Signals and Systems
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Lecture - 2
Domain and Range of Signal

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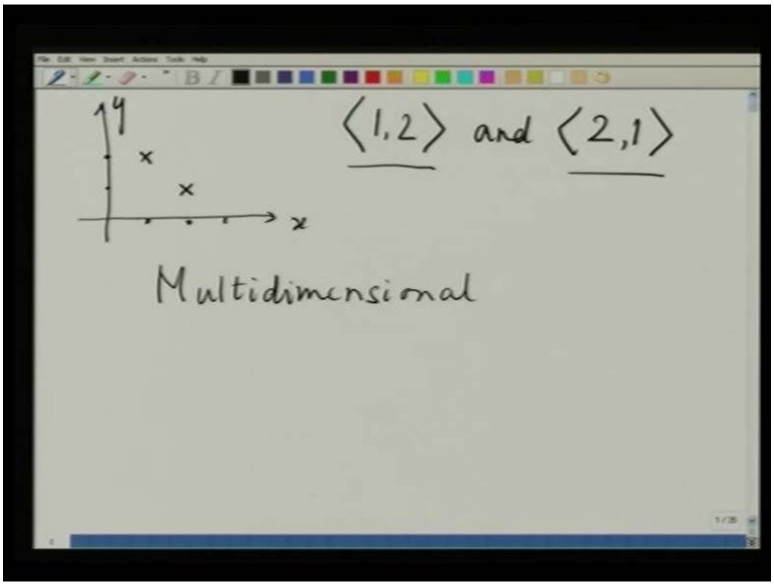

Actually this fact is quite evident from the following diagram, where I have drawn a pair of axis, let us call them x and y. And I plot the points on the plane corresponding to the two ordered pairs - 1 comma 2 and 2 comma 1; 1 comma 2 would be a point between x co-ordinate of unity and a y coordinate of 2. So, if this is 1 and this is 2, 3, so fourth; 1, 2 etcetera, then the point, the first point 1 comma 2 would be around here; the second point 2 comma 1 by the same procedure would land up here. And these two are obviously different points on the plane.

Since our signals are supposed to be functions, which have different values at different points of the domain; and since both these two are clearly different or distinct points of the domain. There is no reason at all, why a signal should have the same value at the first point and at the second point. In short therefore, we mean to exclude the possibility of confusion and consider the most general situation by making a distinction between the points 1 comma 2 on the one hand and 2 comma 1 on the other hand; or more generally the point a comma b on the one hand and b comma a on the other hand. These are always different points.

There is one more occasion, in which we will be concerned with ordered pairs of number, and that is about to be discussed in a moment. But let me wind up my remarks about the domain of a signal, by saying that here we have considered a domain of two dimensions. We could also consider a domain of three or more dimensions, and define signals on domains of higher dimension, such signals where the domains is not a point on a line, but either a point on a plane or a point in three-dimensional or higher dimensional space is called a multi-dimensional signal, multi-dimensional signal.

A picture such as a photograph for example, is an example of a multi dimensional signal, because at each point on the surface of the photograph you have a certain colour value or gray level value depending upon whether the picture is colour or black and white. And since the points on a picture, the points on the photograph constitute a subset of the points on the plane, this is an instance of a two-dimensional domain and therefore, the picture is a two dimensional signal, a multi dimensional signal.

You could correspondingly have three dimensional signals for example, suppose you are in a your room, consider different points of space in the room; each point of space in the room could have a different temperature at a given point of time; points closer to the roof in summer would have a higher temperature than points closer to the floor this is just a fact that everybody knows. And so if we define a signal as the function that takes every points in the room, in the volume of the room to a value equal to the temperature at that point at a given instant of time, this is a three dimensional signal, a signal whose domain has three dimensions. So, signals can be one-dimensional or multi-dimensional and both these terms refer to the dimension of the domain of the signal.

Now moving on, we will now talk about the range of a signal. The range of a signal is the set, out of which the signal takes values. The temperature at a point in a room is the number, which has the units of degree centigrade or degree Kelvin whatever. Now this is a member of a certain set which we called the domain. The range set contains all possible values that the given signal can take at different points on the domain; the set of all possible values that the function can assume.

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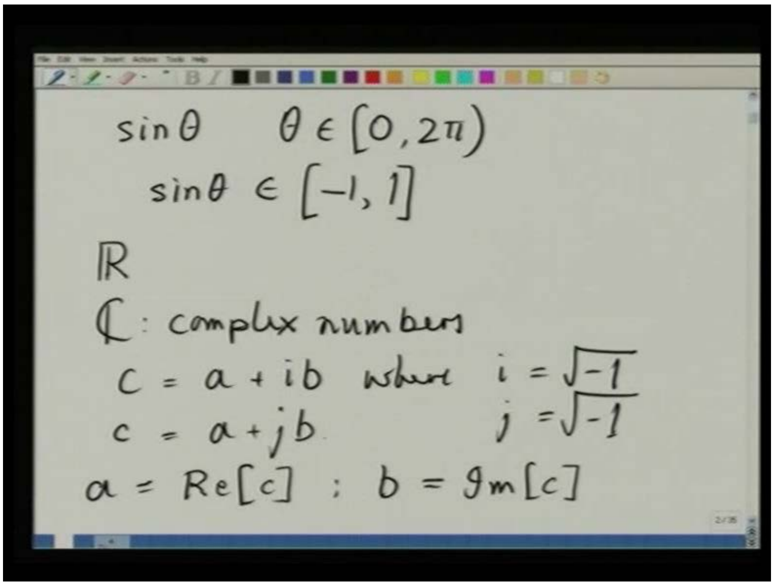

Thus for example if we take a concrete instance such as the signal sin theta, theta now is the domain, and we know what principle values theta can take; theta the domains, the set of all values theta is the interval 0 to 2 pi on the set of real numbers. What about the range of sin theta? The range of sin theta is the set of all values that sin theta can assume when theta runs from 0 to 2 pi. This we know is the set minus 1 to 1. In short if theta lies in the range 0 to 2 pi then sin theta lies in this set. This is the range of a set.

In general the range of a set for the kind of instances we would be considering in this course will be the set of the real numbers. And therefore, we would denote the range by this symbol, but for mathematical considerations which cannot be fully explained at this early stage it is more convenient for us to consider a larger set than the set of real numbers. And we actually chose the set of complex numbers denoted by c complex numbers. Now I would quickly like to go through a review of complex number, complex numbers as we know are represented by ordered pairs of real numbers, we had back to ordered pairs once again as you can see.

And so a general complex number c would be written as a plus i b, where i is the square root of minus one something we know, somehow for historical reasons in electrical engineering the letter I has been used for the other proposes than representing the square root of minus 1, and in electrical engineering instead of i, we use j square root of minus one. So, is the same thing, name alone is different.

So, we would write c as equal to a plus j b; a is called the real part and is denoted by the real part of c, b is called the imaginary part and is denoted by this. So, this is the real and imaginary part representation of a complex number. Since a complex number is also an ordered pair any complex number would be represented as point on a plane.

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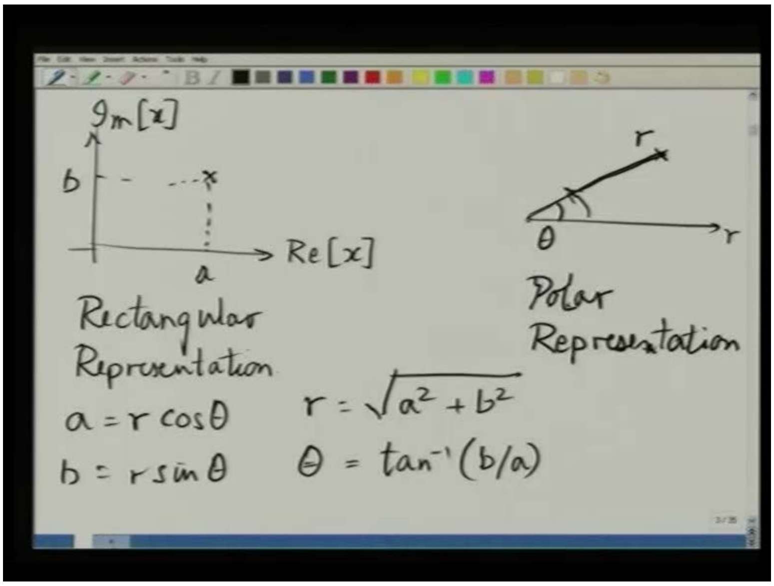

If we use the horizontal axis to denote the real part of any complex number x, and the vertical axis to represent the imaginary part of the same complex number, then a plus j b would be say a point over here, where this would be a and this would be b; different complex numbers would result from using different values for a and b and again it is evident that a plus j b is certainly not equal to b plus j a, they are deferent complex numbers, so the order matters, what we just plotted is often called the rectangular representation of a complex number.

There is the another way in which complex numbers can be represented, and that is in polar coordinates and we can do that to get the polar representation of a complex number. In the polar representation, we convert a and b into polar coordinates using the equations r equal to the square root of a square plus b square, and theta equals tan inverse b by a clearly a and b can be recovered from r and theta, and that can be done by using the equations a equals r cos theta, b equals r sin theta, anyway all this is by review, by way of a review, you already all know of this.

Now using R and theta we could represent points on the plane, but now the coordinate frame would look a little different, I would just draw a horizontal line like this. This is the R direction and the theta direction would be an arc in this form. Any point let us say the same point a b plotted alongside would now lie somewhere over here, its location on the plane would not change, but it would now be represented by 2 numbers, namely this angle which is theta, and this length of this the length of this radial line which is r. So, this R and theta would be given by this equation as well as this equation for r and theta given below.

So, you could use either the polar coordinate system the polar representation or the rectangular representation to represent complex numbers to address a particular complex number. And there are times when using one is more advantageous than using the other this is the point we will be just coming true now.

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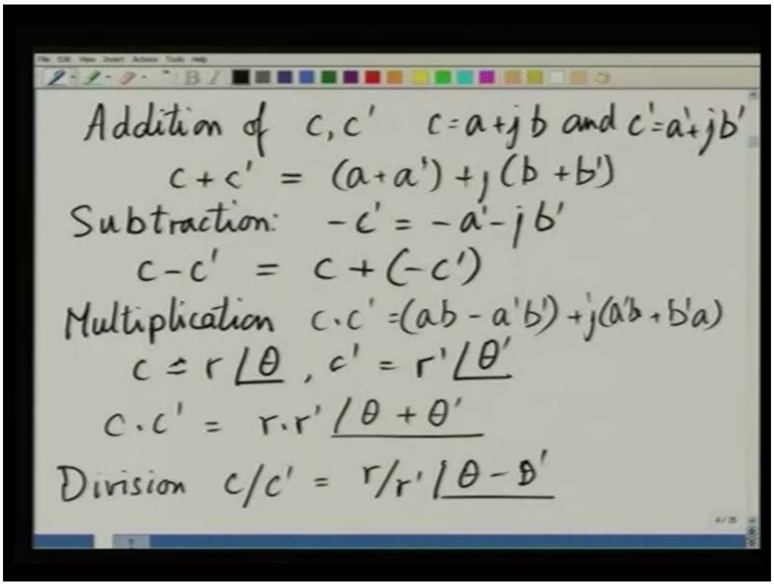

When you want to add the two complex numbers, you would find the rectangle coordinate representation more convenient, because c plus c prime is defined as a plus a prime plus j plus b plus b prime where of course, c equals one second a plus j b and c prime equals a prime plus j b prime. So, this is how we define addition. And clearly it is very easy to add two complex numbers if they are in the rectangle coordinates subtraction is no different from addition really, except that c prime is replaced by minus c prime; minus c prime is defined as minus a prime minus j b prime; this is the first stage of defining subtraction. And subsequently this, we will say that c minus c prime is simply c added to minus c prime, no big deal. Cleary subtraction is also easier to do if we use rectangular coordinates.

Coming to multiplication and division, the scene is a little different. Multiplication here it is not convent to use the rectangular representation, because if we use the rectangular representation, we would get c times c prime as equal to a b minus a prime b prime plus j times j time a prime b plus b prime a; that is not so convenient. Instead if we look at multiplication in polar coordinates is much easier let us say that c is r at an angle theta this is how we would denote the function in polar coordinates, and c prime equals r prime at the angle theta prime. Then the multiplication of c and c prime is given by r times r prime at an angle theta plus theta prime. Likewise division is very straight forward in polar coordinates, c divided by c prime is simply given by r divided by r prime the usual prime is how that r prime is not 0 must be full filled.

The angle of this is given by theta minus theta prime. So, this summarizes the four things we normally do with complex numbers, addition, multiplication, subtraction and division. And you see that for addition and subtraction, the rectangular coordinate system is more convenient, multiplication and division the polar coordinate representation is much more convenient. However, there are a couple of other things we do with complex numbers; and as it happens in the study of signals and system we will be doing this pretty often.

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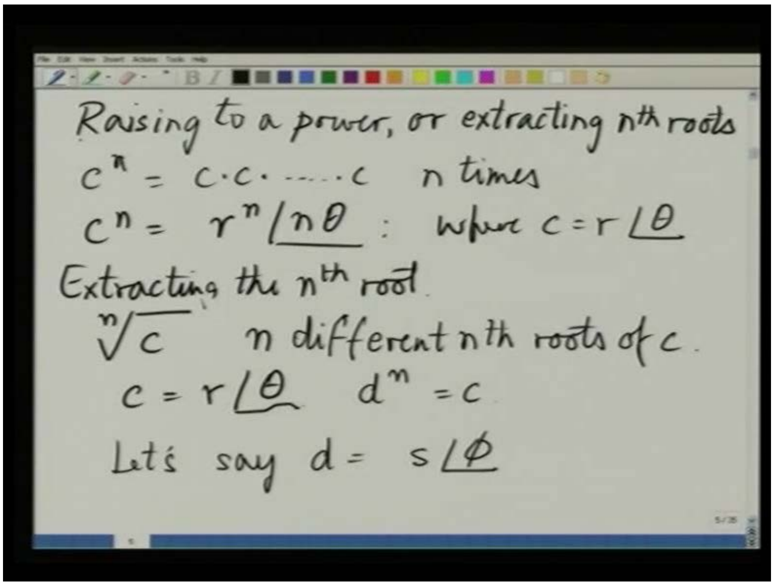

And these things are raising complex numbers to a power or extracting the n th root, n th roots of a complex number. And for this, it is important to just use the rules of division and multiplication that we have already revised. So, how do we raise a certain complex numbers, say c to the power n. Recognize that c to the power n is nothing but c times c times itself n times, it has to be multiplied with itself n times. And therefore, c to the power n is just equal to in polar coordinates r to the power n at an angle n theta, where of course, c is n theta, c is r theta. Extracting the n th root, now this is more dangerous, it is a little complicated, because unlike c to the power n which is a unique number. If you extract the nth root of c denoted by this, then from our knowledge of algebra, we should expect not 1 root, but n n th roots. So, there are n solutions to the n th root of c.

So, how do we find the n different n th root? Let us try to understand this carefully, we have c given again just to remind ourselves by r and theta, fine. Now we want to find a number let us call it d such that d to the power n equals c. Since we have just seen how to raise a number to a power in the earlier lines, we can immediately see that one of the n th roots is very easy to find, what is that one of the nth roots that is most easy to find, it is simply d equals the n th root of r that is the magnitude of d the length of the radial vector of d that is clearly just extracting the nth root of a real number; that is the straight forward. Now this is the angle and we could always take this angle to be theta by n.

But then what about the n th root of r? The n th root of r in this case we will only choose the positive number which is n th root of the r, we are not concerned with complex numbers or negative numbers at this stage, because we know that if this nth root of r has to represent the magnitude of a complex number, then since the magnitude of a complex number is always positive or at least non negative, this has to be the non negative root of d, non negative n th root of d, and finding theta by n also should no problem. However, this is just one of the different n th roots of c. What about the remaining roots? In order to understand how we find the remaining roots, let us first plot c on the complex plane.

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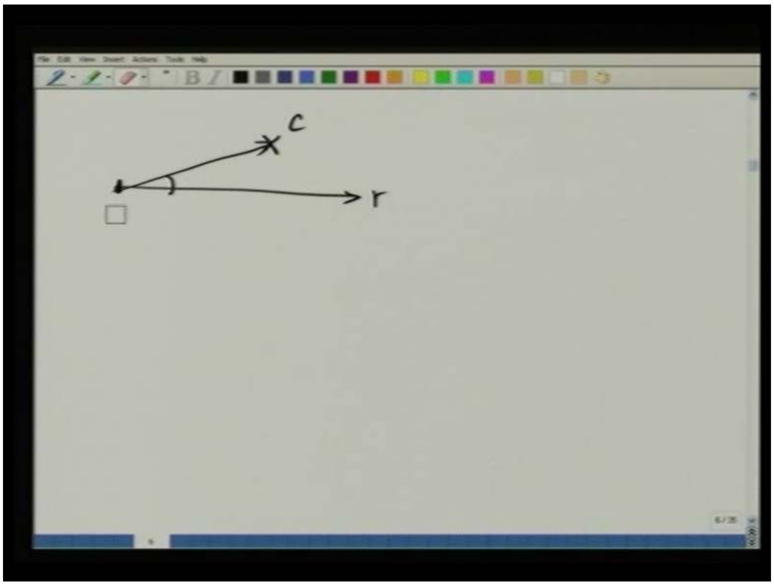

Let us say this is the origin, this is positive direction in the radial direction and let us say that c is over here this is the point c. Now you see what we have just obtain as the nth root of r with a phase or angle equal to theta by n is one of the roots; it is the most straight forward root, it is obtained by just taking this radial vector, finding its nth square root and then taking this entire angle, and dividing it to n equal parts.

So, you would get a much shorter vector normally assuming that R is greater than 1 and you would get a much smaller angle. But because of the fact that theta plus 2 pi is also equal to theta, you have a means of extracting more roots. What we will essentially do now, is to see how we will get the remaining roots. Can you break here for a minute? So, let us say d equals s at an angle phi, d equals s at an phi, our task is now to find out what is s, and what is phi.

Now remember that s which is the magnitude of d, quite like the magnitude of c, which is r has to be a positive number. And therefore, from d to the power n equals c we already know that s to the power n must be equal to r. And therefore, s is nothing but the positive n th root of r there is only one positive n th root of r, and we can easily find that out so much for s.

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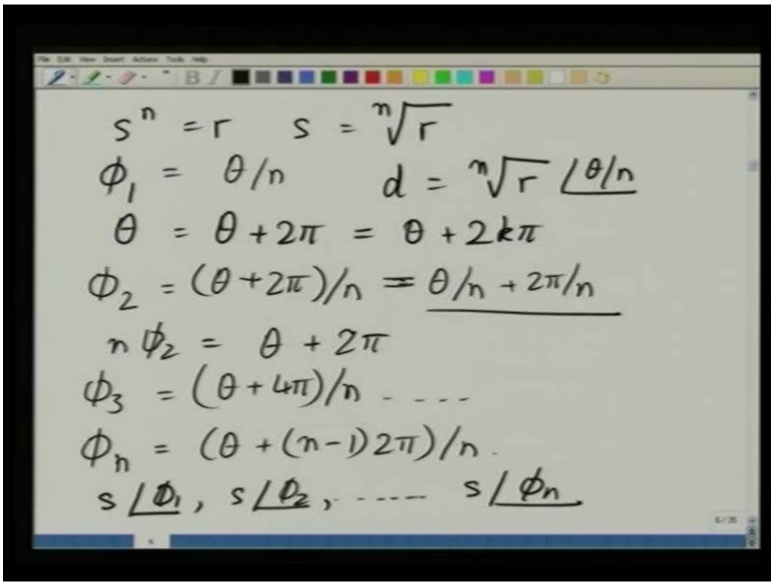

Now for phi, what value can we give to phi? The straight forward answer is take phi equal to theta by n; this of course, is one of the correct answers, but it unfortunately leaves out the remaining correct answers. If phi is chosen to be theta by n then clearly n phi is theta and this particular choice of phi and s would clearly give us one of the nth roots of r of c as follows; d equals the n th positive root, the positive n th root of R at an angle theta by n.

Now, we have to look for the remaining n minus 1 roots, and these can be found as follows; there is little we can do with the magnitude, because s has to remain the positive n th root of r. However, there is room to play with the angle of d namely phi. We have just chosen phi to be theta by n, are there other values of phi, which will when multiplied by n yield the same value as theta. On the face of it seems impossible, but we must recall that angle in a polar representation repeats itself every 2 pi radians; in short theta is the same as theta plus 2 pi. In fact, it is the same as theta plus 2 k pi for various values of k; once we recall this we can see a means of extracting the remaining roots of c by finding other values for phi.

We will now see that we can take, we will call the first value of phi that we have found phi equal to theta by n as phi 1. Now we can say that phi 2 is just theta plus 2 pi, the entire thing divided by n, which means phi 2 equals theta by n plus 2 pi by n. Now this quantity theta by n plus 2 pi by n is quite certain to be different from theta by n, because we have another angle 2 pi by n added to it. But what happens is that when we multiply phi 2 by n, we will get n phi 2 equals theta plus 2 pi which comes out to be the same angle as before.

Thus phi 2 is another legitimate angle that can be assigned to the n th root of c or equivalently to the another choice d, for the angle of d. Now we clearly see how we can get more values for phi phi 3 could be taken as theta plus 4 pi by n and so on, until we come to phi n equals theta plus n minus 1 time 2 pi by n. Each of these gives us different values for phi, and we would say that the n nth roots of c may be enumerated as s phi 1 s phi 2 and so on until s phi n.

So, there are n n th roots of the complex number c; in general most of this would be complex, but there could be certain specific instances where the choice of n or the choice of c would yield more than one root; that I think summarizes what we need to know about the theory of complex numbers as we will be using in signals and system theory. The last thing I would just like to touch upon relating to complex numbers is how we have complex valued functions, how we represent complex valued functions.

A complex valued function is a function whose value at each point in the domain is a complex number. So, the complex valued function can take complex values at the different points of the domain, and adding two complex valued functions would require you to add the values at corresponding points and therefore, add complex numbers, you would similarly require to multiply complex numbers sometimes and so on; this was the reason for the brief review that we just did.

This was of course, the instance where we have chosen the range to be the set of complex numbers there could be situations where we want a more and more intricate set for the range of our signal. Just like for the case of the domain we started with a one-dimensional domain namely the set R or the set z, which was the set of the integers, we could use a higher dimensional range this we would have to use, where the functions we want to represent are actually more complex quantities than just complex numbers. What can be more complex than complex numbers, well here are a few examples of signals where we want to have a more intricate representation than its possible with just a one-dimensional complex range.

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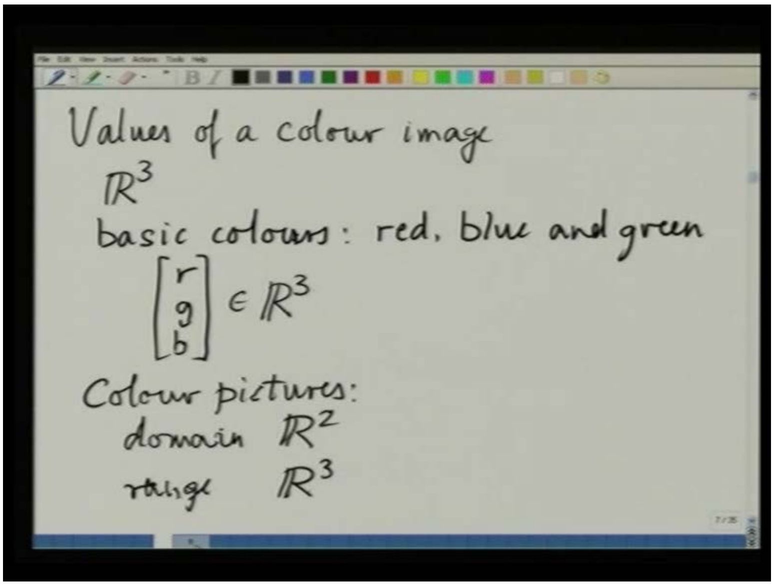

The most common thing that comes to mind is how we represent values of colour image. A colour picture is one, which has different colours at different points on the surface of the picture. Now while it is out of the scope of this course to discuss this in depth, it can be shown that in order to represent different values of colour or rather to distinguish different colours, we need a representation that goes beyond just the set of complex numbers. What is most frequently used is not the set of complex numbers, but a three n three-dimensional real number set; that is we use R 3 to denote different values of colour. This follows from some theory of colour representation, which was done quite some time ago, which showed that human machine can be represented by a set of values called the tri-stimulus values of any colour.

This theory says that if you want to reproduce any natural colour that you absorb with your eyes, then you only need to mix appropriate quantities of three so called bases colour or basic colours. Very frequently we use the colours as red blue and these are chosen as the basic colour. So, any colour that you can see with your eyes would be represented by a certain combination a certain proportion of red blue and green. This therefore, means that in order to label the colour properly in order to represent a colour accurately, we need to specify three numbers; the three numbers would essentially be the amount of red, the amount of blue and the amount of green that have gone into making that colour.

Thus a colour value would actually be a three dimensional vector consisting of the quantities r, g and b and such three dimensional vectors we know are elements of r 3. Thus in the case of representing colour pictures as we find that the domain is two dimensional, it is r 2 and the range is r 3; functions or signals where the range is more than one dimensional are set to be multivariate.

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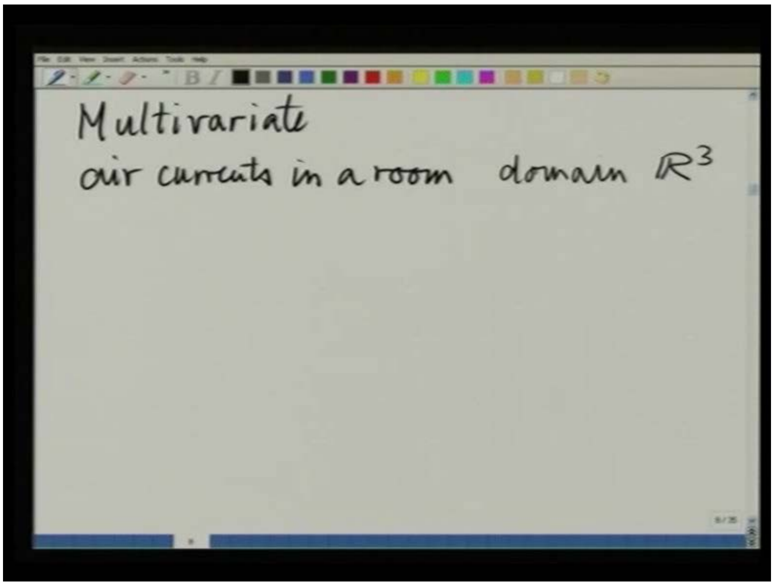

Multivariate functions, the first example of which we have seen as colour pictures can be many more we can find many more examples for them for example, let us say the direction of air currents at different points in a room. So, different points in a room constitute points in r 3. So, for example, air currents room would have a domain as I just said of R 3 because the room is a volume of space. And at each point on the domain which means to say at each point in the room, we would actually have to representation the direction of the currents. That means, we would have to give two angles to give the direction of the air current at that point. If you wanted the magnitude as well as the direction as the air current, then we would acutely have to give the x component, the y component as well as the z component of the air current magnitude at the each point. Functions where the range of the signal requires more than one dimension for representation are called multivariate functions.

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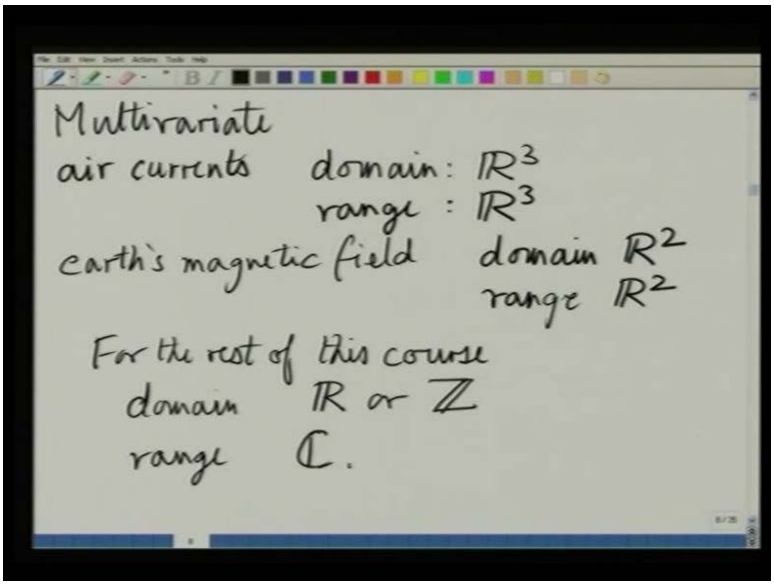

Now, the colour picture is one instance of a multivariate function. We can think of more examples consider for example, air currents at different points in the volume of a room. Suppose we wish to represent this as the signal; that means, at each point within the volume of a room we would like to know the magnitude and direction of the air currents. Now the magnitude and direction of the air currents is essentially the magnitude and direction of the velocity of air at different points. For example, close to the ceiling fan you would have lot of air current magnitudes, mostly aiming downwards; and other points you could have different air currents depending upon local conditions and all kinds of circumstance.

So, to represent air currents in a room, you would need a three-dimensional domain; simply because the volume of the room is three-dimensional, you would also require a three dimensional range; and this three dimensional range would simply be used to represent the three velocity components along x, y and z of the air current at the respective point of the domain. So, at each point on the domain, at each point in the volume of the room, we would have to place a three-dimensional vector.

So, this is another case of a multivariate function, we can construct various multivariate functions in the same manner. For example, another example is the magnitude and direction of the earth's magnetic field at various points on the earth's surface. Now the earth's surface is a sphere, so maybe it is a little confusing to understand what the kind what is the domain is, what is the dimension of the domain. Now though it is spherical and not a flat plane it is quite clear that to identify a specific point on the earth's surface we need two numbers namely the latitude and the longitude; and latitude and longitude are both real quantities. So, the domain becomes two-dimensional; one for latitude one for longitude.

The range is another question here we want to represent the magnitude and direction of the earth's magnetic field if we assume that the earth's magnetic field is confined to the horizontal plane at each point on the surface. Then we only have to give the direction of the magnetic field as so much components in the northern direction. So, much component in the eastern direction or western direction whatever and this is all we would require to do. If therefore, we assume that the magnetic field is confined to be horizontal plane something which is certainly not strictly true, then we would simply have a range which is also two-dimensional. For this course, we will confine our self to the simplest kinds of signals; signals, which have a one-dimensional domain and a one-dimensional complex range.

So, for the rest of this course, the domain we will assume is either r or the set of integers z that the latter is for discrete signals; signals defined on what are called discrete time, I will come to that in a moment. The range in both cases will just be the set of complex numbers. We do not have to worry about multivariate, multi dimensional signals in such an early course. But I did have to motion them just to tell you that there are more complex signals possible and that almost any phenomenon that one can think of can be represented as a signal right. So, we have multi dimensional and multivariate signals, and signals can be represented as maps from domains to range.