Signals and Systems
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Lecture - 18
System Described by Differential Equation

Yesterday, we addressed the question of the multiplicity of solutions thrown up by the mathematical solution of the differential equation. We have found that if certain additional information, which is outside the scope of the differential equation, is also incorporated into the knowledge that we have of the various solutions. Then it becomes possible to select out of all the possible mathematically acceptable solutions, the one which corresponds to the actual physical system.

Thus we are able to get using the mathematics of solving a differential equation, and using the additional information available, a unique solution to the equation, to the behavior of the system, which will match to the actual behavior of the physical system. Having followed up this matter to this extend, we can now go on to studying certain other relationships, between the mathematical model of physical systems and the system themselves. So, one of those important issues that we will now address, is what is already known to us is called linearity.

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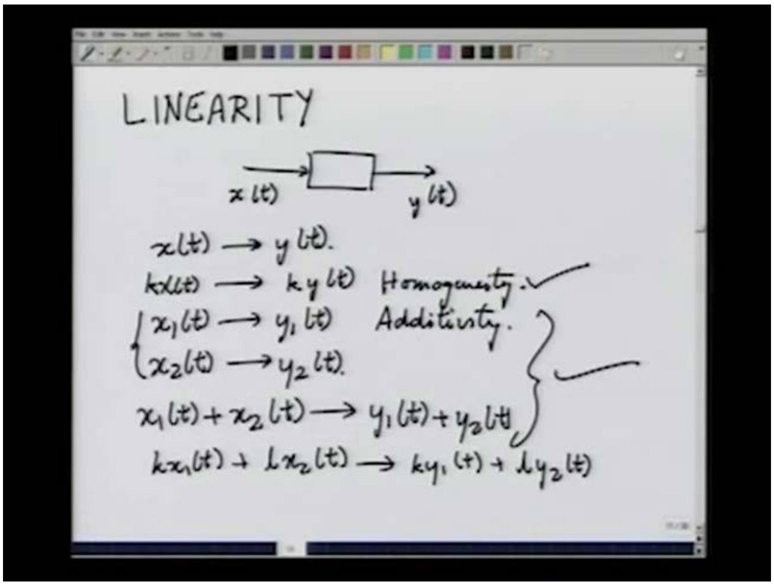

Earlier in this course we have made a definition of what we have the linearity, we said that if we have system, that has an input and an output and if the input is say x of t and output is y of t, then we will say that, given that the systems transforms x t to y t. This same system if it is linear, can also be depended upon to map certain other signals related to x t, to other signals related y t, in short instead of x t suppose I applied k time x t. Then the property of linearity would ensure that the output corresponding to this new input would simply be k times y t, this is the property we called homogeneity.

There was another aspect to linearity which we called additivity, this was concern with applying to entirely different signals or the sum of to entirely different signals, simultaneously to the input. In short here suppose you have x 1 t, which when applied along to the system heals a response of y 1 t. If you had x 2 t, which when applied alone to the system, yielded an output of y 2 of t, then the combination of the two inputs by which of course we mean the some of the two inputs, not any other kind of combination. Namely x 1 t plus x 2 t would elicit a response equal to y 1 t plus y 2 of t, this is what we know in the case with any linear system.

So, a linear system should follow both the homogeneity property namely this, as well as the additivity property which is this. If it follows both these, then we will say that the system is linear. We can summaries both these properties into a single defining equation, by saying that give these two inputs x 1 and x 2 which correspondingly yield outputs y 1 and y 2. Then if we applied say k times x 1 of t plus l times x 2 of t as the input, then the property of linearity would ensure that the output corresponding to this combination would be k times y 1 of t plus l times y 2 of t, this was just a recoup of what we already know.

Now, let us see how a system described by additivity equation, which we have already solved will behave and see how that behavior relates to this definition of linearity. Let us take the same old circuit we had, which involved a resistor or capacitor and a battery with a switch which was close to t equal to 0. If we go to that we will recall the equation we got for this solution, the solution in that context was the equation for the capacitor voltage as a function of time, the expression for the capacitor voltage as a function of time.

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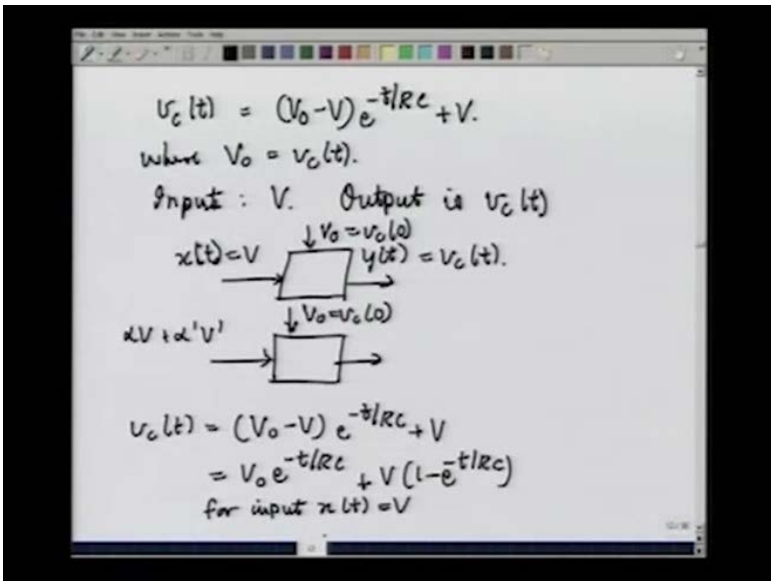

We had V c of t given by V 0 minus V, e to the minus t by R C plus V, where V 0 was the voltage of the capacitor at the reference instant that is at in this case t equal to 0. By the references instant, I simply mean the instant at which the auxiliary conditions have been supplied, so with this we have over here. Now, let us see what is the input over here? Input is the d c signal V, the output is v c of t. So, let us related to the block diagram we had over, here by making another block diagram, what we call x t is now equal to V and what we call y t is now equal V c of t.

In addition we know that some more information needs to be supplied and that information we have called the auxiliary condition, that to needs to be into this system to get the unique solution y t. Let us plug that in somewhere from the top by saying that the initial condition, namely the auxiliary condition for the time instant used as the reference is V 0 equal to V c of 0, this is the auxiliary condition.

Now, this diagram captures everything there is to be captured. Now, let us see is the system or is this equation capable of being linear with respect to x t, namely the input voltage V, the battery voltage V, it is easy to check. We will use the general definition of linearity by taking a linear combination of two battery voltages v and v dash, let us scale the first one by say alpha and second one by alpha dash.

So, let the new input be alpha V plus alpha dash V dash, dash the input. They also condition remains V 0 equal V c of 0 and the output, we will see what the output is?

Going by the equation we have which was that V c of t equals V 0 minus V times e to the minus t by R C plus V, which is equal to V 0 e to the minus t by R C plus V times 1 minus e to the minus t by R C, this is the expression for the output, when V of the input. Now, what about the response for the new input?

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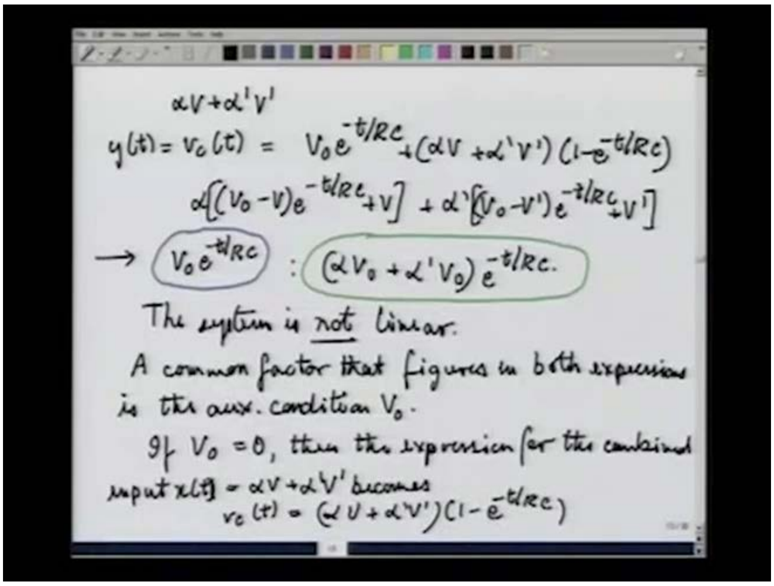

The new input is, if you substitute this quantity in place of V in the expression for V c what do we get? Let us see, we get V c of t namely y of t equals V 0 e to the minus t by R C, which has no room for the input V, this term is independent of V and then we have alpha V plus alpha dash V dash times 1 minus e to the minus t by R C, this is what we have. If this system have nearly been linear, according to our definition of linearity which we had couple of slides ago, namely this, this definition of linearity. Then we ought to have got alpha V 0 minus V, e to the minus t by R C plus V plus alpha dash V 0 minus V dash, e to the minus t by R C plus V dash, if the system have the linear we ought to have got this.

What is the difference between these two expressions, the expression above and this second line that we have written? Simply that on the one hand we just have a term like V 0 e to the minus t by R C, in the actual physical system. Whereas, this stands in contrast to alpha V 0 plus alpha dash V 0, e to the minus t by R C, in the assumed linearity of the system. There is a clearly a problem, because these two things do not match, are we to say that this system described by this equation by the differential equation, which we solved and have found an expression for the output y t equal V c of t, are we to say that the system is not linear? Clearly the answer is yes.

So, it is not linear, what can we do about this? Let us take a close look at the discrepancy between these two equations. You will see that in the discrepancy which is indicated by this particular line where on the one hand you have this in the actual system and on the other hand you have this in the theoretical expression for linearity. In both these expression you find that a common factor that figures in both expressions is the auxiliary condition V 0, this is the cause of the discrepancy.

If V 0 had not been there, which is equal and to saying that if V 0 were equal to 0, then let us see what happens? The expression for the combined input x t equal to alpha V plus alpha dash V dash becomes V c of t equal to the term now on the right side corresponding to V 0 will be 0, because V 0 itself is 0, so V 0 into the minus t by R C will be 0. So, you just get alpha V plus alpha dash V dash times 1 minus e to the minus t by R C, this is what you get, let us just expand this a little.

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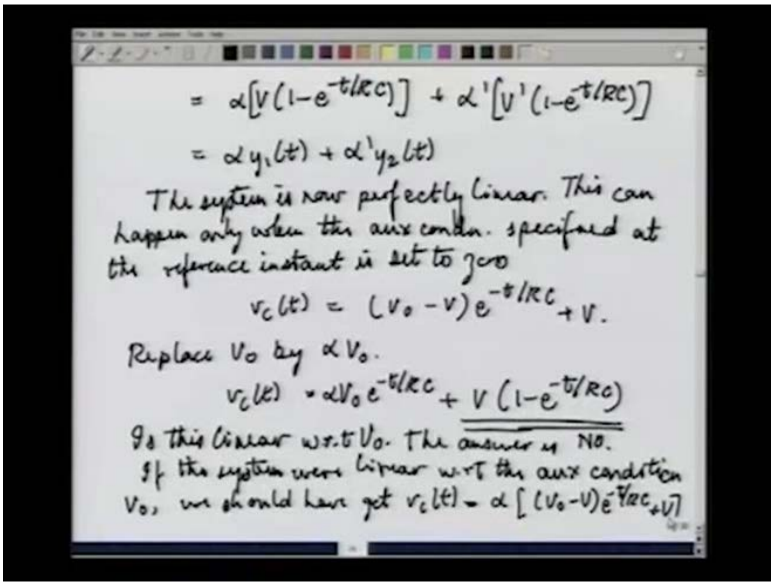

This equals alpha V times 1 minus e to the minus t by R C plus alpha dash V dash times 1 minus e to the minus t by R C, which is exactly of the form alpha y 1 of t plus alpha dash y 2 of t, where y 1 was the response to V and y 2 was the response to V dash. So, the system is now perfectly linear. That is nice to hear, but this was possible only when we made a certain sacrifice or introduce a certain constraint. The constraint was that we had to keep the auxiliary condition at the reference instant equal to 0, that is to say that this can happen only when the auxiliary condition specified at the reference instant is set to 0, otherwise the system is not linear.

Now, let us just explore this from a different angel, suppose we say that in some sense since the output, the ultimate unique output y t is determent or affected both way by the input signal v or x t as well as the auxiliary condition. Suppose, we try to explore the relationship between possible changes we apply in the auxiliary condition and the corresponding outputs that result. Going back once again to this petty equation of ours V c of t equal V 0 minus V to e the minus t by R C plus V, we shall now see what happens when we replace V 0 by some alpha times V 0. We are not changing the choice of reference instant; all we are doing is changing the value of this auxiliary condition at the reference instant.

Suppose, we make this change, then we have to go back to this equation and replace V 0 by alpha times V 0, then the output V c of t will evaluate to V 0 alpha times V 0, e to the minus t by R C plus V times 1 minus e to the minus t by R C. Now, we find that there is a certain change in the output, but there is a change only in that part of the output, in that term of the output that contains the auxiliary condition as a factor. There is no change in the other term in the output where the auxiliary condition is not involved. So, on the whole, suppose we again as the same question, is the system linear or not, is this linear with respect to V 0? The answer is no, not linear.

Taking a cue from what we did last time, let us see if there are any constraints you can impose upon the system in order to make it to force it to be linear with respect to the auxiliary condition. Again the answer just stays right attach in the face, you can see that there is a separate term over here that involves the input namely V. It is this term which is refusing to scale, because after all if the entire system had been linear with respect to the auxiliary condition, then what would we have got, that is if the system were linear we should have got V c of t equals alpha times V 0 minus V e to the minus t by R C plus V.

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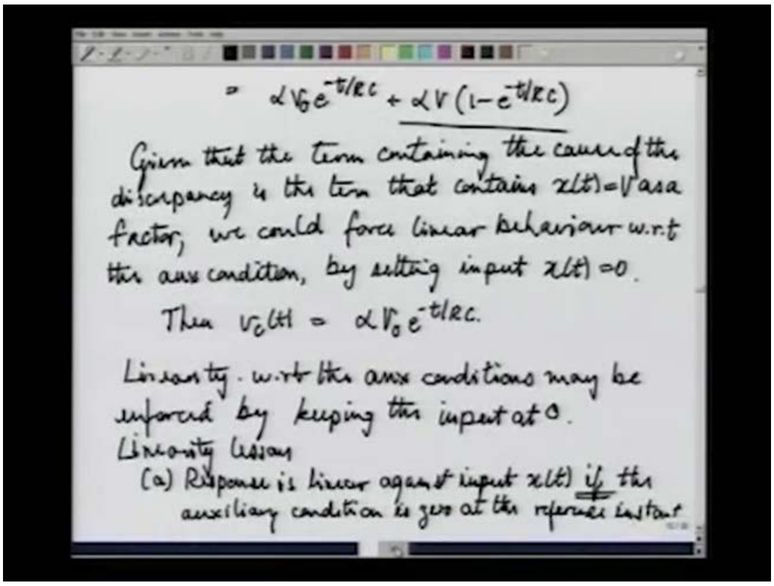

That would expand to alpha V 0 e to the minus t by R C plus alpha V 1 minus e to the minus t by R C, this is what we should have got if it had been linear with respect to this, auxiliary condition, but that is not what is happening. What we are seeing is that there is this term which is different in the two cases, in the case when we are assuming linearity with respect to the auxiliary condition and in the case where we actual equation is being evaluated by replacing V 0 with alpha V 0, this is the cause of the discrepancy.

So, given that the term containing the cause of the discrepancy is the term that contains x t equal to V as a factor, we could force the system to behave linearly with respect to the auxiliary condition by just setting x t equal to 0. By setting the input V equal to 0, we would get perfect linearity, because then V c of t would simply be equal to alpha times V 0 e to the minus t by R C it would be linear. Hence, the summary is as follows, linearity with respect to the auxiliary condition may be enforced by keeping the input at 0. So, to summarize all that we have observed so far, let us put down both these lessons that we have learnt, a, response is linear to input, if the auxiliary condition is 0 at the reference instant.

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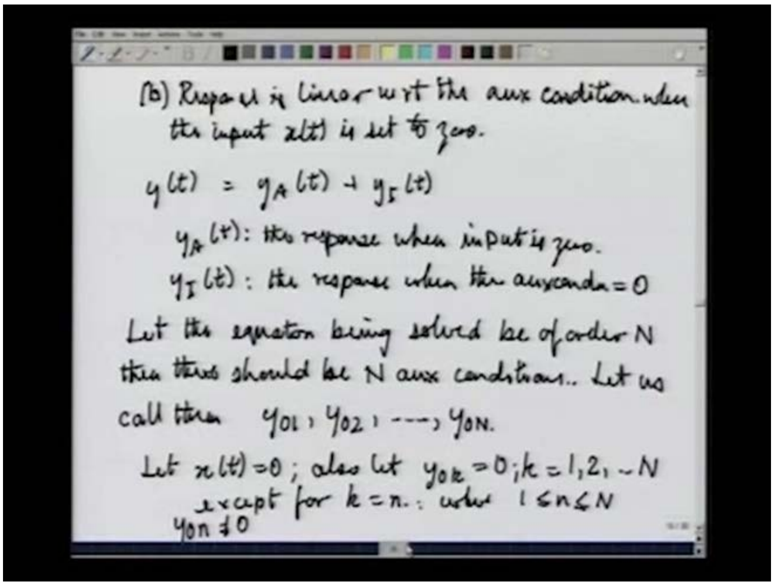

Lesson b, response is linear with respect to the auxiliary condition, when the input x t is set to 0, clearly the problem of the whole system not being linear, either with respect to the input or with respect to the auxiliary condition is happening, because we are adding two different things. In short we can almost say that the output or the response consist of two components, one component that is reverentially linear with respect to the input and the other component which is similarly linear with respect to the auxiliary condition. But when you add the two together there, the combination is linear neither with respect auxiliary condition, nor with respect to the input.

So, this makes it very attractive for us to actually write out the solution y t or the response y t as a sum of two separate components, let us do that. Let us write that y t equals y A t plus y I t, y A t is the response when input is 0 and y I t is the response when the auxiliary condition is 0. So, this has brought us to a certain position of understanding to a certain level of understanding, we only need to put all these things to together and also do a little generalization. What happens when you have more than auxiliary input? This is what we need to look at next.

So, what do we think will happen to a system which is a higher order and therefore, requires multiple auxiliary conditions? Without trying to prove these things let me just tell you that the most natural thing will happen, since the circuit are the systems which are of our concern are linear. Each auxiliary condition will just have to treated separately and the system will turn out to be linear to each auxiliary condition individually, but of course not to the combination of them. In order to explain what this means and to put it down in terms of clear equations, let us just make a few notational announcements. Let the equation being solved be of order N, then there should be N auxiliary condition.

So, let us call this auxiliary condition y 0, y rather y 0 1, y 0 2 and y 0 N, these are the N auxiliary condition. Further let us say that we can force all the auxiliary condition except to 1 to be 0, as well as the input to be 0, when we choose to do so and observe the response of the system to just one particular auxiliary condition not being 0. In short let x t be equal to 0, also let y 0 k be equal to 0 for k equal to 1, 2, N except for k equal to small n, which is some number 1 lying between 1 and capital N. So, that particular auxiliary condition is not 0, that is to say that y 0 n is not equal to 0, all other auxiliary condition are 0. Then it will turn out that the response, please note that x t is also 0, the input is also 0.

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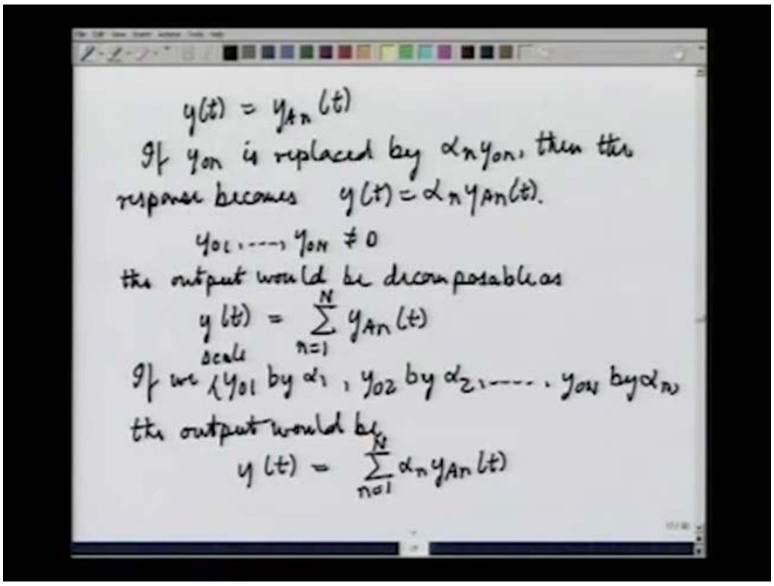

Then the response I will denote by y t equal to y A n of t, that is to say that is the response when only the n th auxiliary condition is not 0 and everything else is 0. Well this is the case, what happens if we scale y 0 n by some factor alpha n, is replaced by alpha n y 0 n, then the response becomes this. However and when we say that the response is this, that seems to be indicate that the system is linear, its homogenous with respect to the choice of alpha n. So, its linear when all other auxiliary condition as well as the input are 0, is linear with respect to the particular auxiliary condition which is not 0.

Now, suppose all the auxiliary condition were not 0, but only the input was 0, that means to say we are just considering the simple situation where only the input is 0 and you have a set of non trivial auxiliary condition y 0 1 to y 0 n. Then the output would be decomposable as y of t equals summation y A n of t, N running from 1 to n. Hence, the relationship of linearity would simply be to say that if we replaced, if we scaled each of these individually by alpha n, then the total output y would be just some of the respective scaled responses.

So, you would get a scaling, if we scale y 0 1 by A 1, y 0 2 by A 2 or let us make it alpha 1 and alpha 2 just to maintain habits. If we scale, we scale y 0 1 by alpha 1 y 0 2 by alpha 2 and y 0 N by alpha N, then what we would get is the output? The output would be y of t equals summation n equal 1 to N alpha n y A n of t, this is the response we would get all right.

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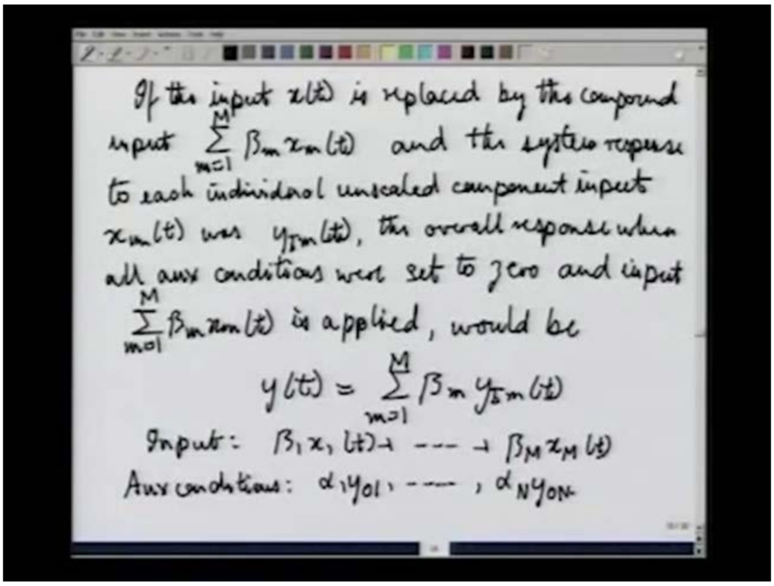

Now, let us keep this in mind and also apply a compound input consisting of several component inputs x 1 t x 2 t x m t, x t is replaced by a compound input, beta m x m of t. The system response to each individual un-scaled component input x m of t was y I m of t, then the overall response when all auxiliary condition was set to 0 and input summation m equals 1 to M beta m x m of t is applied would be y of t equal to summation beta m y I m of t, for m running from 1 to capital M. So, this would be the complete response to all the inputs where each respective input x m was scale by beta m, you just get the sum of the scale corresponding outputs.

Now, suppose you put everything together into a single equation and say that we also have a set of auxiliary condition running from y 0 1 to y 0 n, which we scale by alpha n or alpha 0, alpha 1 to alpha capital N and a set of inputs x m, m running from 1 to m which are all scaled by beta m respectively, then let us see what would happen? We can now write the complete grand equation relating to, that describes the linearity behavior of the system. Auxiliary conditions alpha 1 y 0 1 onto alpha capital N y 0 capital N, this is the complete description of the data that should go into the solution, all the auxiliary condition and all the component inputs.

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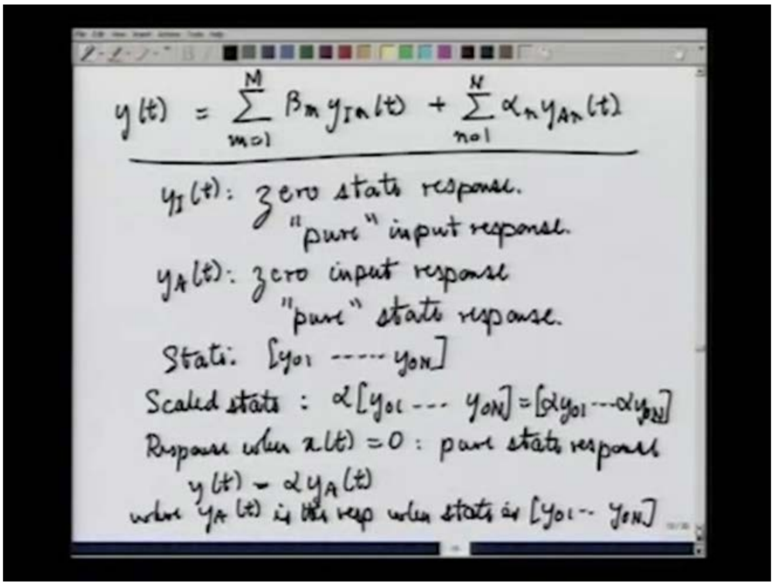

Then we would get y t in the form of the sum of two parts, m equals 1 to m beta m y I m of t plus summation small n equals 1 to capital N, the set of responses due to the individual auxiliary condition, alpha n y A n of t. This is what we would get and this essentially summarizes the entire linear linearity behavior of the linear constant coefficient virtual equation that represents the physical system.

We can see many things in this, but the simplest way of describing this whole thing is just that this consists of capital M plus capital N different components. And while they entire response is linear with respect to each component of this m plus n different components, is that linearity is visible only when that is the only component, which is non zero and all other components are zero.

When more than one component is non zero, then linearity has to be defined in this manner that we have put down over here, this completely describes the linearity behavior of the system. One of short of what we have learnt today is that it is instructive and attractive to consider the output of a system described by a differential equation to consist of two component parts. The component part arising out of the auxiliary condition, and the other component part that arises out of the actual input being applied, so this is a certain kind of d decomposition.

Now, there is a different terminology from for what we have done and I will just now mention this terminology to you, y I of t which I called the response due to the input with the auxiliary condition being set to 0 is called the zero state. Now, what does it really means? It simply means that the system, the condition of the system as described by the auxiliary condition is what we are called in the state and we are saying that if all the auxiliary condition are 0. Then the response of the system due to the input x of t would be the zero state input, the zero state response, sorry this is the zero state output or the zero state response, that is to say it is the response due to the pure input.

When the state is 0 and the response is only due to the input, y A of t is termed the zero input response, because it is the response you observe, when the input is 0, when x t is kept as 0. This can alternatively and instructively be termed as the pure state response. Now, we can simply say that the overall system behavior is linear in two different senses, it is linear with respect to the input when the state is 0, that is to say then y i of t scales with the input. The second statement of the linearity is that the system is linear with respect to the state, where we can now describe the state by a single vector y 0 1 to y 0 n.

Now, it is to say that it is linear with respect to the state is to say that all the arbitrary, sorry all the auxiliary condition are scaled by the same component say alpha, by the same factor say alpha, if we do this. Then the overall response y of t would just be in the absence of course of any input would just be equal to y a of t also scaled by alpha.

In short, if you put this down, we will say if the state is given by y 0 1, y 0 N, then we will say the scaled state is alpha times y 0 1 to y 0 N, which is equal to alpha y 0 1 to alpha y 0 N. If we say this we find that the response when x t equal 0, which is to say pure state response would be y t equal to alpha times y A of t, where y A of t is the pure state response, when the state is y 0 1 to y 0 N, and the input is 0.