Signals and Systems
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Lecture - 16
Solving Differential Equation

So, to retrace our steps, let us look at one of the solutions of the algebraic homogeneous equation. Let us say we take s k; some k, which is less than N, of course.

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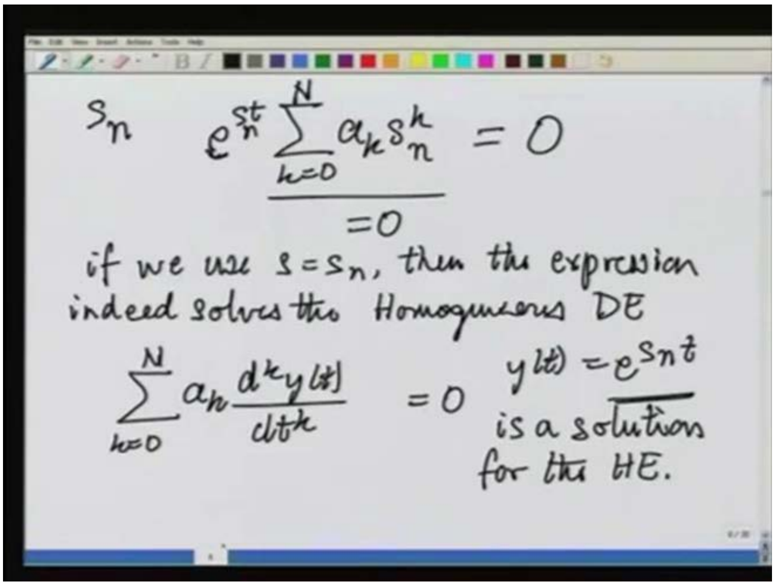

So, what we do with this s k? We know that it is a solution to the algebraic equation. Hence, we go back to the factorized form of the solution of the homogeneous differential equation namely e to the s t summation k equals 0 to N. Let me not call this s k. There is confusion can be caused. I will call it s n. Let say, I call it s n. So, this was our factorized form the solution e to the power s t times k equals 0 to N a k s to the k. This was the factorized form.

Now, if f n is the solution for the algebraic equation, then if I put s n over here, that is I would subscript this by s by m. So, I have chosen s to be the nth root of the algebraic equation. Then, we already know that the right side factor the polynomial factor in this solution of the homogeneous equation will become equal to 0. Hence, this entire expression will become equal to 0. So, if I write n in more these places that is if I use f equal to s n, then the expression indeed solves the homogenous differential equation.

Thus, after I have put f equal to s n in this entire expression e to the power s n t times summation k equals 0 to N a k s n to the k. This entire thing evaluates to 0, because as I said the polynomial factor evaluates to 0.

Thus, we have seen that I have found a solution to the differential equation. The original differential equation; if we recall they are taking now, our second step backwards. We have moved from the homogenous algebraic equation and reach the homogeneous differential equation, the solution of homogenous differential equation. Now, we want to go further. But to recall where are exactly, let us now see that when you have the homogenous differential equation as a k d k y t to d t k. k equals 0 to N. This was our homogenous differential equation. When the right side is set to 0, then we know that if we use y t equals e to the power s n t. Then, this equation is indeed solved. So, e to the power s n t is a solution for the homogenous equation. That is nice.

We may next to solve something at least, after having on round and round is solved something at least. But this that we have solved this part that we have solved as a lot a ramifications; what we have already done will take us another 15 minutes at least to understand the implications of it. Now, from what we have written over here, the first thing I can say is that I just chose f n as one of the N roots of the polynomial algebraic equation. If e to the power s n t is a solution for this differential equation for the homogenous differential equation, then I can make the following remarks immediately. Each of these remarks we done out to be self evident or will require very little explanation first.

If e to the power s n t solves the homogenous equation, then inevitably so will e to the power s 1 t e to the power s 2 t and e to the power s subscript N t. All these will solve the homogenous equation. So, we did not get one solution for the homogenous equation like we wanted. We have already got N solutions. Fortunately or unfortunately, the story does not even end there. It gets much bigger than that as we will see in our little one. We have a n solutions for the homogenous differential equation. Let us choose any one of them and study further. Consider the solution e to the power s small n t. This is one of those solutions. n takes some value between 0 and between 1, N.

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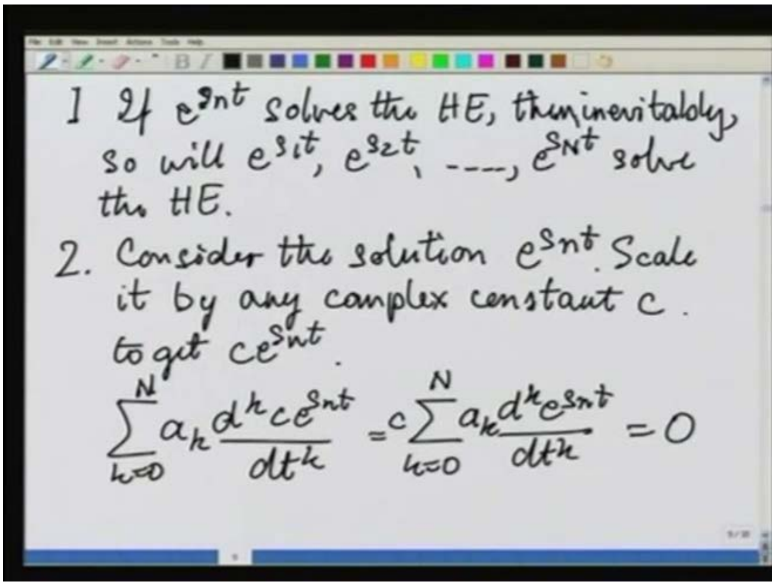

If this is a solution, let see what happens. If we use a scaled version of this same function, scale it by any complex constant say c to get c times e to the power s n t. Substitute this c e to the power s n t in the homogenous differential equation. What do you expect? You will get this. This is what we have. In words, this is equal to every time you differentiate a function that scaled by a constant, the constant factors out of the differentiations.

We know that differentiation is linear with respect to scaling as well as to proposition. Furthermore, we just have some of these terms; c is going to be a common factor in all these terms. So, we will simply get this; c times k equals 0 to N a k d k e to the power s n t and d t to the k. This is c times this same differential equation we had earlier, for which we know that e to the s n t is a solution. So, this whole thing is just equal to 0.

What is that mean? That means that not only are e to the power s n t, n equals 1, 2 up to N solutions for the homogenous equation. So, are other also c times e to the power s n t are solutions. Remember, we can put any value for c. c is the constant, but it is what we called an arbitrary constant. c is an arbitrary constant. Now, there is no room to get confused. What we mean by saying that it is arbitrary and constant at the same time? We simply mean when we say it is a constant, we mean that it is not a function of time.

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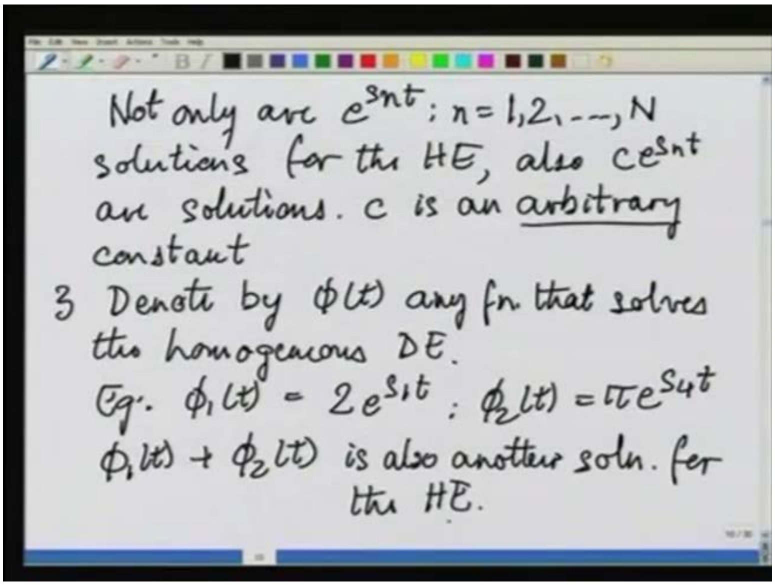

It is once you set a number for it, a value for it, it stays is the same number irrespective of what time it is. On the other hand, you can choose any number you like. So, it is not a function of time. In that sense, it is a constant. It can be any number you like. Therefore, it is arbitrary. It is an arbitrary constant. You can choose c to the any number from plus infinity to minus infinity, if you want it to be real. You could even choose any number on the complex plane if you want it to be complex.

For every one of those choices of c, you will have c e to the power s n t as a solution for the homogenous equation. So, that means that we do not have just one solution for the homogenous differential equation. We do not have just N solutions as we thought a little while ago for the differential equation. We have now an infinite number of solutions for the homogenous differential equation.

In fact, if you pick each one of this s case or s n's, each one of the roots of the homogenous algebraic equation associated with each of those roots. You will have an infinite number of solutions, which you can generate by changing the value of c over and over again on the complex plane. So, in fact you can of course, usually say that you have in infinite times N number of solutions. Infinity times N is still infinity. So, any way, we can say that there are an infinite number of solutions. So, we have all these solutions. Now, the matter does not even end here. That is what it so surprising. Let me now, make the final most general statement that will truly define the scope of the solutions of the homogenous differential equation.

In order to say this, I will first denote by phi t, any function that solves the homogenous differential equation. So, we already know that there are multitudes of these phi t's. I can have phi 1 t. I can have phi 2 t. I can have phi 3 t, each for different from the other. For example, I can choose phi 1 t equals 2 e to the power s 1 t. I can choose phi 2 t equals pi times e to the power s 4 t. Pi is a constant, arbitrary constant. So, I chose it to be pi. Then, 2 is another constant. Here, I have chosen the root s 1 of the algebraic equation. Here, I have chosen the chosen the root s 4 of the algebraic equation. So, in spite of all these variability and all these freedom, I have phi 1 t. It solves the homogenous equation. Phi 2 t also solves the homogenous equation.

So, I have just taken two different solutions of the homogenous equation. Since, I have already known at this stage of the argument that there are lots of solutions; I can choose two of them. I can choose hundred of them very easily. Now, what we will wait for? We can see very demonstratively, very easily that phi 1 t plus phi 2 t also is a solution for the homogenous equation. Then, where that phi 1 t and phi 2 t has certainly distinct functions, they do not look the same.

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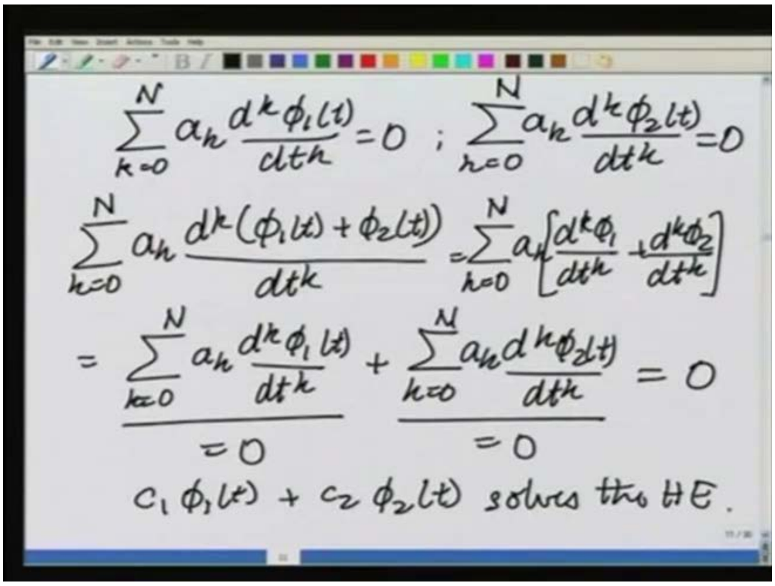

They are not the same function. Furthermore, phi 1 t plus phi 2 t is different from both phi 2 t and phi 2 t. It is non trivially distinct from both. Yet this new creature that is phi 1 t plus phi 2 t also turns out to be solutions for the homogenous differential equation. We can demonstrate this very easily. Let us just do it. Let us take, well, what are our input facts for our argument?

The facts are that we have this says phi 1 t is a solution by assumptions for the differential equation. This equals 0. Likewise, this is also equal to 0 because phi 2 t is also a solution. Now, let see what happens if we apply phi 1 t and phi 2 t simultaneously; that is add, superpose them. Consider them to be a solution for the equation. Let see if it still established. Very obviously it is because what we have a hint is k equals 0 to N a k d k phi 1 t plus phi 2 t by d t to the k. This is what we want to examine. Is this a solution? We suspect it is. We want to prove this.

What is this equal to? This is equal to the summation. This is what it is. So, each term has split into two terms. What we will now do in the summation is to split the entire summation into two summations. Now, we already know that phi 1 and phi 2 are solutions for the homogenous equation. In short, each of these two summations is individually 0. So, the 0 and this is individually 0. So, the entire expression is 0. This proves that if phi 1 and phi 2 are taken out of the set of the homogenous solutions. Then phi 1, phi 2 is also a member of the class of homogenous solutions; solutions for the homogenous equation.

We can even in fact easily establish a slightly more general result. If phi 1 and phi 2 are solutions for the homogenous equation, then in general, c 1 phi 1 t plus c 2 phi 2 t solves the homogenous equation. That is every evident because all that will happen can be demonstrated in the expressions we have already written over here. All that will happen is, we will have d k by d t k of c 1 phi 1 plus c 2 phi 2, which will split into two derivatives a k times d k c 1 phi 1 by d t k plus d k c 2 phi 2 by d t k. By the linearity, the homogeneity of the differential operator, we would get a k times c 1 d k phi 1 by d t plus a k times c 2 d k phi 2 by d t.

Then, we could split into two summations as we have already done. Each such summation would be 0 because c 1 would be a common factor for all the terms in the first summation. c 2 would be a common factor for all the terms in the second summation. The whole thing still comes to 0. Now, this means us to a realization of the vastness on the collection of solutions of the family of solutions. For a homogenous differential equation, we do not have one solution. We do not have two solutions. We have an infinite number of solutions, a huge infinite number of solutions for the homogenous differential equation.

That is the state of the story up to now. Now, we want to add a few refinements to our understanding. We want to see whether we can somehow understand the relationship between the different solutions. It is not the case that though there are an infinite number of solutions, all these solutions are arbitrarily related to each other or arbitrarily different from each other. They are all related to each other in a very nice manner. In order to show you how they relate to each other, I will have to invent the notion of a vector space, the vector space of solutions to the homogenous equation.

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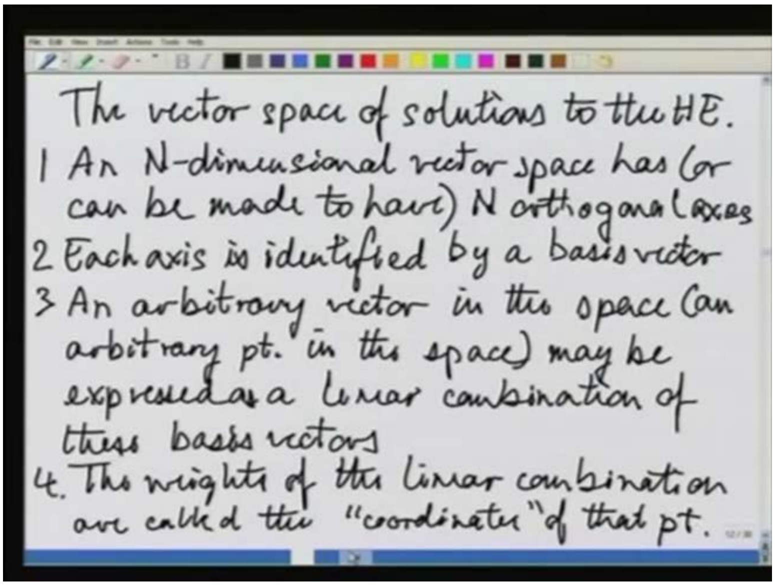

Well, if we do not know what it a vector space, do not worry too much of it. What I am about to say? If you do, you will be happy to understand that this set of solutions to the homogenous equation constitute of vector space. A vector space resembles our three dimensional physical space, n dimensional vector space has N orthogonal axis. It can be made to have each axis is identified with so called basis function, which you would normally call a unit vector in that direction by a basis vector and arbitrary vector in this space. It is to say an arbitrary point in this space may be expressed as a linear combination of the basis vectors.

So, I will just number the points and an arbitrary vector in the vector space, which is to say an arbitrary point in this space. Each point in the vector space is a vector in this shown. It can be may be expressed as linear combination of the basis vectors. Let us make a new point. The weights of the linear combination are called the coordinates of that point or of that vector. Now, what is this? I will just take this vector space, got to do with the set of solutions of the homogenous equation. What it has got to do is just this. We can organize all these huge number of solutions in to the structure of a vector space simply as follows. Let there be an N dimensional vector space and the N basis vectors or the N basis functions. We have chosen to be e to the s 1 t e to the s 2 t and so on.

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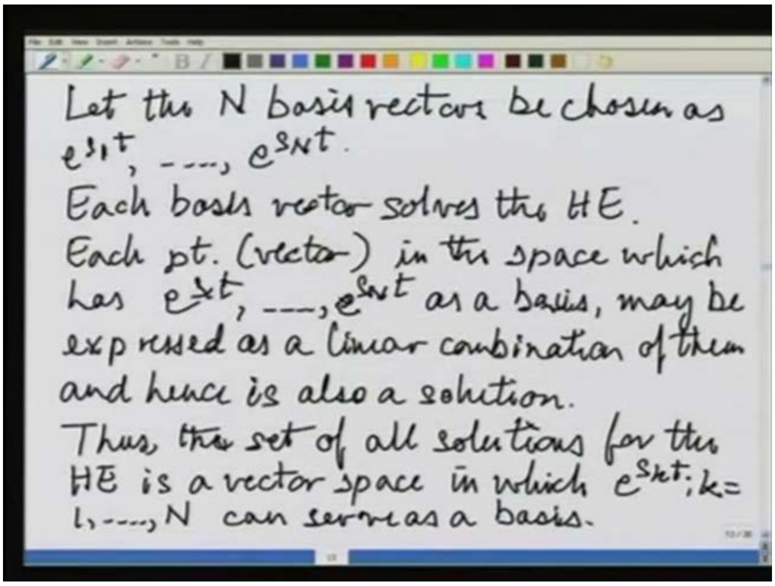

So, if we choose these as the basis vectors, then clearly we can see that not only or each of these basis vectors solutions for the homogenous equation. But any linear combination of these basis vectors is a point to that lies in the same vector space on the one hand. From our knowledge of whatever we have just learnt in the last half hour is also a solution for the differential equation for the homogenous differential equation. Then, in chart, we can say that each basis vector solves the homogenous equation. Is basis vector shafts the homogenous equation and by what? We have found each point by I mean vector in the space, which has e to the s 1 t on to e to the s N t as a basis.

It may be expressed as a linear combination of them. Hence, it is also a solution. So, we have constructed a vector space whose basis vectors are the fundamental solutions, the most basic solutions. We got e to the s 1 t e to the s 2 t on up to e to the s N t. We have found that every point in the vector space, which has this set of vectors as a basis will also be a solution for the same homogenous equation.

So, the concluding remark, the enlightening remark is that the set of all solutions for the homogenous equation is a vector space. e to the power s k t k equals 1 to N can serve as a basis. This gives as a good picture of the manner in which the solutions of the homogenous differential equation or organized relative to each other. This is also probably the last remark that we can make about the homogenous solution alone. We are therefore, in a position to take the next step forward solving the non homogenous equation.

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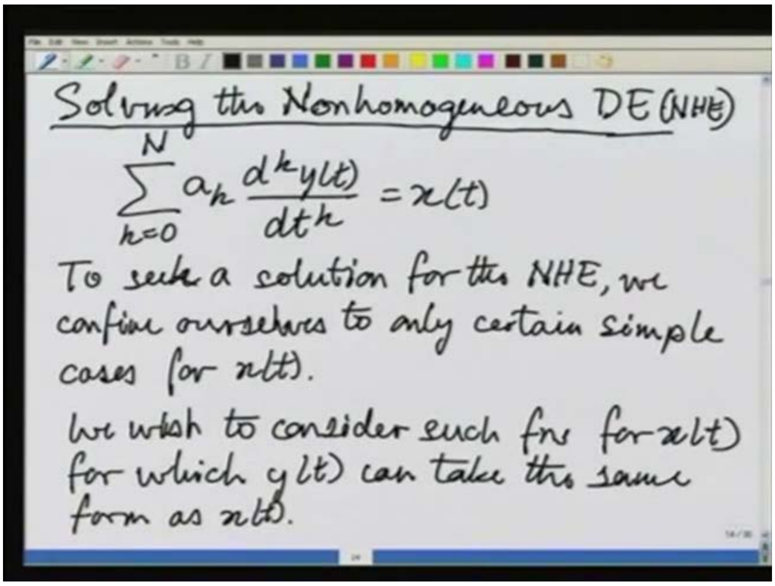

What was the non homogenous differential equation? This was the original differential equation, where we had a non zero forcing function x t. So, let us write it out. This is going to be a different kind of game altogether. We do not have 0 on the right side. There are lots of things we read earlier, which we cannot do. Now, our approach and trying to understand what sort of function can be substituted for y t. We ensure that the left hand side evaluates to x t for every instant of time is going to be governed by other considerations. We want to know how we can do this. Before we even begin, in order to simplify the exercise, we will have to make certain assumptions, some constraining assumptions which will allow as proceeding.

Probably, we can relax these assumptions later. But at the initial stage, when we want to get an understanding, let us constrain our forcing functions to be one of a few different kinds of possible functions. To seek a solution of a non homogenous differential equation, we will abbreviate to n h e, the non homogenous equation. To seek a solution for the n h e, we have to consider only very simple forms of x t. We confine simple cases for x t. What are these simple cases?

What constitutes of simple case? The answer is the following we want to consider such x t's for which y t will be of the same form as x t. We want to consider such functions x t for which y t can take the same form as x t. Further, we want such functions where the derivatives of y t has the same form as y t. So this, what we have just written, we shall call the first constraint. We want to write the second constraint we will say.

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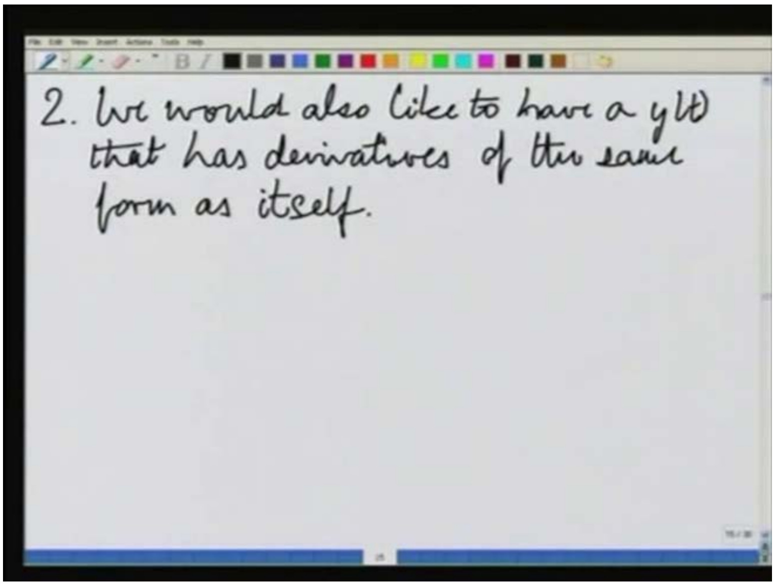

We would also like to have such a y t that has derivatives of the same from as itself. So, two constraints x t and y t should be of the same form y t. Its derivatives must be of a same form. Now, what sorts of functions satisfy these constraints?