**Signals and Systems**

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**Lecture - 21**
**LTI Systems Described by Difference Equation**

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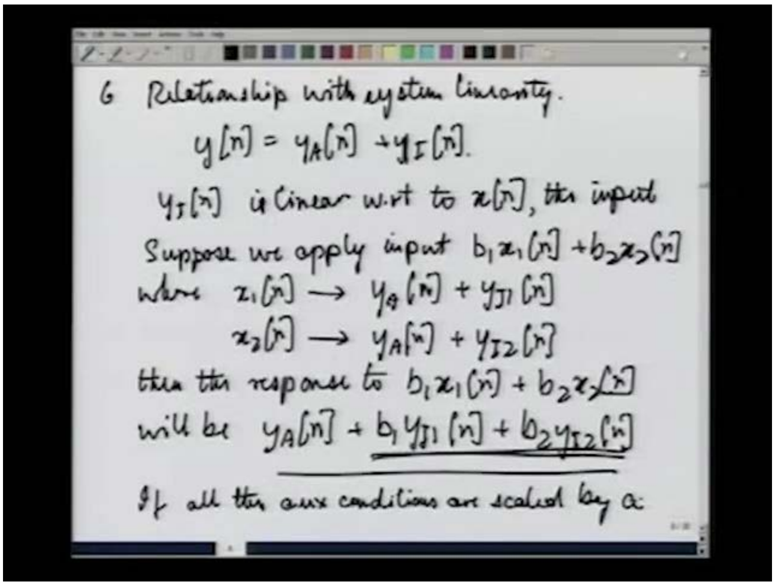


In number 6 relationship with system linearity, relationship with system linearity again follows very similar principles as in the case of the differential equations. The system is linear with respect to the input, if all the auxiliary conditions are 0. That is because again it is convenient, and it is very, very informative to represent the complete solution as its sum of a of a response to the auxiliary conditions plus a response to the input. So, we will do that again here, and write y(n) equals y A(n) plus y I(n) where the notation carry the same meaning as we had at the previous suggestion right.

So, now we have this then it becomes abundantly clear, that y I(n) is linear with respect to x(n) the input. So, that in place of x(n) suppose I apply x 1 (n) plus x 2 (n) or even more generally, suppose I apply say b 1 x 1 (n) plus b 2 x 2 (n), where x 1 (n) yields y A(n) plus y I 1 (n), and x 2 (n) yields y A(n) plus y I 2 of n. Then the response to b 1 x 1 (n) plus b 2 x 2 (n) will be y A(n) plus b 1 y I 1(n) plus b 2 y I 2(n). Clearly this entire thing taken in one place is not linear with respect to the input, but this part which corresponds to the pure input response is linear with respect to the input. Likewise y A(n) will be linear with respect to the auxiliary conditions, if all the auxiliary conditions are uniformly scaled by the same factor say c, then instead of y A(n) you will get c times y A(n).

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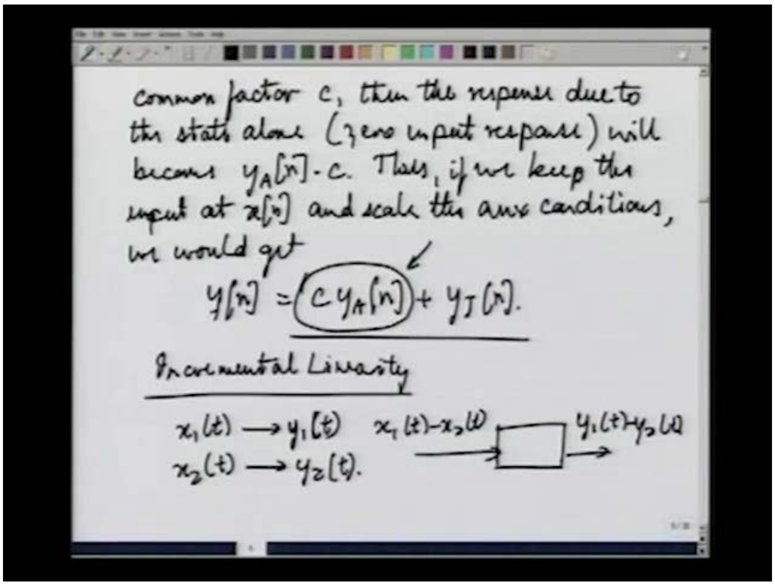


Common factor c, so that you have a response well, then the response due to the state alone, that is to say the 0 input response will become y A(n) times c, thus keeping the input same if we keep the input at x(n) and scale the auxiliary conditions, we would get y(n) equals c times y A(n) plus y I(n) which taken as a whole again is not linear with respect to the auxiliary conditions either. But this component taken alone is linear with respect to the auxiliary conditions, just like in the previous which this component alone was linear with respect to the input. It is always the other component that is spice the linearity.

So, this more or less summarizes our discussion of the relationship between the linearity of the differential equation and linearity of the system. Having understood all these, we will just make one closing comment. The presence of the auxiliary conditions and the effect they have on the linearity of the system with respect to the input is sometimes captured by some authors using the term called incremental linearity. This common sense about to make will apply equally for the continuous time and the discrete time system, the argument they make use the following.

Suppose you have x 1(t) as a possible input, and x 2(t) as another possible input and the corresponding outputs are y 1 t n, y 2(t). What they claim is this? Let us now look at whether the output y y 1(t) or y 2(t) is linear with respect to the input x 1 or x 2(t), let us instead ask whether the difference between y 2(t) and y 1(t) is linear with respect to the difference between x 2(t) and x 1(t) right.

This is a very interesting way of posing the problem. What we are saying is let us pretend that we have the system, and the input is x 1(t) minus x 2(t). So, that the corresponding output is y 1(t) minus y 2(t), then they claim that system which are subject to auxiliary conditions whose linearity is now hostage to this term over here which prevent it from being linear with respect to the input, such systems will still be incrementally linear that is to say y 2(t) minus y 1(t) will be linear with respect to x 2(t) minus x 1(t).

This is easily demonstrated all you have to do is go over here, and write what happens when you write when you apply the input x 2(t) minus x 1(t) by the linearity of the system the response should be the corresponding combination of the individual responses. So, for x 2(t), you have y 2(t) as a response x 1(t) you had y 1(t) as the response.

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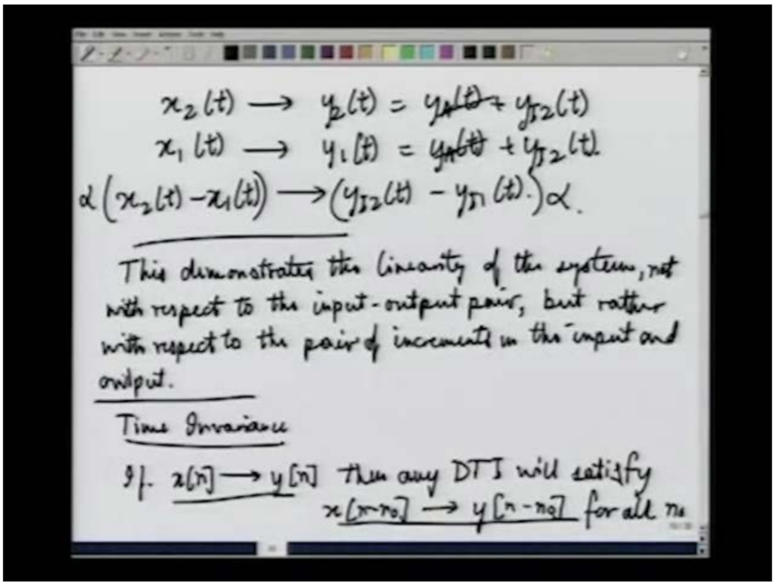


So, let us write x 2(t) yield y 2(t) which is basically y A of t y A remains the same, because it is due to the auxiliary conditions, the input (( )) are being changed plus y I 2 of t x 1(t) will yield y 1(t) which is y a of t again the same y a of t plus y i 2 of t. So, x 2(t) minus x 1(t) would give you the difference of these 2 expression on the right where y A(t) would automatically vanish and you would just get y I 2 of t minus y I 1 of t.

Clearly, if you scale this by a certain factor alpha for example, this would also scale by the same factor alpha, it was realized that this linearity we have just spoken of is really not, the linearity of a system with respect to input and output. It is more precisely the linearity of the system with respect to increments of the input and the output. So, let just put this down as closing comment not with respect to the input output pair, but rather with respect to the pair of increments in the input and output. So, this closes the discussion about the linearity properties of system described by linear constant coefficient ordinary difference equations.

The next step of course, for going along the same lines as we followed in the case of the differential equation linear constant coefficient ordinary differential equations. Is to study the time invariance properties of the system, we saw we the properties of the differential equation difference equations which describes it. So, let us look at that next. Again we only proceed by analogy we do not start a discussion from the fundamentals, we only go by analogy with the corresponding study for the case of the differential equation and show how practically the same results can be modified to get the corresponding results for this case. Time invariance again is the property by which the output shifts along the time axis by the same amount that the input shifts.

So, if if x(n) yields output y(n) then any discrete time invariant system will satisfy x(n) minus n naught must yield y(n) minus n naught. So, this is the property we are looking for. We essentially going to ask the question what should we do, what should we ensure, in order that a system which yields this input output pair also yields this input output pair for all values of n naught, once second (( )). The answer as may be expected is very similar to what we had for the case of the time invariance of the continuous systems described by differential equations, is simply the following. When you shift the input, shift the reference time at which the initial conditions or the auxiliary conditions are given also by the same amount.

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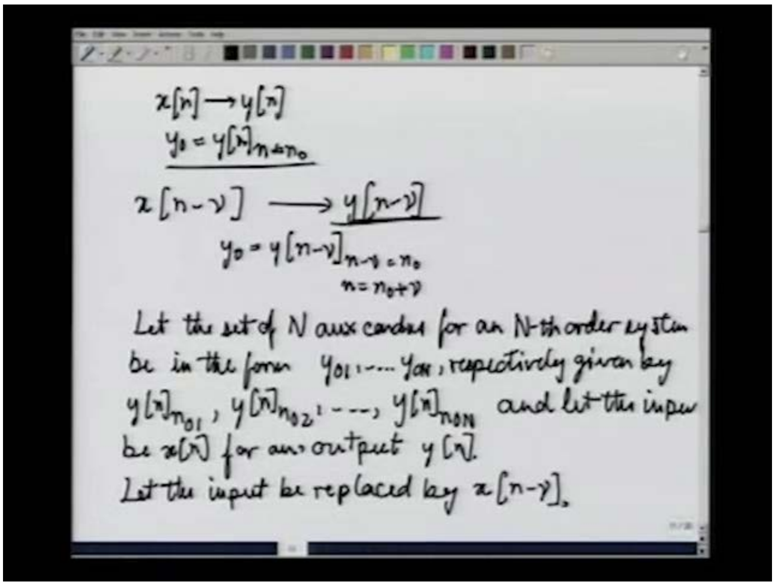


In short there what we are going to say is if the system yields for input x(n), the output y(n) then I mean and the auxiliary condition was y(0) which was essentially y at y of n at n equal to n 0, which would be the case for a first order system which requires only one auxiliary condition. Then if we shift the input by a certain quantity nu, let us say that instead of x(n) we apply x(n) minus nu, where of course nu must be an integer. Then in order for this to result in y(n) minus nu, this condition which we have applied over here must now be the same value y naught, but it must apply at y(n) minus nu at n minus nu equal to n naught that is to say at n equal to n naught plus nu, if we ensure this then the output will indeed be this quantity.

Now, going further to generalize for the for the case of higher order system, higher order system described by higher order differential equations, it simply this. All the time instants at which the different auxiliary conditions have been specified must be shifted by the same amount namely nu in the same direction as the input has been shifted, that is to say if you had let us consider the case when only points have been given at capital N at different places, that is to say.

Let the set of initial conditions, let the set of n auxiliary conditions for an n th order system be in the form y 0 1 to y 0 n, which are essentially respectively given by y(n) at n 0 1 y(n) at n 0 2 and so on to y(n) at n 0 N. This is the initial case, with these initial conditions and let the input be x(n) for an output y(n), this is the first part of the formulation. Now, let us say let the input be replaced by x(n) minus nu, if the input is replaced by x minus nu n minus nu.

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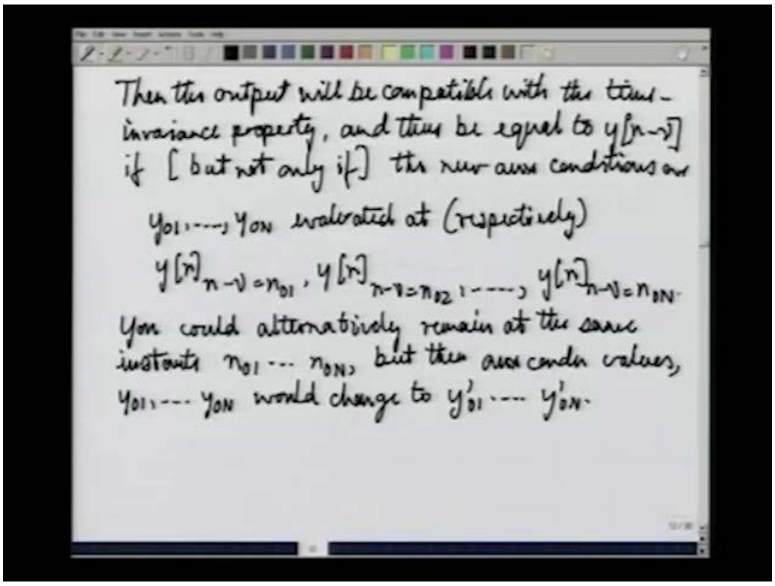


We can now say that then the output will be compatible with the time invariance property, output will be compatible with the time invariance property and thus be equal to y(n) minus nu, if but not only if, it is just a sufficient condition that we are providing here, it is not a necessary condition. As was the case in the discussion on differential equations the time invariance property in the relation to differential equations also, we are only providing a sufficient condition here. If the new auxiliary conditions are y 0 1 onto y 0 n evaluated at respectively y(n) at n minus nu equal to n 0 1 y(n) at n minus nu equal to n 0 2 and so on to y(n) at n minus nu equal to n 0 n. So, the values remain the same, but these value should hold at n equal to n 0 1 plus nu, n equal to n 0 2 plus nu, n equal to n 0 n plus nu respectively for y 0 1 to y 0 n.

If this sufficient condition is met, then we are going to actually observe a shifting of the output just in the manner that the input was shifted by the same amount and in the same direction that the input was shifted. But again there are other ways of specifying it, instead of shifting the points at which the auxiliary conditions has specified, you can instead shift the auxiliary condition values that you expect to find at those same instants that you use earlier. In fact instead of using n minus nu equal to n naught that is to say n equal to n naught plus nu or n equal to n 0 1 nu plus n equal to n 0 2 plus nu, you can stick to the same time instants, but then you should be the correspondingly shifted values; that is to say you should alter the values appropriately.

You could for example you could alternately, alternatively remain at the same instants n 0 1 to n 0 n, but then the auxiliary condition values y 0 1 to y 0 n would change to y dash 0 1 to y dash 0 n, right. What are these values y dash 0 1 to y dash 0 n? There the values you would see for the curve after it has been shifted. So, finally let us look at the issue of causality, we only had a cursory discussion of this will be dealt with the difference equation differential equation, and in fact I said at that time that this issue of causality would be better explained in the case of the difference equation, and that is why I promised to postponed this discussion to that time well now the time has come.

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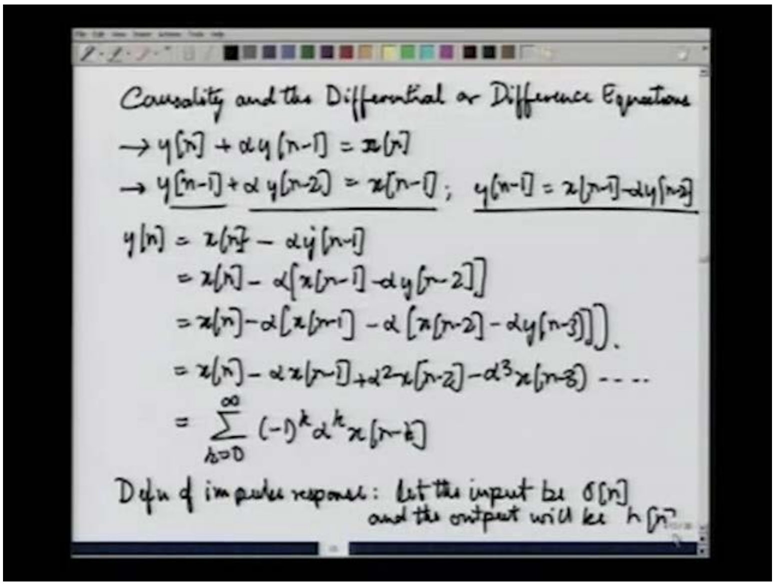


Causality and the differential or the difference equations, before I actually plunge into this discussion, let me just go through a brief exercise for the case of the difference equation. Let us in fact handled them simplest of possible cases the first order difference equation is there a way to find the impulse response of the system from the difference equation this is the question I will rise. So, let us try to answer this question.

Suppose we have let us take a very concrete example, suppose we have a difference equation like y(n) plus alpha y(n) minus 1 equals x(n). Suppose we have this, can we find the impulse response of the system from this. On the face of it our experience our intuition everything tells us, that this must be a causal system. And we also know that we are free to replace the independent variable over which the signals are given by some other quantity.

Thus for example, if y(n) plus alpha y(n) minus 1 equals x(n), we can we are perfectly write in in writing y(n) minus 1 plus alpha y(n) minus 2 equals x(n) minus 1, and so on we could write this for whatever. Amount of time shift we wanted we could write the equation like this and it would still be valid. Now, I am going to process as follows I am going to write this original equation as y(n) equals x(n) minus alpha y(n) minus 1 which equals x(n) minus alpha what is y(n) minus 1? Y n minus 1 can be found from this second equation over here, this 1 by just rearranging the terms and writing that y(n) minus 1 equals x(n) minus 1 minus alpha y(n) minus 2.

So, that is what we are going to replace for this over here and write y(n) minus 2 equals, we already have y(n) equals x(n) minus alpha y(n) minus 1. So, again we would write alpha over here and then write in brackets x(n) minus 1 minus again alpha y(n) minus 2 you can do all more round of this and you would get that this equals x(n) minus alpha x(n) minus 1, fine minus alpha what is y(n) minus 2? It would have the same form as y(n) minus 1 that we have written over here, except that all time instant would be shifted again by 1 further point. So, you would get x(n) minus 2 minus alpha y(n) minus 3, this is what you would get?

Now, in principle you can keep on doing this, and every time you would do this an additional term would appear and you would keep pushing the time instant y on the right side further and further away, You could keep doing this indefinitely as I said and so ultimately what will you get? You will just get a sequence of terms on the right which involve x. Let us see the terms that we have already collected they are x(n) minus alpha alpha minus alpha x(n) minus 1 plus alpha squared x(n) minus 2 minus alpha cubed x(n) minus 3 and so on.

Now, can we find the compact expression for this a compact formula for this, I suppose we can on the one hand you have all the members of the x sequence x(n) x n minus 1 x(n) minus 2 x(n) minus 3, then you have alpha to the power 0 alpha to the power 1, 2, 3, and so on. So, in general you have something of the form alpha to the power k x(n) minus k, fine, but the sign there is a problem with the sign; first term has a positive sign, the second has a negative sign, the third has a positive sign, and so on.

So, what we really should is to get a minus sign for the second term, you again see that there is place where k is equal to 1. So, you just write minus 1 to the power k, this is the general form of any typical term, but we have the, we need the summation of all these terms. So, let me just rewrite this to get the final form of the relationship between y and x expressed purely in terms of x recursive equation in terms of x is equal to the summation as k goes from 0 to infinity minus 1 to the power k alpha to the power k x(n) minus k.

Now, that you have this relationship between output and input, where the output has been exercised from the right hand side, you can apply the definition of impulse response; definition of impulse response. Let the input, the definition is let the input be the discrete impulse delta of n and the output will be h of n. So, let us do that over here, you had y(n) on the left side that would become h n, on the right side you will just get minus 1 to the k alpha to the k delta n minus k summation.

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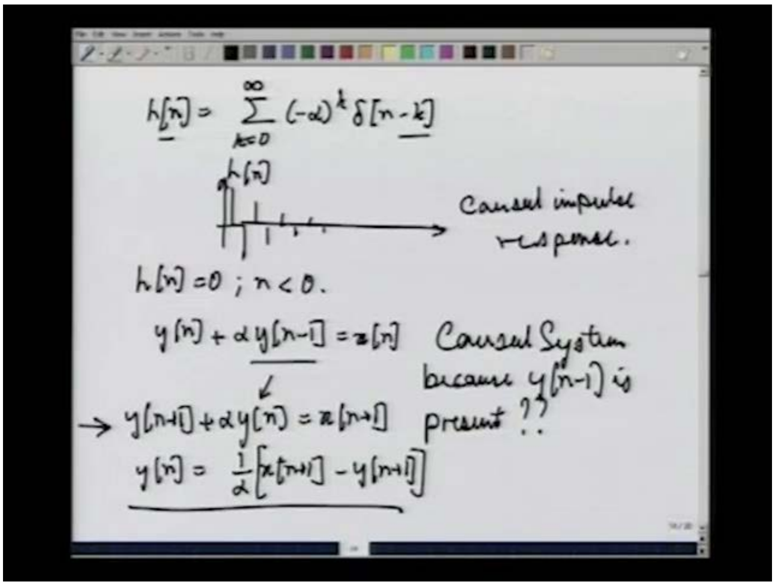


H n minus alpha to the k delta n minus k. So, this has given us the impulse response of the system, starting from the difference equation that we had over here this was the difference equation. If you look at the form of the impulse response you have, you clearly see that the present value of h(n) depends upon a lot of past impulses, that is consist of a lot of impulses that occurred at various points in the past n minus 0, n minus 1, n minus 2, n minus 3, and so on.

In short you have an impulse response, which is what is conventionally caused called causal. You will have a sequence like this minus 1 to the alpha k delta n minus k would give you an impulse response that is may be like this. This would be h(n), and this indicates a causal impulse response, because h(n) equals for n less than 0, fine.

So, this is fine, does this mean that the difference equation inherently implied that the system was causal, this is the conclusion that most of us come to. Let me just rewrite the difference equation once again that we used, we had y(n) minus sorry plus alpha y(n) minus 1 was equal to x(n). If we go back to this equation, we can probably trace what we believe is the reason why this system is causal, we will say that y(n) recursively depends upon y(n) minus 1, where y(n) minus 1 represent the point of the output to the past of y(n). Since you have y(n) minus 1 no over here, clearly this must be a causal system, but I will put a question mark on this is? This really true.

And in order to turn the whole thing upside down, let me make some very, very simple changes in the way this difference equation has been written. Instead of n I will write n plus 1. So, I will get y(n) plus 1 plus alpha y, now n minus 1 becomes n becomes I have added one to the time variable at all the points plus alpha y(n) equals x(n) plus 1. This is the equation I have.

Now, let me transform it suitably to get an expression for y(n), what will I get? I will get y(n) equals 1 by alpha x(n) plus 1 minus y(n) plus 1, fine. Already over here where I rewrote this equation, you would began to wonder wait a minute, is this anti causal system all of a sudden, because clearly now y(n) is present over here in this term, but it seems to depend not on y(n) minus 1 as it did before, but on y(n) plus 1.

That means, to say depend recursively not on the past input on the not on the past output, but rather on the future output in order the kept this further, let us again go through the same exercise of extracting the impulse response for the system. We have this form over here that we will repetitively and recursively use, we have y(n) equal 1 by alpha x(n) plus 1 minus y(n) plus 1 in this we will replace y(n) plus 1 by an expression for y(n) plus , just like we did last time.

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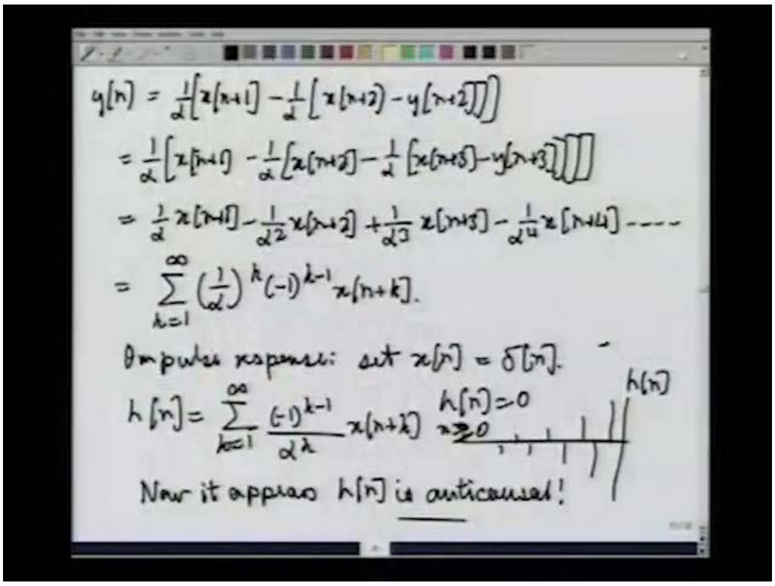


y(n) equals 1 by alpha x(n) plus 1 minus, what is y(n) plus 1? The expression is the same just like you have over here, just like you had over here. You have y(n) plus if you wrote y(n) plus 1 on the left hand side you would get 1 by alpha x(n) plus 2 minus y(n) plus 2. So, let us write that over here, minus 1 by alpha x(n) plus 2 minus y(n) plus 2. Now, we are going to handle this in another round to get 1 by alpha x(n) plus 1, fine minus 1 by alpha x(n) plus 2 fine, minus y(n) plus 2. What is a y(n) plus 2? It will be equal to 1 by alpha open brackets x(n) plus 3 minus y(n) plus 3 right.

So, this is what it looks like? You could keep doing this until infinity if you wanted, but already the pattern has began to emerge, let us just expand this terms is equal to 1 by alpha to the multiplied by x(n) plus 1 minus 1 by alpha squared x(n) plus 2 plus 1 by alpha cubed x(n) plus 3, again you would get minus 1 by alpha to the power 4 x(n) plus 4, so on, you would get terms like this.

Now, again it is important for us to put this into a proper compact form get it into a formula, because that makes us see things more clearly. So, what is this? You have x(n) plus 1 x(n) plus 2 x(n) plus 3 x(n) plus 4 in consecutive terms, so that part is straight forward. You have 1 by alpha minus 1 by alpha squared plus 1 by alpha cubed 1 by alpha to the power 4. In fact, with x(n) plus k you have 1 by alpha to the power k, fine.

Finally the first term has a positive sign, the second term where k is 2 has a negative sign and so on. So, I could write this as summation k equals 1 to infinity 1 by alpha to the power k, then I have minus 1 to the k minus 1 this is the only way I will get the even terms to have a negative sign, the by even terms I mean terms containing even values of k. And then I have x(n) plus k this, if you plot this you would get an expression that goes this way without prejudice to what choice of values, we what choice of value, we take for alpha.

Let us say that alpha is equal to 2, then 1 by alpha is equal to half and you would get a sequence which diminishes for larger values of k diminishes and magnitude, but alternates in sign, because there is minus 1 to the k minus 1 over here and the first term starts with k equal to 1. So, you would get may be a term like this and a term like this, and something like this and something like this, and something like this and something like this, this is what you would get? Clearly, now it appears that well this is a little premature.

So, let us let us just take this all the way and write the impulse response, first impulse response for this you set x(n) equal to delta n, then you would get h(n) equals summation k equals 1 to infinity minus 1 to the power k minus 1 minus 1 to the power k minus 1 divided by alpha to the power k x(n) plus k, which is what I was trying to plot actually h(n). So, now it appears h(n) is anti causal. Look at that same equation in the previous page gave you an h(n) which looked causal h(n) looked causal over here, where that expression for h(n) yeah, because h(n) is 0 for n less than 0, whereas here clearly you have h(n) equals 0 for n greater than equal to 0, this is what you have over here now.

So, it appears to be anti causal over here very much in contrast with what we what we just discovered a little while ago, now what can we say about this is either this derivation wrong or previous derivation wrong the point is that neither was wrong both were right. And in fact that is the most important thing, we have said in the last few minutes both are right, the difference equation alone is neutral with respect to the direction of time the point that I made long ago, but did not substantiate until, now the way I have done it with an example over here is a sufficient indication, that the differential equation or the difference equation is neutral with respect to time you could solve it with towards positive time, you could solve it towards negative time, it has itself no preference for direction of time.

That is as far as the mathematics is concern as the mathematics, the difference equation is concern or the mathematics of the differential equation is concern, we have given you a demonstration here of how the same equation with the slightest of changes can be made to look like a causal a system, describing a causal an equation, describing a causal system or an equation describing a anti causal system. You could make it look either way you could go ahead and actually derive the impulse response, which could also look causal or which could also look anti causal.

The fact therefore, is that what you want to do depends upon your choice pretty much like the case of the auxiliary conditions, where it was additional information about the physical system that compelled us to or that helped us to pick the correct solution out of possible solution here. Again you can say you have multiple possible solutions, you have causal solutions, you have anti causal solutions which one do you want the answer again has to come from the outside world from the world outside the equation itself, and the answer that comes is this.

As far as one is concern with real physical systems; physical systems are the would use the word physical certainly, we cannot have anti causal system, because there is now way you can predict the future. You cannot anticipate what the value of the output is going to be at a future instant of time, you cannot certainly anticipate what the value of the input is going to be at a future instant of time.

So, actual system that you can implement be the continuous system that you implement with capacitors, inductors, and the assistors or the discrete system that you implement which shift, and adders and multipliers and so on. Both this kinds of systems would be hostage to the fact that the real world around that is causal. So, it is the causal solution that you would choose, but it is always important to keep in mind that you are only choosing this by conscious choice, the theory has no preference for direction of time it could be this, it could be that there is no, no problem either way fine.

So, this completes our discussion of causality, and its relationship to the differential equation, that describes the system both as I said for the continuous differential equation case as well as for the discrete equation case. The discrete time difference equation case causality or anti causality has to be chosen by us depending upon constrain of the actual system, the important thing is that there are some situations in modern signal processing where everything is done completely by software. You do not have capacitors and inductors, anymore you just have an array of data completely available the entire input data is available for all time from starting instant to the closing instant before and after which we will assume there are only zeroes in the input data, you have the entire array given to you.

And if you this entire array is given to you, and you are asked the question whether the competition that is involved as described by a differential equation or difference equation, sorry is a causal computation or not you could say that I could construct a computational, which is causal I could construct a computation which is anti causal. Since all these are the numbers, which are being taken in and out of the memory of a computer, it really does not matter. You can compute the performance the output of an anti casual system, just as easily as you can compute the output of a causal system, because it is an artificial world, anyway it is a virtual world; it is a world of the computer you could always do it either way.

So, while the actual physical world around as is compelled to be causal, that is not really true with the virtual world around us it is possible to find pockets of or to construct pockets an anti causal universe inside our computers, and so all these theory until the time. Computer were widely available for stimulation and computation until that time all these theory was considered to be unnecessarily general dealing with possibilities, which had no physical meaning no longer true these possibilities too can be exploited under certain situations, you can actually make an anti causal filter, you can make an anti causal system to do certain things. Now, that we have more or left more or less left the difference equation and differential equations behind us let us just carry the essence and not the details.