Signals and Systems
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Lecture - 17
Physical System Relation with Differential Equation

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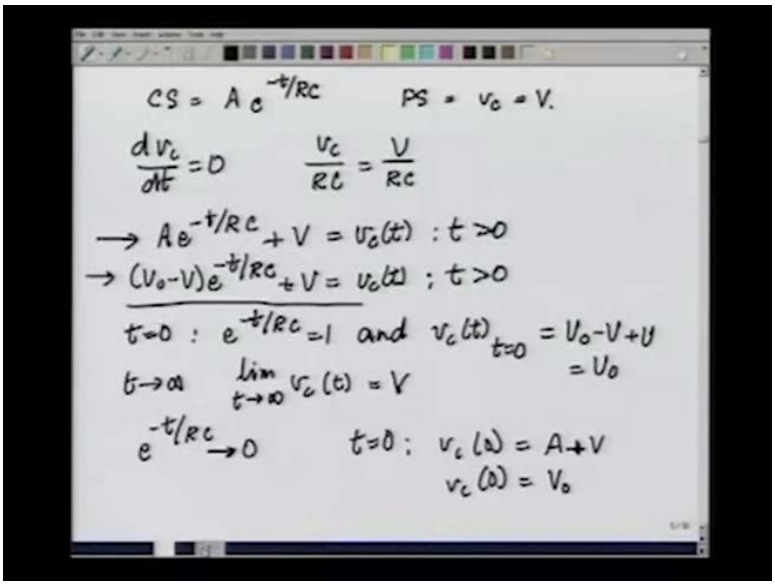

So, now let us look at the other end t equal to 0. The math teachers equation will say that at t equal to 0, you get v c of 0 equals A plus V, whereas the circuit t i, the circuits teacher will automatically tell you that it is V 0. ((Refer Time: 01:02)) the different between the two. Now, clearly the circuit teacher know something, that the mathematics teacher does not. Let us see what it is. In order to understand where we are going, let me look at the math equation for the solution which is that it is A e to the, e to the power of minus t by r c plus V. Let us plot this graph for various choices of A.I will make a nice big graph so that everybody can see it very easily.

This is time and this is V c, right. Now, the equation we wants to plot is V c of t equals A e to the minus t by r c plus. Right, this is what we want to plot. Alright, now A is an arbitrary constant, so I can choose different values of A and get different curves on this graph. Let me start with choosing A equal to 0. If I choose A equal to 0, the entire first term disappears and if for example, I choose v in to v equal to this much. Let say that, this is the value of the battery voltage, then the first curve I get which corresponds to the case when A equal to 0 is this straight line. So, I will write, I will put an arrow over here and say that this corresponds to A equal to 0, this corresponds to A equal to 0.

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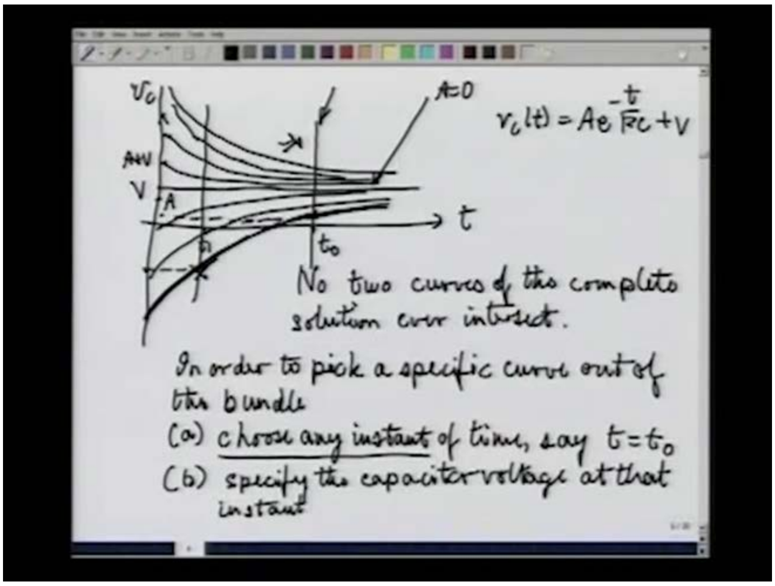

Next let me choose A to be some other number, A plus B is the answer we will get. So, suppose I choose A equal to 1, then I should get V plus 1 as the value of V c at, t equal to 0. And subsequently as t goes to infinity, the capacity of voltage must end up at V. So, it essentially takes an exponential trajectory starting from A plus B and ending at V. Let us say if A is equal to, say this then, approximately here you will have A plus V and from here you will get a curve like this, this for one value of A. Now, let us make another larger value of A, if A is largest then A plus V we would be somewhere over here and you would get a cure like this. Still larger value of A, you would get a curve like this and there absolutely nothing to stop A from becoming negative so I could chose A equal to minus V.

For example, choose A equal to minus V and A plus V will just be equal to 0.The initial condition, the initial value of capacity or voltage will be A plus B equal 0. Now final value would of course, remain equal to V and you would get a curve like this. More values, negative values, another negative value. So, you get this bundle of curves for various values of A. So, the mathematicians says I need some more information to give a single unique solution as a function of time and that information is not captured by the equation itself. Remember that I said that while the differential equation certainly summarizes certain aspects of the behavior of the physical system, it is however not an exhaustive description of the physical system.

We are now coming to the point of finding out what is the part that the differential equations misses and that is exactly what we are coming to over here. What the differential equation tells you is, the rate at which it tends, the rate at which the voltage on the capacitor changes with time. That is completely captured by the route of the characteristic equation namely 1 by r c, that tells you in that part.

Also, the differential equation is capable of telling you what is the final value of the capacity or voltage, we know that that going to V. However, you can have a lot of curves such as, the curves that we have drawn over here on this board. All of which satisfy both this properties, they are all exponentials with time constants of one by r c, they are all exponentials which end or tend towards V, but they still differ from each other they are not all the same curve.

So, we will have to see how they differ from each other. Clearly one very important point you can observe with this set of curves is that, no two curves in this set intersect. Though of course, the way I have drawn it them may appeared to be intersecting in some places ((Refer Time: 06:50)) two curves in this bundle ever intersect. Let me even actually write this down, this is a very important observation. Though it is only valid for a set of solutions for a first order differential equation, that is that, no two curves of the complete solution never intersect.

So, what do I have to do if I to pick a particular curve out of this bundle, what do I have to do to pick a particular curve. Very simple in order to pick a particular curve out of this bundle, I have to do two things. A choose any time, choose any instant of time say t equal to t naught. Let me put this on this graph somewhere over here let us say that, this is t naught and then at that instant of time, let me just draw a vertical line. If I look at all these curves in the bundle and remember that there is an infinite number of curves in this bundle.

Though, we have drawn just a few examples of the different curves that can be there. At every point on this surface of this sheet you can put your figure on any point here and there will be a curve going through that point like this, Sorry, I can choose any point like this and there will be a curve going through that point that curve will also be a solution to this equation. So, there are infinite number of curves here and they literally cover the entire plane of this graph. There are no places where there are no curves, but if you put your pen down at a particular point on this surface then at that point, through that point there will be only one curve passing, one unique curve passing through any point on the surface. This ((Refer Time: 10:28)) an important thing and therefore, what I have said here can continue now. Step a was to choose any instant, step b is to specify the capacitor voltage at that instant. Now, moment I specify a certain value of capacitor voltage, I will uniquely pick a particular curve.

So for example, on this vertical line which represents different voltage values on this vertical line over here, I can choose say this as the capacitor voltage, this. If I choose this then I immediately get this curve, which I have already drawn. That gets fix for all time so I only choose a particular instant of time and specified the capacitor voltage at that instant of time and I have immediately been able to isolate a particular curve for all time. ((Refer Time: 12:04)) important and that is interesting.

Now, if I wanted to pick ((Refer Time: 12:09)) same curve, but instead of t 0, I was working at some other instant t 1. Let us say this is t 1, then I would draw a vertical line here and in order to pick the same curve that I did before I will now choose, I will now need to specify a different capacity or voltage namely this. I will choose this capacitor voltage now. So, this means that depending up on which instant you want to specify the capacitor voltage you are probably going to have to specify a different capacitor voltage.

This is however not true for all curves. Remember the curve we got for A equal to 0, the perfect horizontal line. That is a unique, very special kind of solution out of this familiar solutions because irrespective of what instant of time you choose you just have to specify the V c equals V. And you will get that particular curve, that particular straight line curve in this case. So, you will probably have to specify ((Refer Time: 13:23)) different capacitor voltage for different instance of time, in order to pick curve of you are choice, but not necessarily.

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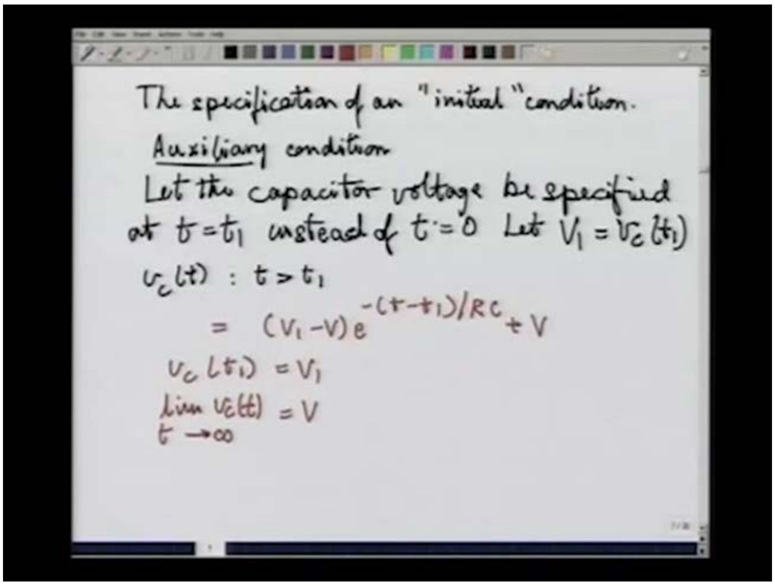

Now let us move on, we have learnt a few things, we have said that we have to choose a particular capacitor voltage what the electrical engineer calls that, is he calls that the specification of an initial condition. However, the short coming of the stunted way of learning how to solve these things from a purely circuit theoretic point of view is that, when the textbook says initial condition. It normally refers to an instant when something important has happen in a circuit like, some switch was opened or some switch was closed.

It is only at that instant of time that the capacitor voltage is suppose to be the initial condition, that means there is an impression conveyed that, there is something special about that instant of time. That represents the initial instant of time and all other subsequent instants cannot be called initial instance. That is why this terminology of speaking of initial conditions is slightly problematic and that is also why I have not use the word initial condition, I have use the word auxiliary condition.

When I say auxiliary condition, I am not confining myself to refer to or to address some particular initial state. Some special state when something interesting happen like a switch was close or like a switch was open, I know and have to confine myself to any such any some special kinds of points of time. It could be any point of time and we could easily appreciate that by going back to these graphs. You have all this graphs over here right. Now we said that for the circuit, which I think is available at an earlier panel, still earlier. Well, let me just read on the circuit over here.

We said that we had a capacitor, a resistor, a switch which was closed at t equals to 0, this was C, this was Rand then we had this battery over here which had a voltage V and we were told that V c of 0 equals V 0. This is the circuit we had over with us earlier. Now, at t equal to 0 we know that the capacitor voltage is V 0. And knowing that we have gone ahead and found the electrical engineers solution to the problem, which was given by this equation, by this particular equation over here, we have this solution. Since, we knew that the switch was closed at that time, we used that and we found the solution. But the point I want to make is there is nothing special about any initial instant, any instant at all can be treated as the initial instant.

For example I can choose that t one shown in this graph as the initial instant then, what will be the capacitor voltage be, it will be some number, which we will call say V c of t one. So, let the capacitor voltage ((Refer Time: 18:07)) specify at t equal to t one instead of, t equal to 0. Let V one ((Refer Time: 18:36)) equal to v c at t 1. Suppose, V 1 is what we use to denote V c at t 1, then can we now write the differential equation, knowing full well can we solve the differential equation, knowing full well that, the circuit switch was not closed at t 1, but it was close earlier, is this going to be a problem for us. Thought the switch was closed at some earlier instant of time.

We want to solve the equation, completely get the unique physical solution with an knowledge not of the capacitor voltage at t equal to 0, but on the capacitors voltage at t equal to t 1. Now, if your knowledge is incomplete, if you are understanding of the theory of differential equations and its application to circuit is in complete, we will say well I really do not know how to use the capacitor voltage at t one to get a unique solution. Though of course, I know how to use it, how to use the capacitor voltage at t equal to 0 to get a unique solution. Now, there is no problem, suppose we want to solve v c of t for t greater than t 1. All you have to do is to get the same curve, but the later part of that curve.

In short instead of finding this entire curve we want to find only, if I could change colors, I will just, instead of getting the entire curve I just wants to get this part of the curve [FL] you just want to get this part of the same curve as the solution. I just done it decided you can see the red curve now that is what I want to get, every solution, so how do you go about it. Now, the initial condition does not correspond to any important event in the life of the capacitor or in the life of the circuit when some switch was close or opened, It just refers to arbitrary point of time in its continuous evolution, except that we know the capacitor voltage at that time. And we know that ((Refer Time: 21:15)) we have given it a name we have called it V 1.

So, we will simply call it, we will now write the equation for the solution. We will write that v c of t, for t greater than equal to t 1 is equal to v 1 minus v e to the minus t by R C plus V. All I did was to replace V 0 by V 1 and now I have the curve, that describes the solution to the equation, subsequent to t equal to t 1. You can verify this, this equation should evaluate to V 1, when you put t equal to t 1 ((Refer Time: 22:07)).

If you put t equal to t 1, it will not, we have miss something over here and that is, that here since the equation has got shifted what we should really write over here is this. Instead of writing e to the power of minus t by R C, what we should really write is e to the power minus t minus t 1 by R C plus V. Now, we have made the correction that was required the intuitive was not right. Now, let us see what we will get, put t equal to t 1 over here the exponent goes to 0 and you get V 1 minus V plus V equals V 1.

So, V c at t 1 comes out to be equal to V 1 and of course, the limit as V c goes to infinity sorry, in the limit as t goes to infinity of V c of t still equal to V, there is no change in that, fine. Now we have demolished the seeming sacredness of choosing the initial instant or the initial condition to happen, only when an event of some importance has occurred in the circuit such as the switch opening or closing. We can choose any instant of time as ((Refer Time: 23:57)) initial instant.

It does not have to be only the instant at which something happen. So that is one important step forward, this important step was possible because if you go back to the curves we note that no two curves in this bundle happen to be intersecting. Since, no two curves are intersecting as I said earlier in order to pick a particular curve out of this set of curves. One does not have to confine oneself to t equal to 0, one can confine oneself to any instant of time, on the positive time access and at that instant of time specify the capacitor voltage.

The moment you specify the capacitor voltage that becomes we so called initial condition, but more generally sine the terminology is not confined to anything initial, the curve as excitant from t equal to 0. But you are specifying a value at t equal to t one greater than 0. We just call this the auxiliary condition, so this is a generalization of the approach that is often used in solving simple circuits. You can specify a capacitor voltage or a certain condition in a circuit at any instant of time. Well, that as for as first order differential equations are concerned, now if you want to go to higher ordered differential equations. Let us just take a brief glance at what happens with second ordered differential equation.

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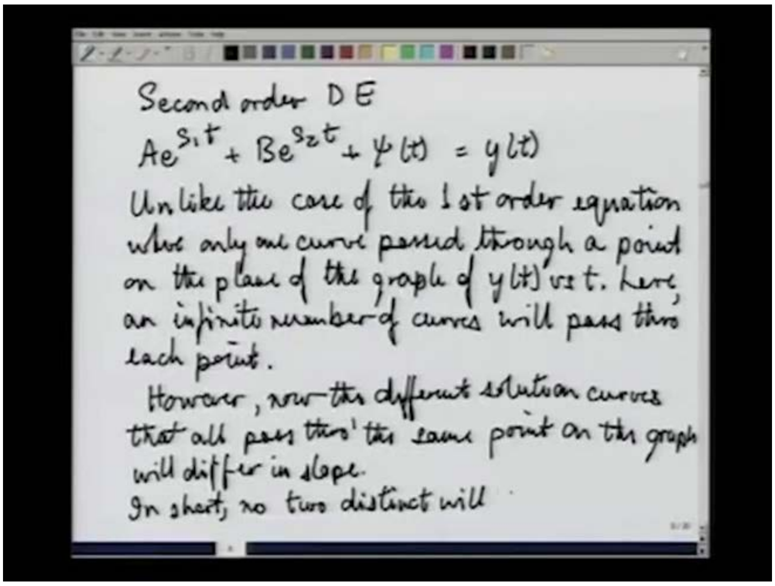

Here, there are two arbitrary constant because you will have a some of two exponentials in the general case provided the roots are not repeated. So, you will have A e ((Refer Time: 26:00)) power S 1 t plus B e to the power S 2 t, as the solution to the homogeneous equation, plus you will have, what you call psi of t, which is the solution to the non-homogeneous equation, this. Since, there are two arbitrary constants if you now try to plot all the possible solutions, the curve of all the possible solutions, then what you will get here is a much bigger mess. You will get, briefly speaking I am not even trying to plot it because this is because the previous graph was messy enough.

What you will get is unlike the previous case, unlike the first order case. Let me write down the difference, unlike the case of the first ordered equation, where only one curve passed to a point on the plane of the graph, of the graph of y of t verses t. Here ((Refer Time: 28:10)) infinite number of curves will pass through the point, pass through each point. And yet there will be something available to distinguish these different curves, the different solution curves that all pass through the same point on the graph, will differ in slope.

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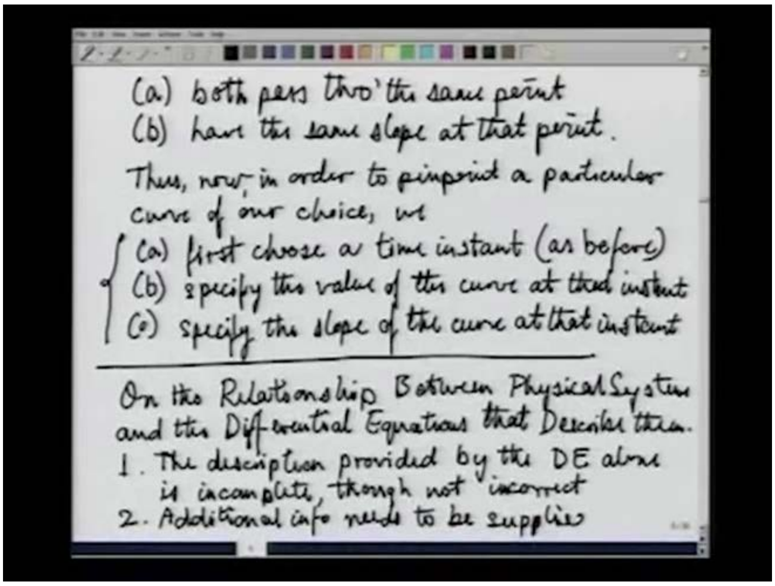

In short, no two different solutions, no two distinct solutions will a, both pass through the same point and b, have the same slope at that point. So, in the first order case we were able to pick up a particular solution out of the bundle of infinite number of solutions by specifying a point of time of our choice. And specifying the value of the curve at that point, at that incident of time that is we just have to pick the point on the graph surface.

Now, we have to pick the point on the graph surface that is having chosen in instant of time, we have to tell the value, we have to specify the value, second we also have to specify the slope. Thus the difference is the following, now in ordered to pin point a particular curve of our choice. We first, choose a, time instant as before that means, as in the first ordered case b, specify the value of the curve at that instant of time, at that instant and c, specify the slope.

If we do all these three things then, we would be pin pointing a unique, a specific solution out of the infinite number of solutions. Here in some sense the number of solutions is the double infinite because there are two arbitrary constants. Each of which can take one out of n infinite set of possible values and ((Refer Time: 34:04)) infinite for a third ordered equation, ((Refer Time: 34:07)) infinite for a fourth ordered equation and so on and so forth.

It gets more and more complicated so for example, for a third ordered equation, I am not even going to write it down. The way you pick specific solution out of the bundle of ((Refer Time: 34:22)) infinite different solutions is, go through step a, choose the time instant as before as step b, specify the value step c, specify the slope step b, specify the second derivative the value of the second derivative .

So, in short for an nth ordered differential equation to pin point a particular solution or a specific solution out of the ((Refer Time: 34:55)) infinite possible solutions that the mathematical solution throws up, you will choose any instant of time of your choice. This is important for whether you are solving a first order equation or an nth order equation, you are always free to choose the instant of your liking. You do not have to choose anything like an initial instant and that is why we just call it the auxiliary point of time at which we are specifying the curve.

Now, choose the instant and then go through b, c, d so on and so forth, where you specify the value, specify the first derivative, specify the second derivative, specify the nth derivative. If you have done all these then you have chosen a particular curve. So that how the importance of these so called auxiliary conditions is brought to our understanding. Earlier, all we have known is that initial conditions have to be specified at t equal to 0, there is nothing special about t equal to 0 as I am saying over and over again. You can choose any t you like and specify a sufficient number of characteristics of the solution, depending up on the order of the differential equation.

Instead of choosing just one particular instant of time and specifying the value the derivative the second derivative and so on, as I just mention for the nth order equation there are several other options available and it is important to understand this as well. For example, in the case of a second order equation, since we have a double infinity of possible solutions, we could take two different instants of time say, t one and t two and specify the value at t 1 as well as the value at t 2. Even this is sufficient to isolate a particular solution, unique solution out of all these solutions that are there.

Even this is possible so many, many options are there and they are all equivalent to each other, they help us specify the conditions of the physical system which is being represented by the differential equation. So, let us summarize what we have learnt today before we close. On the relationship between differential equations that describes systems and the system themselves. One, the description provided by the differential equation alone is incomplete, though not incorrect. Two, additional information, additional information needs to be supplied.

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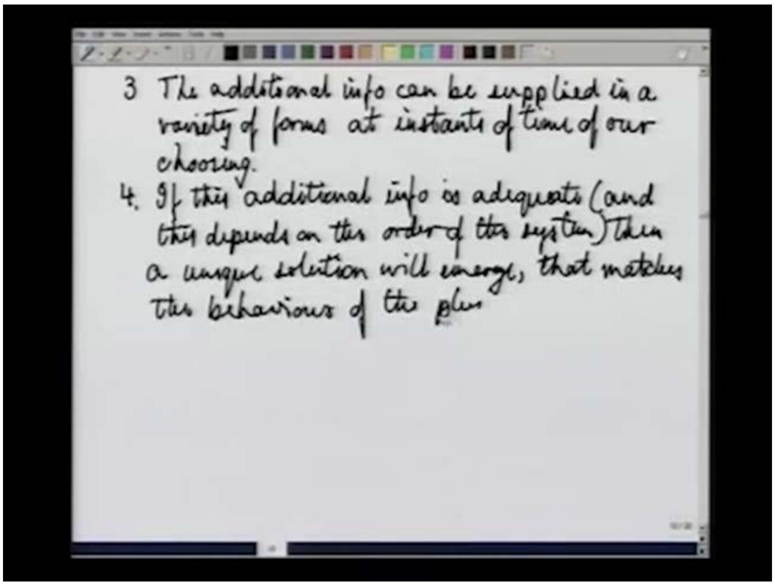

Three, the additional information, additional information can be supplied in a variety of forms at instants of time of our choosing, four if this additional information is adequate and this depends on the order of the system. Then a unique solution will emerge, that matches the behavior of the system, the behavior of the physical system. This only addresses the issue of multiplicity of solutions. What we are going to take up in the next part of this course is, how other issues like for example, the linearity of a system or notions like, the timing variance of a system all these things that we are conventionally ((refer Time: 43:30)) to use on this terms, that where conventionally ((Refer Time: 43:34)) to use to describe physical systems, how they relate to the nature of the differential equation itself.