. Then those component which going to the representation of x(t) must be harmonically related to x(t), otherwise it is impossible for x(t) to have a period of T. Let us put this down if a T periodic x(t) is to be represented then all representing components must have must be periodic.

Secondly or additionally all the representing signals must be harmonically related to x t; that means their periods must be of the form T by n for sum n, this clearly rules out if we wish to seek representation for a T periodic signal this clearly rules out all signals which are not harmonically related to a T periodic signal. So, the representation the class of signals which can represent x(t) has been narrowed down, but the question is can we narrow it down further. We are familiar with the particular property of linear time in-variant systems and the property is that complex exponentials in short general exponentials are preserved under linear and time in-variant processing, if you do not recall this very important result, we will just derive it in a couple of steps.

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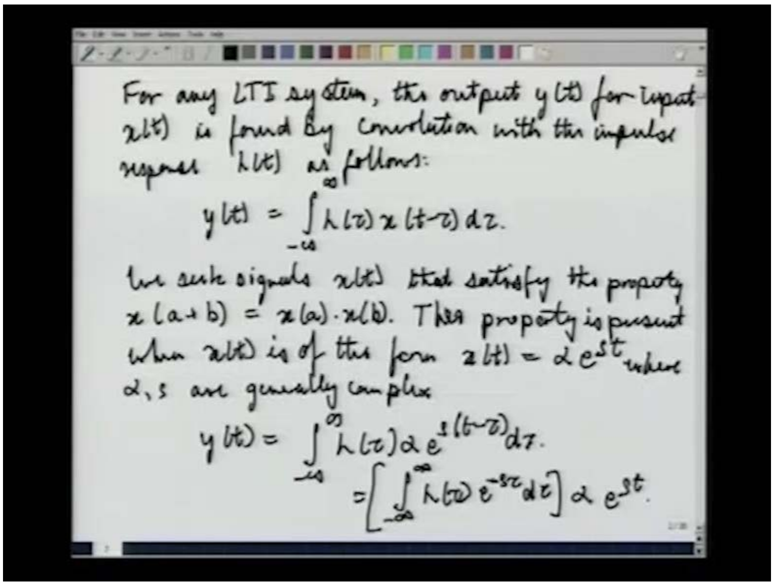

So, let me go to a fresh page for any LTI system, the output y(t) for input x(t) is found by convolution with the impulse response h (t) as follows, y(t) equals integral minus infinity to infinity h of tau x of t minus tau d tau. So, tau is the variable over which integration is carried out and this is convolution right. Now, is it possible to find signals x t, with the property that under processing by LTI systems, the form of x(t) is preserved in spite of reprocessing, do their exists a signals that is the question. And the answer at least is already known we see signals x(t) that satisfied the property x of a plus b equals x a times x b, fine. These are the kinds of signals for which we will get what we want namely form preservation under LTI processor.

Now, what are the signals what are the signals x(t) which have this property, the exponentials the general exponential, this will happen or rather this property is present when x(t) is of the form x(t) equals alpha times e to the power s t, where alpha and s can generally be complex, alpha s are generally. Why this happens, very easy to show, you just substitute an x(t) of this sort in the convolution equation, you get y(t) equals integral minus infinity to infinity h tau e to the power sorry, alpha e to the s tau t minus tau d tau, since it has the property that x of a plus b equals x a, x b being an exponential this can be written as an integral minus infinity to infinity h tau alpha e to the s tau d tau multiplied by e to the s t.

In fact, we could do better and keep alpha outside, fine. So, that this can be taken away from here and we just write e to the power minus s tau d tau, and write alpha e to the s t off set. So, this shows that for a choice of a signal of this sort. We have the output of the same form as the input expect for a scale factor, and what is that scale factors?

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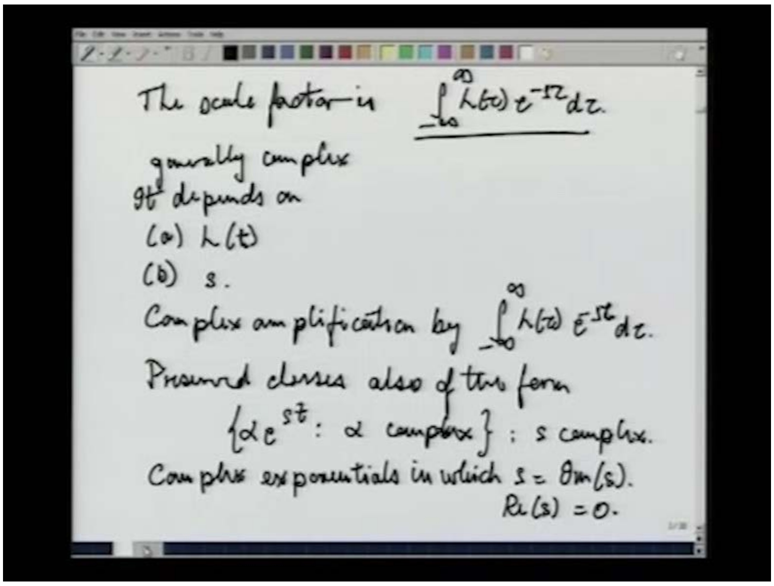

The scale factor is integral minus infinity to infinity h tau e to the power minus h tau d tau, this with the scale factor; given the fact that s is complex this is generally complex. Furthermore it depends on the impulse response h. It also depends on the value of s, it depends on these two things, but apart from depending upon these two things, if you keep if you are concerned with a particular processor h t.

Then this part is fixed then dependent on s, we just get a scaling of the input signal to get the output signal this scaling is tantamount to a complex amplification by this quantity, right. So, all we are all these happening to a signal e t to the s t when applied to LTI system is that is subjective to a certain complex amplification; that means, that if you take a particular LTI system, then it has preserved classes also of the form. So, to put this more precisely, that is to say there are lots of different preserved classes each preserved class is indexed by the s that associated with the members of that class; all members of the of a given preserved class have the same value of s, but take on different values of alpha, different complex amplification.

So, we have far more compact preserved classes than what would get by considering merely periodic signals; however, these signals are generally not periodic, but certain kinds of complex exponentials are periodic, we know this and precisely which complex exponentials are periodic those for which s is imaginary, that is to say. Complex exponentials in which s equals the imaginary value of s that is to say real value of s equals 0, fine; these are preserved classes, they are also periodic.

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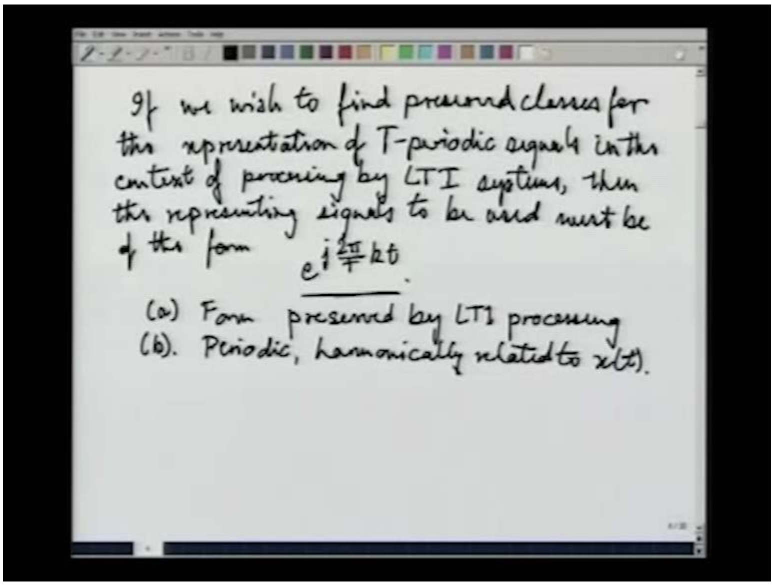

Finally, then we can say the following collecting all these results set we have obtained, so far that if… We wish to find preserved classes for the representation of periodic signals t periodic signals in the context of processing by LTI systems, then the signals to be used the representing signals to be used must be of the form e to the power j 2 pi by t k t. What property does these signals have first of all it is a complex exponential. So, its form is preserved by LTI processing b, it is periodic and harmonically related to x(t), we know that this is an essential requirement as we have just shown, if a signal is not harmonically related it has no business to be presenting their representation.

So, it is possible and harmonically related to x(t) h is also possessed of the property of its from being reserved by LTI processing, fine. So, these two things are there, and hence these are the signals that we will now use to construct the first representation. The first case of representation, which we will call the Fourier series.

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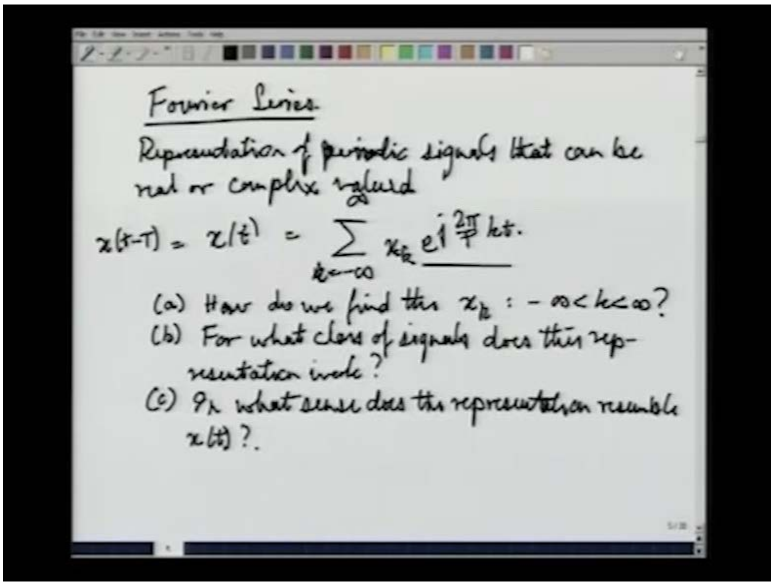

The Fourier series is concerned with the representation of periodic signals real or complex valued, that is the objective of the Fourier series now given the fact that we have identified the kind of representing signals, that we will used. And the fact that is the linear representation in this sense that the combination is by addition, we can tentatively construct the equation for representing for the synthesis of x(t) using components, which are complex exponentials, that are periodic and harmonically related.

So, x(t) will be of the form a linear combination k equals minus infinity to infinity x k e to j 2 pi by t k t, fine. We still are not sure, if there are any constraints that must be placed on x(t) for these whole thing to work for this representation to work, all we know is that it is important that given that x(t) is periodic with period T, that is x(t) equals x(t) minus t.

And given that we are concerned with LTI processing a convenient representation would evolve functions of this sort, right. So, let us see there are several questions waiting to be answered first of all how do we find the x k, second for what class of signals does for what class of signals x(t) does this representation work, three in what sense does the representation resemble x t. Now, is difficult to say which of this question should be answered first, what I just follow the policy of answering the relatively easy questions, first even though they are not the relatively important questions. We really should be answering question b, first then move on to question c.

And then finally answer question a, but I will take the liberty of trying to answer question a first. See, if we look at the representing equation representation equation, we have over here, we have x(t) represented as a linear combination of complex exponentials, that is to say that like, when you produce a dish in the kitchen, you want to know how much salt to add to it, how much lemon juice to add to it, how much sugar to add to it? Clearly, at a very intuitive level at a lay level, you can say that if the dish that you have produced is very salty, then it probably has a larger component of salt in it if it is very sharp, then obviously it has too much lemon juice in it.

On the same basis we are going to see what is the overall flavor of x(t), where different flavors are represented by different representing periodic complex exponentials on the right side, that are present on the right side. So, we are essentially going to say ask the question is x(t) more sour is it more sweet, is it more salty or more precisely how salty is x t? Because the moment we know how salty x(t) is we will know how much salt has been added to x(t), the has been added to construct x(t), if we know how sour x(t) is then we can say that this much lemon juice must have been added to make x(t), fine. So, we have to essentially compare x(t) or do a resemblance study of x(t) with each of these different flavors or each of these different complex representing exponentials on the right side. This process of comparison or measurement of resemblance is carried out by a process called correlation.

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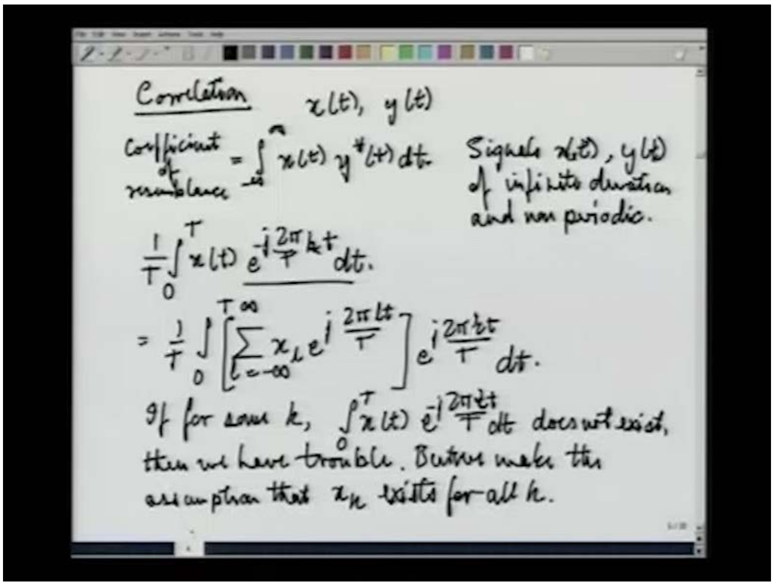

Correlation. In fact, though the name may sound unfamiliar is nothing but an inner product. Now, the inner product of two vectors or of two functions is defined in a very simple way, and let me just give it to you over here, we are assuming the vectors or functions to be generally complex value. Then if you have let us say let us write correlation for functions, let us say in this case correlation for two functions, which are both of finite period of the same finite length say length capital T, then if x(t) is 1 function, and y(t) is another function. Then the correlation will be given by integral minus infinity to infinity x(t) y star t d t, where y star represents complex conjugation, this yields the coefficient of resembles between x(t) and y(t). Of course, this is not the word you will find in self respecting book using to describe correlation, but these gives the student a very good feel of what we really doing we are trying to see how much x(t) resembles each of these components signals.

If it resembles a particular component more than the corresponding weight x k for that particular component k th component must be larger, if it does not at all resemble a particular component. For example, suppose you made lemon lemonade with no salt in it at all then resemblance is zero, that simply means do not add any salt or other you did not add any salt. Since we are doing this post factor, if you forgot to add sugar; of course, then you would find the resembles with sugar to be very low, if you to forgot to add lemon juice. Then of course, it does not lemonade at all, but in any case it would have no resemblance zero on correlation with lemon juice.

So, now let us see, let us see what happens? If we take x(t) and construct the coefficient of correlation for a periodic signal, we have oh sorry, we said that this is for signals of infinite period infinite duration and non periodic. There actually some inter case is involved with this definition, which I will not going to write. Now, let us leave it. What it is it needs to be refined it needs to be constrained by certain ifs and buts right. Now, we will write this correlation expressions for a periodic signal for a periodic signal, we carry out the correlation for a pair of periodic signals, we carry out the correlation only over one fundamental period.

And hence we will write integral 0 to t, where t is the fundamental period the period of repetition of x(t), then we will write x(t) e to the power j 2 pi by t k t d t, we also average or weight this quantity by the duration itself. So, we actually write this as 1 by t times, this over as you can see this is not a very serious matter, you just get the coefficient of correlation or the coefficient of resemblance in a different set of units, if you did multiply by 1 by t, but that is alright. Since, everybody does it lets go ahead, and do it. Now, let us see what happens? This is what we have if indeed x(t) is that sort of a signal for which a representation exits, which is an assumption we are making, because we are answering question a before questions b and c, then x(t) can be replaced by its representation.

So, we could write, this as equal to 1 by t integral 0 to t summation l equals minus infinity to infinity x l e to the, there is a minus sign over here which I forgot to add, because we have taken the complex conjugate of this expression of this function over here. And therefore since its exponent is purely imaginary, it is just becomes e to the minus j 2 pi k t by t. So, here we have e to the j 2 pi l t by t. This entire thing multiplied by e to j 2 pi k t by t d t, this is what we have. Since, again we have not yet discussed very serious issues, such as when this representation will exit what are the constraints and so on.

We will just make things easy for ourselves by assuming that there are no mathematical problems along the way, one of the mathematical problems that could exits along the way is if for some k. Then we have something to worry about if this does not exist, there is something to worry about, but we shall assume that it exits for all k, but we make the assumption that x k exits for all k, fine. Since, we have made this assumption life has become little easy, and we can write the next step for this coefficient of resemblance or coefficient of correlation, we will see what this coefficient of correlation comes to we have we can rewrite this now as 1 by t summation.

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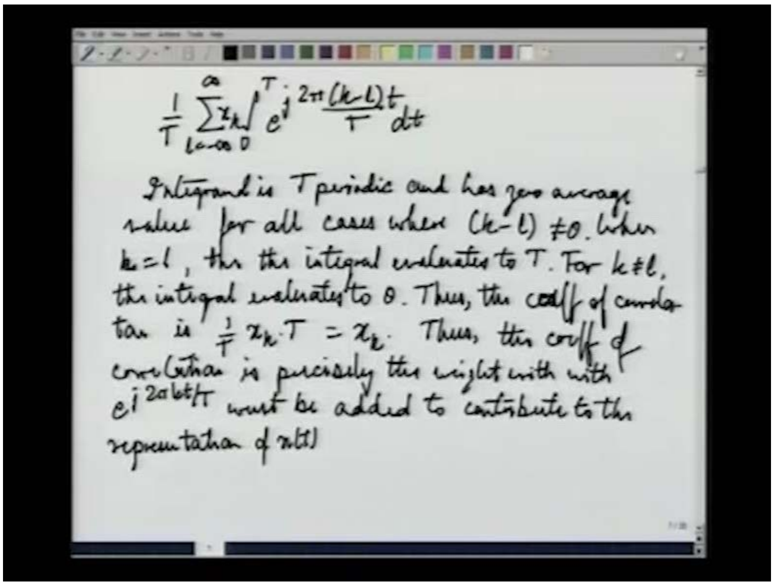

We are able to exchange the summation on integral, because we assumed that x k exits for all of the not x k, this we assume that this exits for all of the, then we can write l equals minus infinity to infinity summation integral x k can be written over here, 0 to t e to the j 2 pi k minus l t by t d t. Now, let us just look at the integrant the integrant except for the case when k equals l is a t periodic function, so integrant is t periodic and has zero average value for all cases where k minus l is non zero, when k minus l is non zero where when k minus l is zero, when k equals l the integral evaluates to to t for all other cases for k not equal to l the integral evaluates to 0, so though we have an infinite sum every term except one.

In this in this infinite sum will be zero and that one term will have a value of x k times t, thus the coefficient of correlation is 1 by t times x k t equals x k in short the weights with, which we must combine the different periodic complex exponentials. That are harmonically related to x t, in order to get x(t) are themselves the correlations, the the coefficients of correlation or the coefficients of resemblance. That I called of these different flavors these different periodic complex exponentials with x(t), thus the coefficient of correlation is precisely the weight with which e to the j 2 pi k t by t must be added must be added to contribute to the representation of x(t).