Signals and Systems

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Lecture - 19
System Described by Differential Equation

At this point let us recall one important remark that was made about the initial conditions or the auxiliary conditions. The auxiliary conditions I had said could be specified in more than one way when we are considering a system of order greater than 1. If you have just have a first order system then there is only a one auxiliary condition that needs to be specified. And in that context you just specified at some instant of time as I said it could be any instant of time say t 0.

So, y 0 is the value of y of t at t 0 and that constitutes the auxiliary condition, the additional information required to make the solution unique. Suppose, we are instead concerned with a system of higher order, so far there are two auxiliary conditions to be specified as would be the case of the second order system. Then as I has said it that time though only in passing there are two ways you could specify the auxiliary conditions. One way would be to fix a certain instant of time t 0 and specify y 0 that is the value of y at t 0 and specify also the value of derivative, the first derivative of y at t 0.

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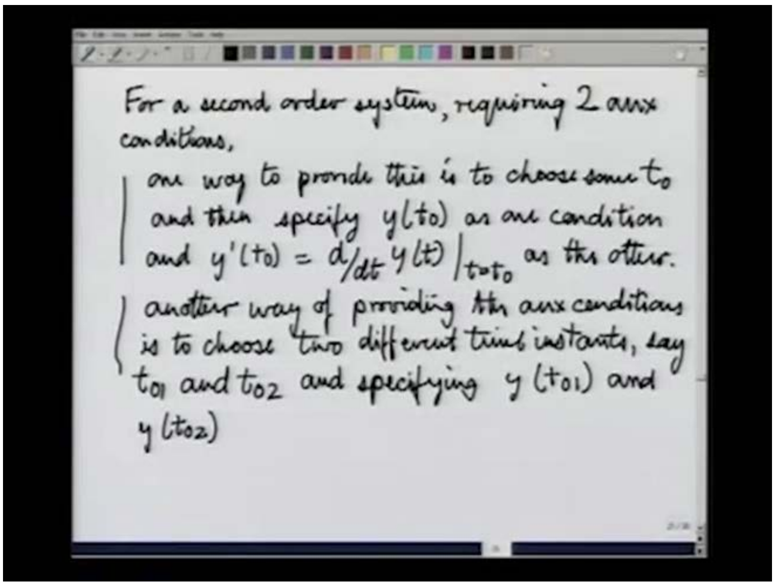

So, for a second order system requiring two auxiliary conditions, one way to provide this is to choose some t 0 and then specify y of t 0 as one condition and y dash of t 0 by which I mean d by d t of y t evaluated at t equal to t 0 as the other condition. So, this is one way of doing it. Another way of doing it of providing the auxiliary conditions is to choose two different time instants say t 0 1 and t 0 2 where the 0 is being used to point out that is a time at which an auxiliary condition is being provided and 1 and 2 are the second substitute meant to distinguish the two time instants.

So, another way of providing the auxiliary conditions is to choose two different time instants say t 0 1 and t 0 2 and specifying y of t 0 1 and y of t 0 2. This second approach would quite (( )) pick a unique solution just as the first methods both of them are equally good. Now, the reason I have had to bring the sub is in order to prepare us better to discuss the next item on the agenda, the relationship between the time invariance of systems that we have spoken of at an earlier part of the course, the properties of the system and the properties of differentially equation that describes the system. In order to bring the sub in its most general format it is required to recognize that auxiliary conditions can be specified in several ways. Now, this was only a second order system.

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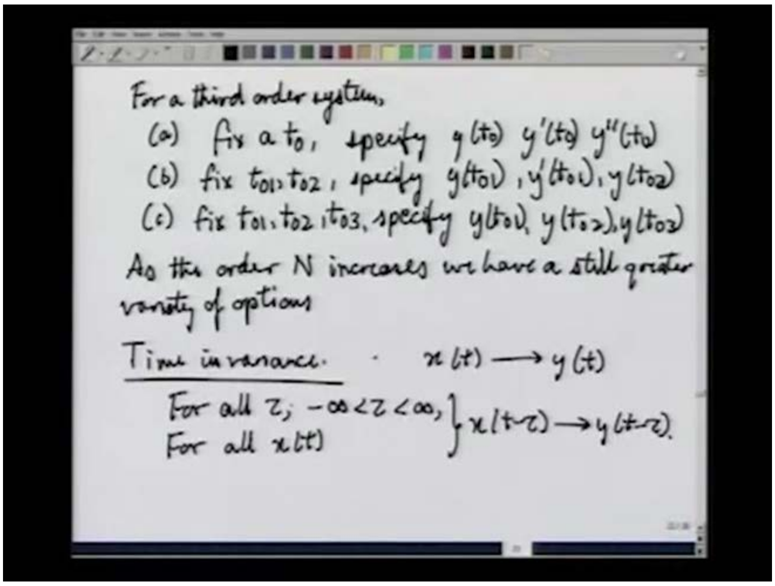

If you had a third order system you could have had for a example either a, fix a t 0, specify y 0 or by which I mean y of t 0 y dash of t 0 y second derivative, y double dash of t 0, this is one way you could do this. You could choose b, fix t 0 1 t 0 2. Specify y t 0 y dash t 0 y t 1 or c, fix t 0 1, t 0 2, t 0 3 specify y t 0 y t 1 sorry something is wrong with the previous line also. I will just going to make all the correction here, y t 0 1 this for the previous line y dash t 0 1 and y t 0 2. And coming back to this line we have y t 0 1 y t 0 2 y t 0 3 and so on.

So, as N increases, the order N increases we have a still greater variety of options. All right, with that point set aside or taken care of, we are now in a position to speak of time invariance. Let us first recall our definition of time invariance. If there was a system for which an input x t yielded and now put y t then the system is said to be time invariant if for all tau minus infinity less than tau less than infinity, for all x t that is to say for all possible signals x t minus tau yields y t minus tau.

This is the definition of time invariance. This means to say effectively going back to our assertion long ago that any system can be completely described by a virtual lookup table where all the possible signals x t are listed on one side and all the corresponding outputs are listed on the right column. This means the, an assumption of time invariance would mean that there is a certain simplification of that lookup table. For example if you specify a certain x t you do not need to specify all the other signals which are merely shifted version are x t in the table.

Instead you can just supply this rule over here and say that given this x t since we are going to get this y t for every other x t which is just a shifted version of this, we will simply get the correspondingly shifted version of the corresponding output. Hence, our lookup table would get a lot more compact, a lot simplified. So, this is the meaning of time invariance. Now, our question in the context of our study of differential equation that describes system is the following.

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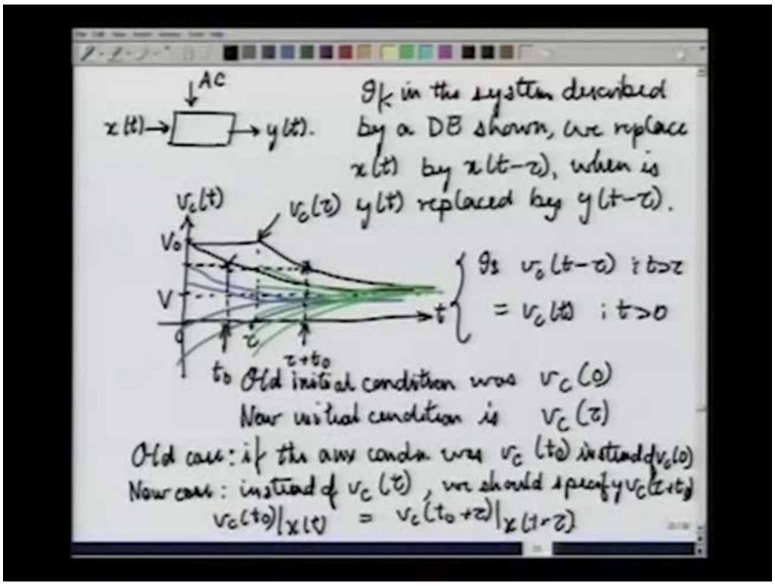

Do you remember we had a block diagram model of a system described by a differential equation which required not only an input x t, but also a certain number of auxiliary conditions which I have just written as A C to get y t. This is the system we have. Now, suppose I wish to ask the question. If to this system I replace the input x t by x t minus tau; what are the conditions under which y t will be replaced by y t minus tau? It would have been so nice if you did not have this problem of auxiliary conditions. If there had been no auxiliary conditions to bother about then clearly x t minus tau would have simply yielded y t minis tau, but there are auxiliary conditions, always there are auxiliary conditions except for the case of a zeroth order system which is too trivial to interest us.

So, even a first order system will have one auxiliary condition to be specified. And higher order system will have even more number of auxiliary conditions to be specified and they can also be specified in various ways as I just pointed out few minutes ago. So, given this complication of auxiliary conditions what makes a system behave like a time invariant system? Let us look, let us go back to our favorite problem of the capacitor and the resistor to throw some light on this fact.

I am not going to draw this circuit again over here, but I will do is I will draw the graph of the solution that we had got, the unique solution that we had got. We had chosen a particular v 0 as the initial condition or as the auxiliary conditions specified at t equal to 0, we had chosen a battery voltage v. So, for this particular choice of v and this particular choice of the auxiliary condition we had got a certain graph. Let us just draw the graph again. This was the curve of v c of t versus time. Given an initial voltage V 0 and given battery voltage V.

Now, what does it mean to say we have shifted the input by a certain time t. In this context it would mean that the switch is thrown at a different time, the capacitor still has an initial voltage of V 0 and we switch on or we close the switch in the circuit at some other time different from 0. Let us say at some time tau. Suppose, we switch this on at some time tau then let us see what going to happen? This capacitor, since the switch was open till t equal to tau the capacitor voltage will be maintained at V 0 till this point of time and subsequent to this it would follow the same track as the previous function did.

What we want to ask is whether v c of t minus tau for t greater than tau equal v c of t for t greater than 0. In this case it is certainly true. This is going to happen, but then there is a change here. Remember, what we specified was the initial condition and in our context the initial condition was that time at which was the voltage of the capacitor at the time the switch was closed. Now, they are saying the same thing. They are saying the voltage of the capacitor at the time the switch was closed, but now it is a different time at which the switch is being closed.

The switch is being closed at tau, not at 0. Hence, the old initial condition was v c of 0 because that was the time when the switch was closed. The new initial condition is v c at tau. Having done this it does turn out that this property is satisfied, that the system is behaving as a time invariance system. This means or rather this at least suggests that certain things need to be done about the auxiliary condition as well. When you shift the input signal by a certain amount which you are now doing by applying the signal at t equal to tau by closing the switch at tau instead of at t equal to 0, then the auxiliary condition that was earlier specified would also have to be specified at an appropriately shifted instant of time.

Thus, in order to understand what I mean by this let us take possible alternate solutions to the differential equation without auxiliary conditions for the old case and for the new case. For the old case if I may choose a different color and to draw alternates curves. For different values of V 0 I would have got a curve like this, a curve like this, curves like this, like this and so on. For the new problem where the signal is being applied at t equal to tau there alternate solutions depending upon variation of initial condition would be given by curves like this.

In short the set of curves use the original set of curves all shifted by the quantity tau in the same direction. So, now let us see if instead of specifying an initial condition that is instead of specifying v c of 0 in the earlier case as specified v c of t 0 at some arbitrary t 0. Let us say t 0 equal to this. Suppose I had specified t, v c of t 0 which would mean this particular voltage. So, for the old case if the aux condition was v c at t 0 instead of v c of 0, what should it be in the new case because we have already said that there is nothing sacred about t equal to 0.

You could specify the auxiliary condition anywhere along the curve you like and therefore t 0 is perfectly valid place to specify the auxiliary condition instead of 0. All that within the old case, but now that we have new case where the input is applied at tau instead of 0, given that the earlier initial condition or auxiliary condition was specified at t 0, where should we it specify now? This is where you should be carefully generalizing properly from the previous piece of information we got.

Earlier we said instead of v c of 0 we now specify v c of tau. Now, what we say is for the new case instead of v c of tau we should specify v c of tau plus t naught. So, earlier we had specified v c of tau which was this. Now, we will specify v c of tau, this is tau. So, tau plus t naught is somewhere over here and we will specify this value which of course, will turn out to be exactly equal in magnitude to the same value. Magnitude and sign and everything to the same value as v c of t naught for the earlier case.

In short v c of t naught when the input is x t is the same as v c of t naught plus tau when the input is x t minus tau. If this point is properly understood we can move on and try to understand what this means for a higher order system where there are several initial conditions or auxiliary conditions being provided. Let us say that we have a set of n auxiliary conditions y 0 1, y 0 2 onto y 0 n which could stand for the value of y and its higher derivatives or it could stand for the value of y at different instants of time, but not the higher derivatives. It could stand for any of these things; I am just using this as a general notation.

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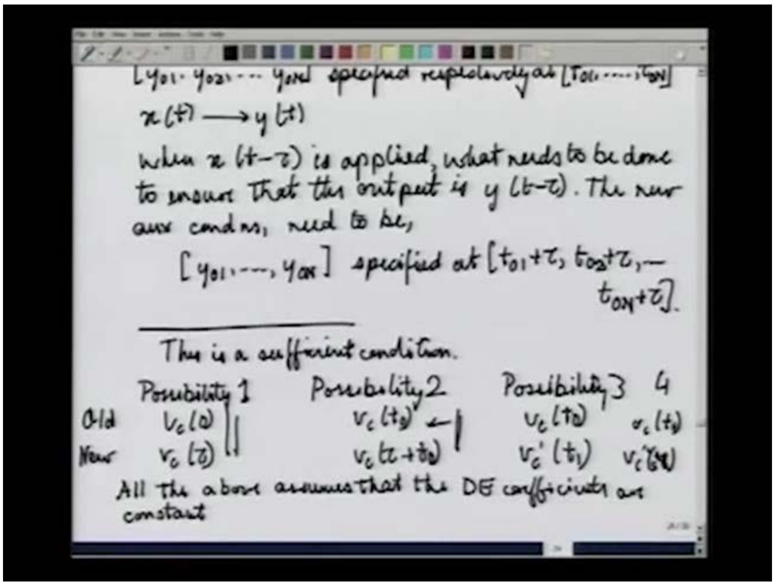

Suppose, we have these and taking the most general case that is they are just the values of y being given at n different instants of time specified at respectively at t 0 1 to t 0 n. So, you have these specified at these instants and you have x t being applied and you get y t as the output. The question we have to ask is what do we need to do to ensure that when x t minus tau is applied instead of x t, what needs to be done to ensure that the output is y t minus tau.

What needs to be done? What needs to be done means of course, in terms of the auxiliary conditions, what do we need to do the auxiliary conditions. And now we can generalize on the basis of our past, recent past experience. What we need to do is to specify the same values; that means, the y 0 1 to y 0 n should remain the same. There values should remain the same, but they will now be specified at shifted time instants, not at t 0 1 to t 0 n. In short the new auxiliary conditions need to be y 0 1 to y 0 n, be the y itself or the derivatives of y or whatever picked at time instants t 0 1 to t 0 n at most.

If you have chosen some multiple auxiliary conditions at same instants of time then you will have a smaller number of time instants. In any case you have a certain set of time instants and values, numbers specified at those times instants. Now, I am saying that the numbers value should remain the same, but the times instants at which these values are specified must be t 0 1 plus tau t 0 2 plus tau and so on to t 0 n plus tau. That means when you shift the input shift all the auxiliary condition times, the times of specification of the auxiliary conditions appropriately by the same amount in the same direction.

Then y t which is what you got with the old set of auxiliary conditions and the old input x t will now become y t minus tau. This completes the specifications that apply with regard to the auxiliary conditions in order to make a system described by a differential equation and a set of auxiliary conditions behave like a conventional time invariant system. This however what we have just derived is only a sufficient condition to make a system behave like a time invariant system.

That is to say shifting the times of specification of the auxiliary conditions by the same amount as we shifted the input signal either sufficient condition to make, to stimulate a time invariant system, it is not a necessary condition. There can be other condition which we could apply alternate to this one and in order to understand what they are, we could just think a little carefully at what we have. Let us go back to the diagrams that we had in the previous pages.

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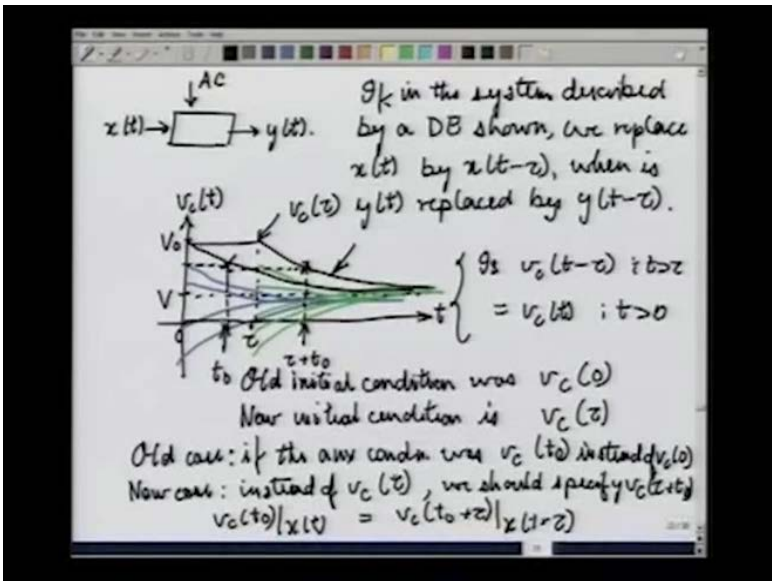

Let us see, let us look at this diagram. Now, here we have shifted the input and we therefore, know that the correct answer is this particular curve. This particular curve is the correct answer because the capacitor voltage has remained the same and now the input has been shifted to a new instant of time. Now, let us since there is no space here I go back to the new page. There we considered two possibilities; possibility one for the old case and for the new case possibility two.

Possibility one was that in the old case we supplied v c of 0, possibility two for the old case was v c at t 0. For the new case the corresponding auxiliary condition positions we have had are v c of tau which would be equal to this. And in this case, v c of tau plus t naught which is equal to this which is equal to this. So, these two are equal and these two are equal. Is there any third possibility? Suppose we had specified v c of 0 initially or say v c of t naught for the old signal x t.

We do not have necessarily have to specify v c of tau plus t naught. Instead we can specify the voltage at some other instant of time, but then the voltage will be different. In this case these two turned out to be equal, in this case these two turned out to be equal, but here since we are changing the time instant, not shifting it by the same amount that the input got shifted we will have a new value v c dash at say some t 1. What v c dash of t 1 should we choose?

Not important, you can just go back to the previous diagram and see that suppose I choose say this as t 1 then I will just have to give this value as the auxiliary condition, this would be v c dash at t 1. After all since it is a point on the same curve the curve that we want to get, we will still get the same unique solution. So, the fundamental problem about auxiliary conditions or the fundamental concept about auxiliary conditions remains exactly the same. There is a bundle of curves and we have to pick a particular curve.

We could do it as I have said again and again in several different ways. Here we are just choosing another different way which still makes sure that we pick the same curve. You could for example have chosen v c of, for the new case possibility four if you want. You can have v c of t 0 over here and here you can have v c dash of tau. That means specify this value over here. So, here you have specified this number, for the new curve you specify this number, that is also fine.

So, there are several ways of doing it, but the simplest of them is to simply shift all the time instants of specification of initial conditions by the same amount as the input signal has been shifted. This I think is a sufficient to discussion of the behavior of systems described by constant coefficient differential equations. Remember, that the system has to be constant coefficient. It does not have to be linear in order to exhibit time invariance, but it has to have constant coefficients.

Otherwise, it would behave differently at different times. The, not only would the system, the signal evolve with time, the system also would be evolving with time and hence we would really not get a system that is time invariant. So, as an important footnote let us write this down. All the above assumes that the D E coefficients are constant. That it is early under that provision that the rest of the discussion is valid. Now, let us quickly move on to a third important property of physical systems that of causality. I am going to keep this discussion brief because I prefer to discuss this not in the context of differentially equations, but in the context of difference equations which I will soon introduce after a brief mention of the causality property.

Now, the question is the following. By looking at the differential equation that describes a physical system can we say whether the system is causal or not? This is similar to the kind of question we asked about linearity and time invariance. For the case of linearity we found that this is, the differential equation has to be a linear differential equation. And we also found that linearity with respect to the input applies only when the auxiliary conditions are all set to 0. So, that related linearity properties of a system to the linearity properties of the differential equation.

Next, for the time invariance property we said that for a system described by a differentially equation to be time invariant we need to shift the times of specification of the auxiliary conditions by the same amount by which we shift the signal. Only then will the output shift in correspondence to the input. Now, we are asking a similar question. Can we look at a differentially equation and say whether the system that is described by it is causal or not. Here, again we find that what a differential equation says about the system is not exhausted. The differential equations description of the system is not exhaustive, it does not constrain the system to be either causal, strictly or anti causal strictly the system is filled to be either of these.

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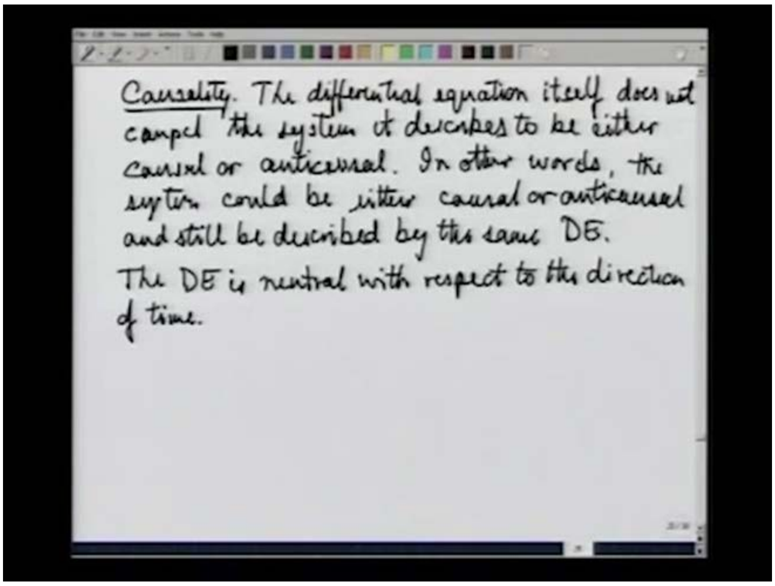

Does not compel the system, it describes to be either causal or anti causal. In short or in other words the system could be both causal or other could be either causal or anti causal and still be described by the same differential equation. The appearance of the differential equation is neutral to the direction of time or the differential equation is neutral with respect to the direction of time. In order to explain this point further I find it convenient to leave it, to leave it b for the time being and move on to the study of difference equations.