Signals and Systems
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Lecture - 20
Difference Equation Intro

When we study difference equations, this mutuality with respect to the direction of time for difference equation is much more easily brought out, and you could by inference conclude that the same whole for differential equation as well. So, let us now start talking about difference equation.

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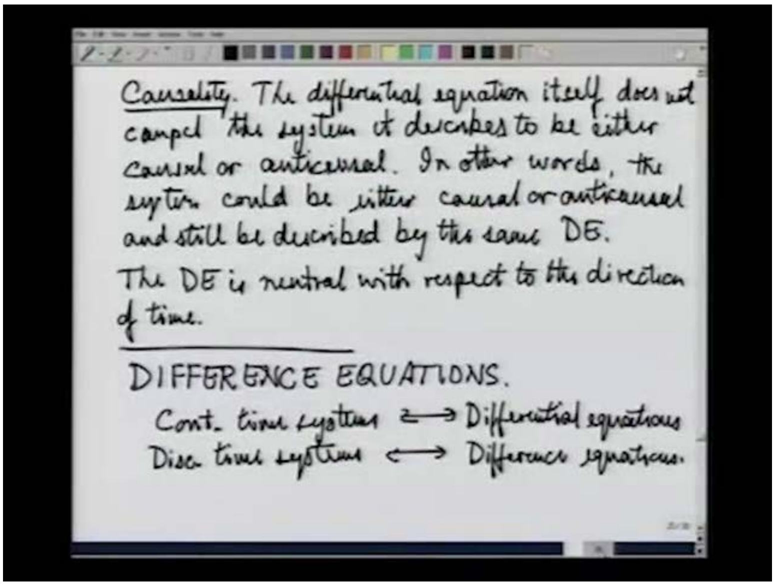

Difference equations relate to discrete systems, in the same way that differential equation relate to continuous time systems. So, if you have continuous time systems, you described them by differential equations, if you have discrete time systems, you would describe them using difference equations. Further more if you had you are if you confined yourself to linear time invariant difference discrete time systems, then you would naturally confined yourself to what are call linear constant coefficient difference equations. So, let us try to see, what is the form of the general linear constant coefficient difference equation.

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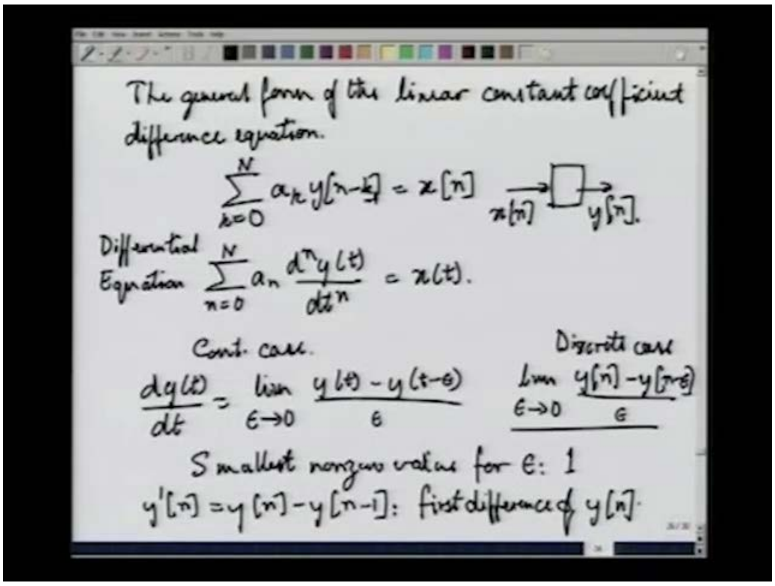

This is the general form of the equation alright. So, we have we believed that this should corresponds to a system where x(n) is the input, and y(n) is the output. Before we proceed further, let us try to understand how these y(n) minus k that we have with essentially shifted versions of y(n) shifted by different values of k and scaling each of them by a k and adding them how this really matches or resembles the earlier differential equation that we had. The differential equation we had was the following, the linear constant coefficient differential equation was n equal 0 to n a n d n y(t) by dt to the n equal to x t.

So, for the continuous time case we have derivatives, for the discrete time case we just have shifted version of the input by shifting an input shifted version of the output, shifting an output or shifting the signal is not the same as or not even similar to differentiating it, so why this discrepancy over here. On the one hand we take gradients in the continuous case, here we are not taking gradients or so it seems, in order to understand this better, let us actually see what we would mean by the counter of a derivative in the discrete context. What is the derivative in the continuous case.

In the continuous time case a derivative is d save y (t) by dt, and it is given by limit as epsilon goes to 0 of y t minus y t plus epsilon or if you like t minus epsilon whichever divided by epsilon. What can we do in the continuous in the corresponding discrete case, suppose we say the same thing. Remember that in the continuous time case, we want epsilon to get as close to 0 as possible, in order to get better and better approximation of the derivative, but we do not want each to become 0. Same thing is true for the discrete case as well, there we could say limit as epsilon tends to 0, is the discrete case of y(n) minus y(n) minus epsilon divided by epsilon. But in the discrete case time which is represented by epsilon can either be 0 or 1 or 2 or minus 1 or minus 2, we cannot have fractional values of time, time is discrete.

So, what is the smallest non zero value for epsilon 1. So, we get and there is no limits to be taken, because there is no continuous variation here, epsilon is clearly going to be equal to this fixed value called 1. So, this entire expression simply becomes y(n) minus y(n) minus 1 divided by 1. So, that you just get y(n) minus y(n) minus 1, this quantity is what we call for convenience now, y dash of n in the language of single processing and system theory, we call this the first difference of y(n) corresponding to what we would have call the first derivative of y(t), here we call the, the first difference of y(n).

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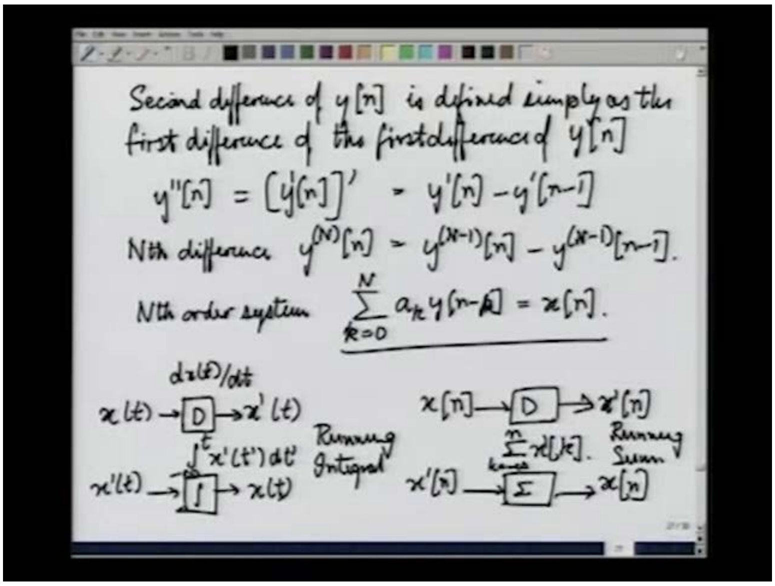

Now, what would be a higher derivative or the equivalent of higher derivative over here, the higher derivative second difference. Second difference or the equivalent of the second derivative of y(n) is defined simply as the first derivative, first difference of the first difference of y(n). So, the second difference of y(n) that is to say y double dash n is defined as equal to y dash of n dash, which is equal to y dash of n minus y dash of n minus 1, this is the second difference. And by another (( )) you can go on and define the n th difference; the n th difference will simply be I will denote this by y(n) of n is equal to y(n) minus 1 of n minus y(n) minus 1 of n minus 1.

So, now it is clear how to get higher differences of any order, we have the first difference, we have the second difference, we have higher differences of all orders right. So, now let us see how we got that equation for the general form of a linear constant difference system described by a linear constant coefficient different system this, this one. How did we get this very simple, after you add the zero th difference which is y(n) itself with some scaled version of the first difference and some scaled version of the second difference and so on up to some scaled version of the n th difference. You will get all kinds of terms, but they will all be y(n), y(n) minus 1, y(n) minus 2, y(n) minus 3, and so on up to y(n) minus capital N, this is all we get. Except that now, the coefficient are all getting a little mixed up you will have several terms with coefficient y(n), you will have several terms with coefficient y(n) minus 1, and so on.

So, you could just collect all the coefficient together and define new coefficients and in fact subsequent to doing that we have got these coefficients a k over here, this completes our justification of calling this the n th order difference equation for a system that describes a system. So, let us re write it over here to acknowledge that we have completed our justification, for a N th order system this is what we have? And just by the side, if you are interested in knowing what is the opposite of differentiation or what is the opposite of differencing. In the case of differentiation its integration, it is the running integral. What is the opposite of the difference of the first difference in the discrete case, that again is best address right now.

What does integration give? Integration after differentiation returns to us the original signal, that is if I had x t and applied it to a differentiator, I would get x dash t where the differentiator is nothing but d x t by d t. Now, the inverse of this would be applied x dash t over here to a system which is an integrator and get back x t, what would be the expression that describe the integrator. We already know this, I think this has been discuss earlier in the course, it is simply integral minus infinity to t x dash of t dash d t dash this would give us x t back.

Now, let us see what happens in the discrete case. In the discrete case, you have x(n) and to apply the first difference, which we again denote by a block with a D written inside.

To get x dash n, if you want to get? If you want to apply x dash n to a system, that returns x(n) then what you really need to do is to replace this or rather to to write this over here, where this actually signifies a summation from k equal minus infinity to infinity sorry to to n, k equals minus infinity to n x dash of k, this would be the inverse of the differencing operation; this is called while this was called a running integral, this is what we just call a running sum.

So, let us now take a look at the difference equation once again, the general difference equation was one with involved a sequence an unknown sequence y(n), which would arise out of the system as a response to an applied sequence x(n). So, the x(n) was the forcing function or the forcing sequence or the input, and y(n) is the solution or the response of the system.

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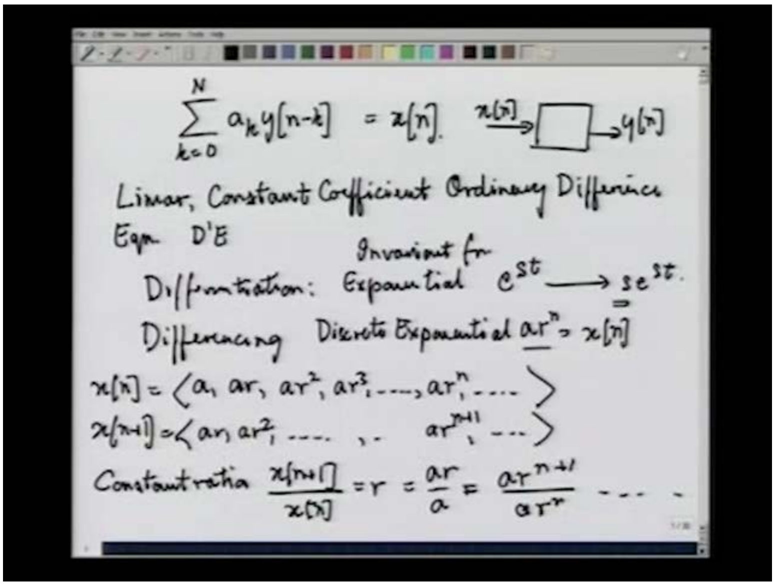

The equation itself if you will recall was, so this was the equation and just as before this we will try to associate with a system in an electrical system, there x(n) is the input and y(n) is the output. In short we wish to determine the output y(n) when x(n) is applied as a input. The way we proceed with this exercise of discovering y(n) given x(n) and given the coefficients a k is going to be very similar to what we did in case of the differential equation, the reason is that both are going to be linear constant coefficient and ordinary. This in fact is something that we should always aware in mind, what we are dealing with is the linear, constant coefficient ordinary difference equation which I will simply abbreviate by D prime E. So, when I write DE it will be the difference equation, differential equation and I write D prime E, it will refer to the difference equation, this will make my writing a likely easier right.

So, how do we solve a linear constant coefficient ordinary difference equation? The steps we have to go through are going to be very, very similar to those that we followed when we solved the linear constant coefficient ordinary differential equation. We start by seeing, if there is any special sequence which is invariant to the process of differencing. If you recall for the case of the differential equation, we did try to seek a function which was not affected by differentiation, and we found such a function. For the case of differentiation, we had an exponential as the invariant function, which are differentiation simply becomes s e to the power s t.

Clearly invariance does not mean a function remains exactly the same, it simply means that a function changes only by a multiplicative constant factor. In this case the constant multiplicative factor is s this, now for the case of differencing we shall claim that what we need is what is called the discrete exponential of the form a r to the power n. Now, let us see what happens to a r to the power n when you difference it. We have to first check the that is indeed invariant is short, that it only alters by a multiplicative constant upon differencing, and subsequently we will see what is that multiplicative constant as well. Having establish that we will be in a position to take the next step forward with regard to discovering the solution of the difference equation.

So, well let us take a typical discrete exponential sequence, if it is of the form a r to the power n that is to say if this is what we call as x(n), then this sequence of members of x(n) would be of the form a x(n) would be of the form a, a r, a r to the power 2, a r cubed and so on a r to the power n and continuing endlessly, this is what we would get right. So, now let us look at x(n) plus 1, this would give us the sequence a r, a r squared and so on, you would get a r to the power n plus 1, and so on endlessly. Now, there are two things to recognize about these two sequences; the first is that there is the constant ratio between corresponding members of the sequence equals for example, a r by a or this is one point, another point is the difference. What is the difference between successive members of the sequence.

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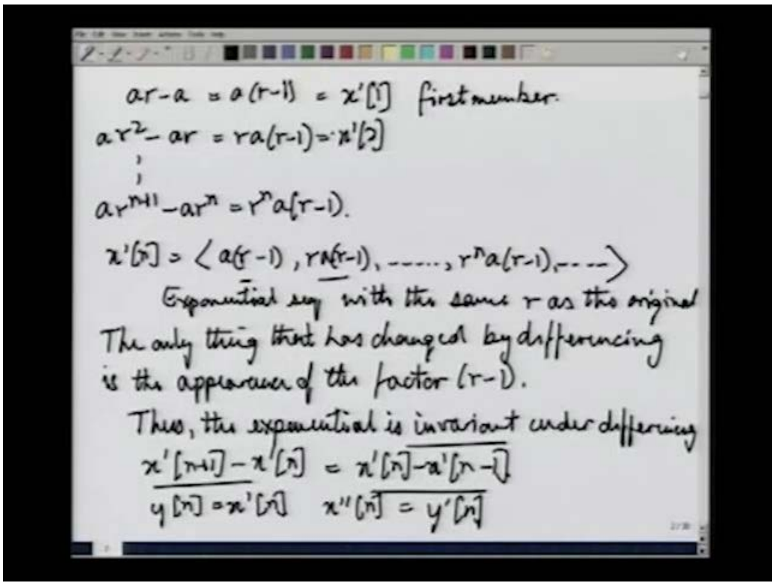

Let us take a look, suppose we subtract the first member of the second sequence of the first sequence from the first member of the second sequence, we would get a r minus a that is equal to a r minus 1. So, this is the first member of x dash n first member, second a r squared minus a r this would give us r a r minus 1 second member. So, this is actually x dash this would be the first we could simply directly write as a x dash of 1, x dash of two. Similarly, if you go on you would get a r to the power n plus 1 minus a r to the power n which is equal to r to the power n times a r n minus 1 r r to the power n a r minus 1.

So, this is the sequence you would get you would get x dash n as the sequence a r minus 1 r a r minus 1 r to the n a r minus 1. So, on which still a sequence that is exponential, why it is a sequence which is exponential, because it matches the first property it still needs the first property that we just discussed namely that there is a constant ratio between a pair of consecutive members. If you take the second member of the sequence and divided by the first member of the sequence, you will get r, if you take the third member and divided by second member you will get r. So, this is certainly still an exponential sequence more importantly the exponential sequence with the same r as the original the only thing that has changed by differencing is the factor is the appearance of the factor r minus 1, that is the only thing that has changed.

So, if since that is only a change by a constant multiplicative factor by our criteria it constitutes or rather it this sequence the exponential is eligible to be called an invariant sequence under differencing, what would happen? If you differentiate once again, that is what would happen? If you took x dash n plus 1 minus x dash n which by the way is exactly the same as x dash n minus x dash n minus 1, this is something that might have worried some of the viewers, because we started by saying that this is what we call the first difference, and later on when x dash was x(n) and x x dash n minus 1 was x(n) minus 1, this is what we call the first difference.

Then what happen was we replace x(n) by x(n) plus 1, and x(n) minus 1 by x(n) does that difference really matter, it does not because whatever differencing operation, we are doing we are doing it for all n. So, if you just write n plus 1 equals m, then you will get the same form as before you will get x m minus x m minus 1 equals x(n) plus 1 minus x n. So, whether you do x x dash n plus 1 minus x dash n at this point to get the second difference or you do x dash n x dash n minus to get the second difference, you can be sure that you get the same result in both cases, you are quacking of the same function in the both cases.

Now, there is really no need to prove that under the operation of second differencing the exponential still remains an invariant function, it automatically will because as far as x dash n minus 1 or x dash n plus 1 or x dash n is concerned, this is also nothing but an exponential sequence, and so if writing x dash makes us uncomfortable. You will completely rename it and call it y(n), then we are just considering then the second difference, that is to say x double dash n is nothing but y dash n, and since y(n) is going to be a discrete sequence discrete exponential sequence as before y dash n, which is x double dash. And will again remain a discrete exponential sequence the only point the only thing that will occur is that another factor r minus 1 will come out. So, that we will now get r minus 1 squared in every member of the sequence as a constant multiplying factor, that is the only change right.

So, we have now established the invariance of the discrete exponential under differentiation or rather under differencing, once we have this very important fact in hand we are ready to go on and look at the solution look at the process of solution of the difference equation. I do not at all intent to do this in full detail, because I have already spent sufficient amount of in fact luxurious amount of time on the study of the differential equation. So, all I will now do is to keep presenting analysis with the differential equation, and very quickly run through the process of solving the difference equation solving the difference equation right.

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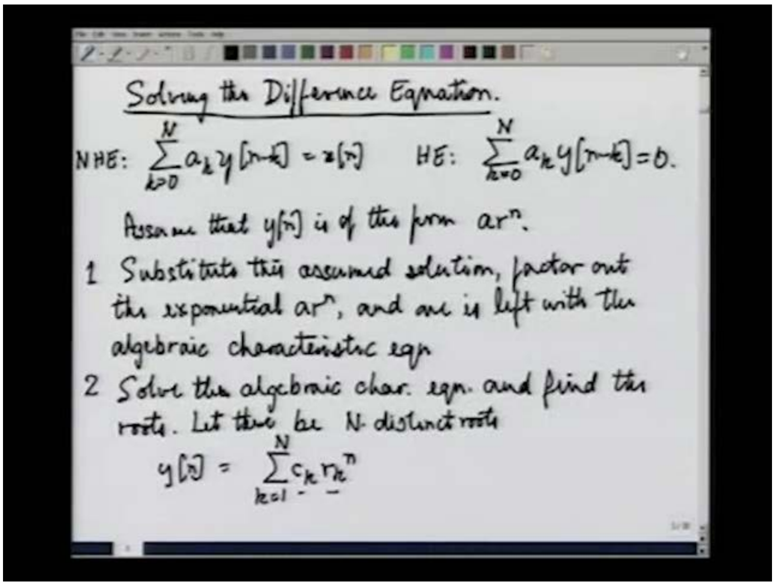

So, if this is the difference equation, we will first look at solving the homogenous equation the homogenous equation the corresponding homogenous equation would be this is, of course what we call the non homogenous equation. So, we have the homogenous equation to look at...

Now, recall what we did last time. We looked we made use of the invariant function under differentiation, and assumed that y(n) would have the form of the invariant function. Hence here to we will assume that y(n) is of the form a r to the power n, if we make such an assumption then is clear that factor a r to the power n will appear in every term of this summation, and hence a r to the power n can be factored out to leave behind just a polynomial, and polynomial in the coefficients a k.

So, what we can say is that substitute, substitute this assumed solution factor out the exponential a r to the power n, and you are left with the algebraic characteristics equation. So, all these is what we will step one? Step two, I am just going to write out the steps, since most of the steps as I said are extremely analogous even step one is analogous to what we used to do for the differential equation, when we try to solve it for the linear constant coefficient ordinary differential equation. So, step two, once you have the characteristic equation solve the characteristic solve the algebraic characteristic equation find the roots fine.

Now, there are several cases that can occur as in the case of the differential equation as it happened with the differential equation, you could have multiple repeated routes, you could have all distinct root so on and so forth. So, in order to avoid getting into matters of details I will assume for the present that we have a set of n distinct roots, then what you will get is a sequence of n different complex exponential each of which would be prefixed with an arbitrary constant. So, what you would get is essentially something of the form y(n), where y(n) is the solution to the homogenous equation place. We are still not even started on the non homogenous equation each of the form summation, let us call them, let us call the arbitrary constant say c k r k to the power n, this is an exponential c k is the arbitrary multiplying constant, and you have. Since, you have n roots you have n capital n possible values of k. So, k goes from one to n this is the form of the solution that you have…

Now, as before we will not stop here, and start examining the arbitrary constant, it is a little too early for that these arbitrary constant do need to be addressed, but not at this point of time before we start looking at them. We have to now look at the solution, we have to first look at the solution of the non homogenous equation.

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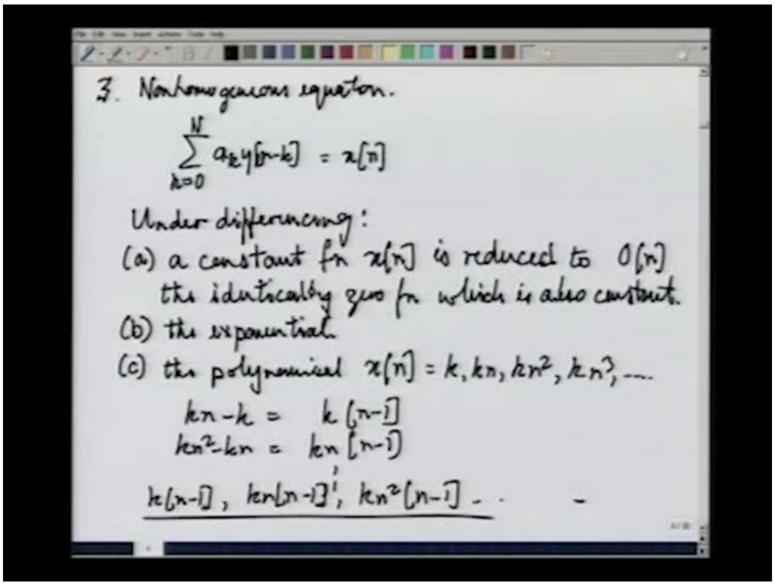

Three well just, just go back for a minute we have solved this, and we have found the roots. So, that there be end district roots then the solution is takes the form given over here, now that we have that we now look at the solution of the non homogenous equation.

Now, again for the case of the non homogenous equation, what we do is very similar to what we did in the case of the differential equation. So, there what we had in hand was only a few cases of possible functions that you going to apply as the forcing function, we confine ourselves to that those forcing functions which had easy solutions namely exponentials constant, and polynomials. We will see here to that these are the three categories of exciting function to our forcing functions for which we will approach the non homogenous solution, solution to the non homogenous equation.

So, the non homogenous equation is of the form a k y(n) minus k k going from 0 to n x(n), well under differencing a constant function x(n) is reduced to the identically 0 function, which you can just call 0 of n. If you like which is just a equation 0, 0, 0 ,0 which is also a constant function, this is obvious because when you do a differencing you take x(n) plus 1 minus x(n) or x(n) minus x(n) minus 1, and when x(n) minus 1 equals x(n) equals x(n) plus 1 differencing would just yield the 0 sequence, that is fine.

So, this is consequence which is invariant to differencing another, we know of course is the exponential, the third we know or we would we guess is the polynomial. So, what is a polynomial sequence? What is a polynomial sequence x(n), it is of the form say k k n k n squared k n cubed. So, on this is a polynomial sequence. We will see, this is invariant under differencing, we are really not seeing if this same sequence appears under differencing. What we are going to see is if we get a function of the same category as the polynomial upon applying the differencing operation, that is what we really have to see, let us if that is the case, k n minus k use u n into k minus 1.

If you take k n squared minus k n, sorry k n minus k will give you k into n minus 1 k n squared minus k n will give you k n into n minus one and so on, this is what you will get? So, you will get the sequence k times n minus 1 k n n minus 1 k n squared n minus one and so on, this is the sequence you will get which clearly is also a polynomial sequence, because it has factors n n squared n cubed and so on. However, it it is to be seen that this sequence is not identical to the original polynomial sequence certainly is not, but that is not a problem all we want is that it remains a polynomial sequence which it does?

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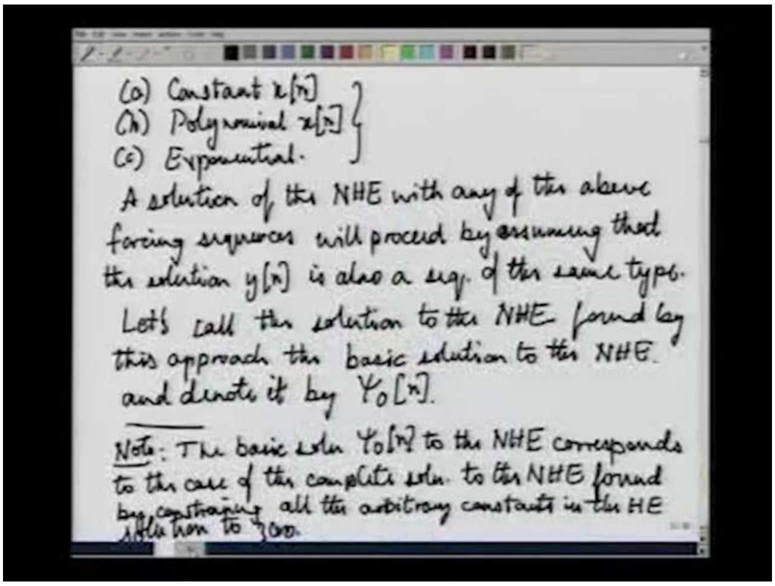

Thus it is only for these three cases of a, constant x(n), b polynomial x(n), and c exponential x(n) that we attempt to solve the non homogenous equation. A solution of the NHE with any of the above forcing function, forcing sequences will proceed by assuming that the solution two is a sequence of the same category, that the solution y(n) is also a sequence of the same type.

So, we assume a solution of this same type, and we go ahead and solve the equation and using this we can find the solution to the non homogenous equation. How we exactly do it is really not necessary to go into it something that you can sort out for yourself its very, very similar to what you do in the case of the differential equation factor it takes similar to that. Let us call the solution to the NHE found by this approach the basic solution to the non homogenous equation and denoted by psi 0 of n right.

Now, let us go and check, what point? We are at this was point three solving the non homogenous equation of point three and this is the end of point three. Now, that we have the basic solution to the non homogenous equation, let us just add a footnote. What do we mean by the basic solution of the non homogenous equation, how is it different from what we have also sometimes called the complete solution. The basic solution to the non homogenous equation is just the complete solution in which all the arbitrary constants to be inserted into the homogenous solution have been set at zero.

So, the basic solution psi naught n to the non homogenous equation corresponds to the case of the complete solution to the non homogenous equation found by constraining all the arbit, constant arbit in the homogenous equation solution to... So, if you constrain the solution the all the constant arbitrary constant to zero in the solution of the homogenous equation, then what you get as the complete solution is nothing but psi naught. So, that is psi naught, and what you get as psi of t is the general case the more general case psi of t. So, we are now at point four.

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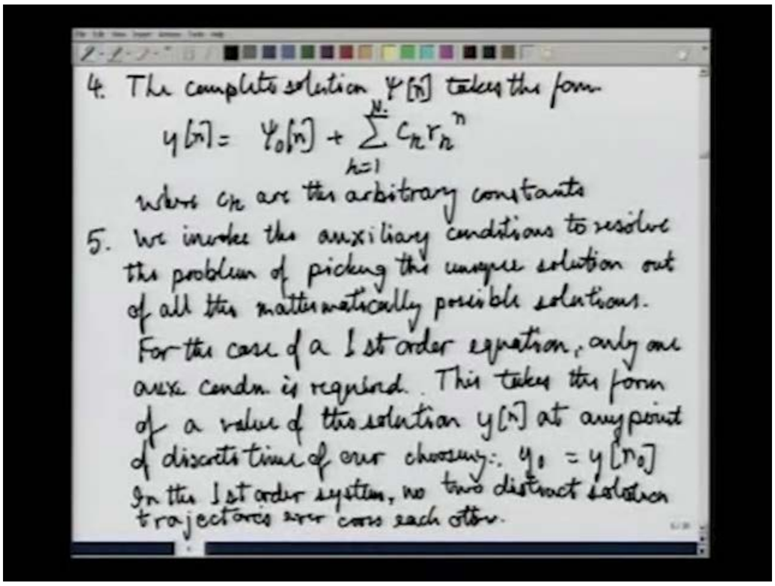

The complete solution psi of t takes the form y(n) is not psi of t its psi of n hangover from our differential equation days y(n) equals psi 0 of n plus summation c k r k to the power n as k goes from 1 to n where c k is the arbitrary constant right. So, as long as you have arbitrary constant, even if you have one coefficient constant, you are about to have an infinity number of solutions, if you have more coefficient constants than one. Then you are going to have more infinites solution, if more infinity makes any sense it really does not actually, but all I want to point out is that you have lots, and lots and lots of solutions just like in the case of the differential equation, but the important part again is how can a physical system have an infinite number solutions. How can even have two solutions for that matter, it should have only one solution.

The answer again is the same as in the case of the differential equations, it is that you have give some external help, help not found in the differential equation itself to pick the correct solution out of the too many that have been supplied to you the physical system has only one solution. And therefore, that one solution must be present in this multitude of solution that the theory is throwing up at us except that you have to know how to pick it up the way, we pick it up is by getting some additional information about the physical solution by in the form of settling value for the coefficient conditions.

So, now come in the coefficient condition, we invoke the coefficient conditions to resolve the problem of picking the unique solution, all the mathematically possible solutions fine. So, we have to invoke the coefficient conditions for the first order case, for the case of a first order equation, only one coefficient condition is required, and that coefficient condition will be in the form of the value of the solution at some coefficient instant of time, just like it was the value of the capacitor voltage at some coefficient instant of time here.

Of course, it is an coefficient discrete instant of time, it does not have to be the initial condition there also it does not have to be here, also it does not have to be it can be any point of time along the discrete time action at which you specify what value the solution has the moment, you do this you have put your finger on just once solution. This takes the form of this takes the form of a value of the solution y(n) at any point of discrete time of our choosing.

So, you choose any n naught and let y(n) take the value y naught at n naught. So, that y naught is the coefficient condition and is given by y of n naught. So, if you do this, you have picked a particular solution this happens in the case of the first order system, because as in the case of the continuous first order differential equation, no to solution trajectories. Though there are an infinite numbers of solution trajectories no to solution, solution trajectories ever cross each other in the first order system no two distinct solution trajectories, trajectories ever cross each other.

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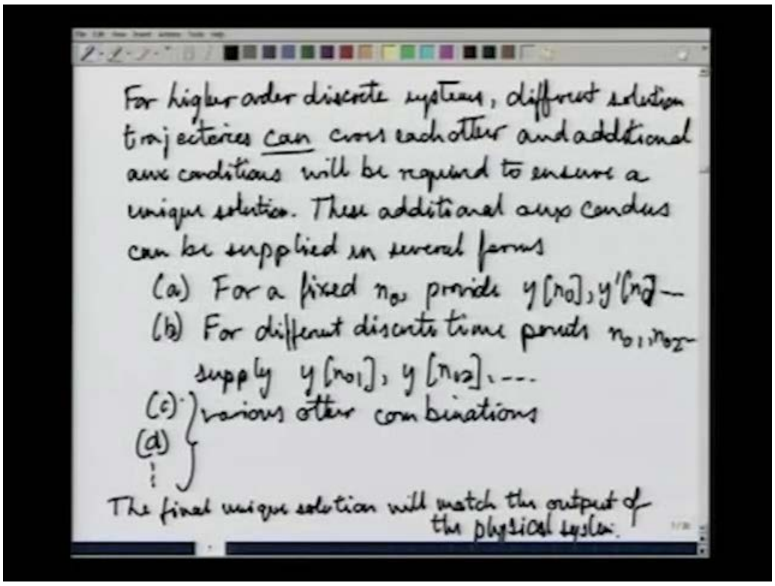

So, since this is the case, one coefficient condition is enough for higher order system. For higher order discrete systems, different solutions trajectories can cross each other and additional coefficient conditions will be required to ensure a unique solution to ensure a unique solution. The additional information could be in several forms can be supplied in several forms, such as a fixed n naught for a fixed n naught for a discrete time instant n naught provide y(n) naught y dash n naught, etcetera. That is one way of doing it or for different discrete time points n naught 1, n naught 2, etcetera, supply y(n) naught 1, y(n) naught 2, so on; c I will guess a various other combinations, this is c, d, and so on all the various other combination.

So, we have multiple ways of specifying the coefficient conditions again very, very similar inform to what we had in the case of the differential equation, and once you set down all the coefficient condition one by one, you tie down the solutions. So, that you narrow down and narrow down, and finally you land up unique solution, this unique solution the final unique solution, the final unique solution will match the output of the physical system right. So, that takes care of the multiplicity of the solutions, I think we were somewhere in the ram of point number four no point number 5, alright. So, point number five, point number four was about composing the complete solution, point number five discussed the multiplicity problem.