Signals and System
Prof. K.S. Venktesh
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 15
Differential Equations

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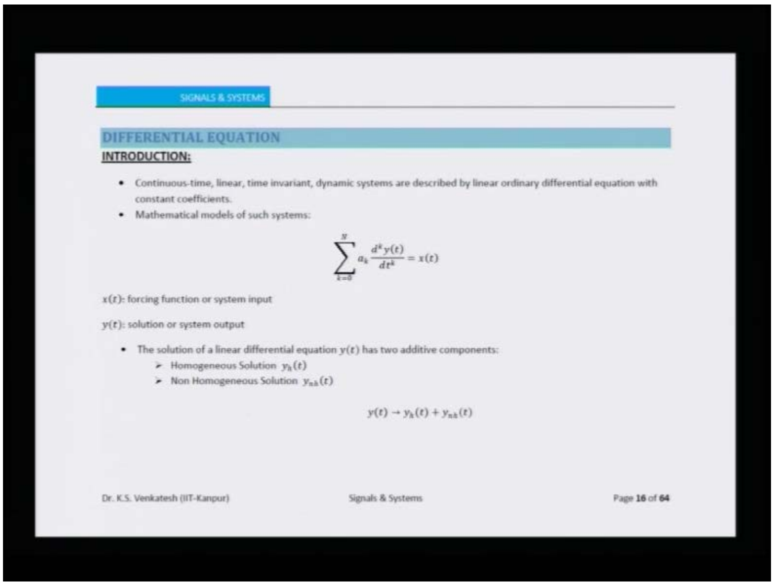

Up to the last lecture, we have studied different points about linear time in variant systems. We concluded with determining from the impulse response properties such as causality, stability, memory etcetera of a linear time in variant system. At this point, we have to take a digression and study differential equations for a while, since I do not expect every student of this course to have a background in differential equations. All I intend to do here is to present the necessary part of differential equations, the part of differential equations that is necessary to proceed with my business no more. So, we will study differential equation for a while then we will see the relationship, we will examine the relationship between differential equations on the one hand and linear time in variant systems which are our main object of study in this course.

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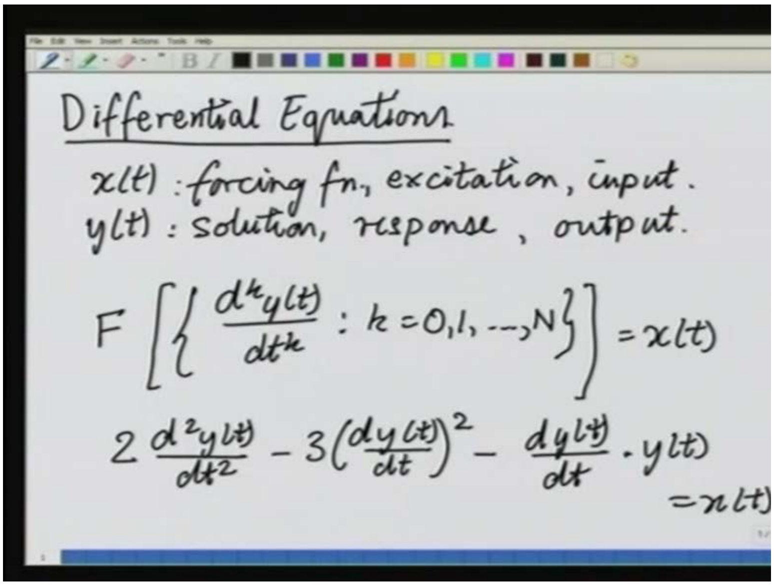

So, differential equations, a differential equation is an equation that relates two functions. In our context, they will relate two functions of time; one function x(t) will be called the forcing function, also sometimes called the excitation. And from the signal systems point of view, we will call it the input, different names for the same thing. The other function that is involved in a differential equation is what we will designate by y(t), the other function that we will concern within a differential equation, we should designate by y(t) and give it names such as the solution the response are from a signal systems point of view the output.

How are these two functions related in a differential equations. On the right side, conventionally of the differential equation will appear simply x(t) - the forcing function. On the left side, we will have a certain combination of the various derivatives of y(t). So, if we denote the k th derivative of y(t) by d k y(t) by dt to the k. Then what is going to happen in the construction of a general differential equation is that some N of these derivatives of y(t) will be combined together to form a function, which is equated to x(t) in the following manner. We first need all the N derivatives of y(t) the first N derivatives of y t in a set, so we shall make a set like this; k equals 0, 1 to N. We construct some function some combination of these entities d k y t by d t k; this function I will simply call f for the time b, this is the left side of the equation. And on the right side as I said we just have x t this makes a general differential equation, let us make a few examples.

Lets say, here is a general example. We have the second derivative and the first derivative of y(t) dy(t) by dt and d square y(t) by dt squared. And we of course, have y(t) itself which is consider to be the 0 e th derivative, hence k equal to 0 is also mentioned in the set of derivatives of y(t) that we will relate it to x(t) in the equation. So, this is one particular example of a differential equation involving y(t) in the context of x(t) with the force function equal to x(t). So, this is rather a complicated equation that we consider, but a typical example, there are let just make one more example.

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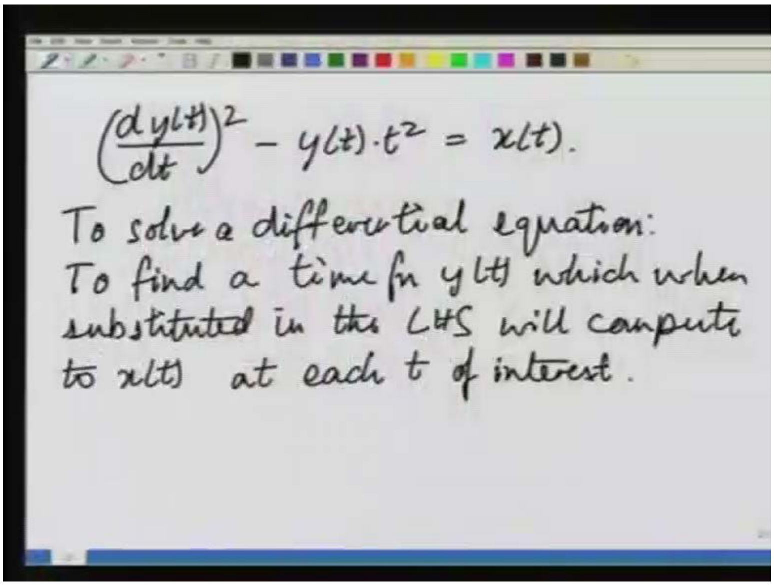

Here we have just the first derivative and the zero eth derivative y(t); y(t) also has is multiplied by a function of time. So, these are examples of a general differential equation and our interest in differential equations is to solve them. So, what does it mean to solve a differential equation. To solve a differential equation is to find a time function y(t), which when substituted in the left side of the equation, will yield will result or will compute to x(t) at each t of interest. So, at every instant of time if you substitute y(t) into the left hand side and compute the function then that function should evaluate to the value of x(t) then you say that y(t) is a solution for the differential equation.

Now there are many surprises and secrets with differential equations even the simple differential equations that we will be dealing with certainly not of the kind of the two examples I have provided. Even those will turner to have lot of solutions unfortunately not just a signal solution. So, uniqueness of the solution of a differential equation will concern as for while and we will have to ensure that in order for differential equation to represent a system. We have to make sure that the number of solution we have or somehow shift through to obtain a unique solution. So, now, that we know what is the solution of a differential equation. Let us move on and concern ourselves with our immediate problem. The immediate problem is to find representations for linear time in variant systems in terms of differential equations.

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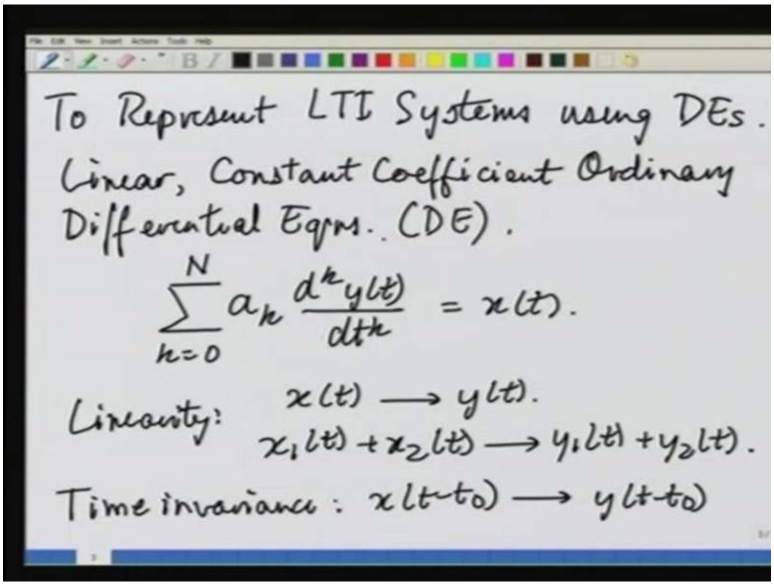

As long as we concern as interest to differential equations only to representing LTI systems; it turns out that we can keep away from concerning with most differential equations except those of a certain narrow kind. This narrow class of differential equations, which can be used to represent LTI systems are linear constant coefficient ordinary differential equations. This class which is linear as constant coefficients and is ordinary. It is only class of differential equations that we need to concern ourselves with for the rest of the lecture and for all of the feature. When I mean a differential equation, I will simply mean a linear constant coefficient ordinary differential equation; simply abbreviated to DE.

What is the general form of this kind of a differential equations. This particular kind of a differential equation the general form we will express just as we did for the general differential equation putting x(t) on the right hand side. But on the left hand side, the general form we wrote earlier will be replaced by a very particular kind of form and that is so this is the most general form of a linear constant coefficient differential equation, and this is all the kind of differential equations that we will bother with. Now we have to understand how to solve such as equation we are not concern with solving other more difficult form of differential equations, only this kind. And in the process of understanding how this is solved we will under able several mysteries of the differential equation.

Before we begin, let us understand a few very very useful properties that this linear constant coefficient ordinary differential equation possesses which other differential equations general do not posses. Let me a numerate them one by one. The first is linearity; suppose y(t) is a solution for the equation when x(t) is the forcing function. Using the other terminology that we just suggested if x(t) is the input suppose y(t) is the output of the differential equation. Then for these kind of a differential equation under certain conditions only under certain conditions, it will turn out that if two inputs are super imposed x 1 (t) and x 2 (t), whose respective outputs are y 1(t) plus and y 2(t). Then response to x 1(t) plus x 2(t) will be y 1 (t) plus y 2 (t) under certain conditions one importantly these differential equations have a very interesting property and that is that if you time shift the input, the output will also get time shifted. If x(t) goes to y(t) then by what we can call the time invariance property, if you apply x t minus t naught then you will have a solution in y t minus t naught, but this thing are best understood first in case of the simple derivative.

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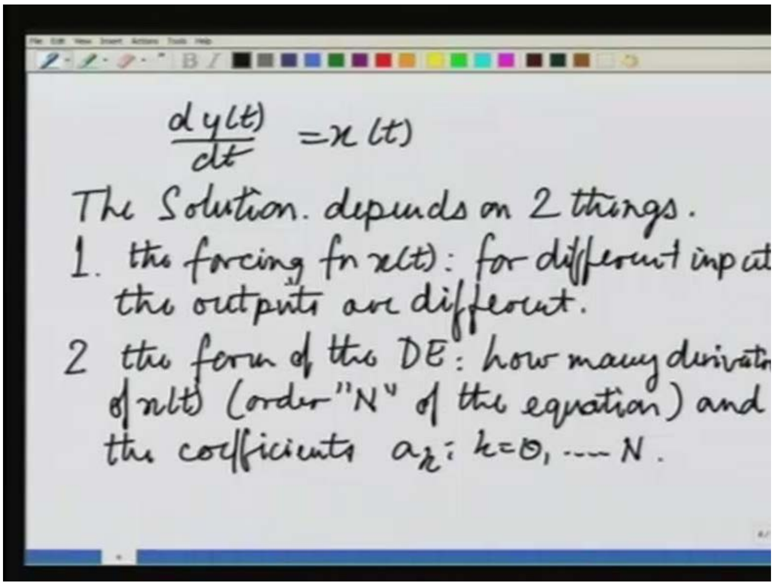

Suppose the differential equation is simply d y(t) by dt equals x(t) this represent a system in some way. And this system is clear the linear and time invariant, because if you scale x t when y t is the solution then the scaled correspondingly scaled version of y(t) will also be a solution. It also supports super position; it also supports time in invariance. If I time shift x(t), so that I apply x t minus t naught as the forcing function then correspondingly y t minus t not will be the solution. Now what we want to do is to go into understanding the solution of the differential equation itself. Now the differential equation has a solution.

Now the solution depends in general on two things. Studying the solution of the differential equation, depends, first it depends on the forcing function. If x(t) is changed, it is natural to except that y(t) will be changed that is to say for different the outputs are different, for different inputs the output are different. Only nature one more thing, but there is another thing on which the output depends the form of the differential equation that is how many terms, how many derivatives of x(t) this is in fact called the order of the equation and the coefficients.

These two things together, the values of the coefficients a k and the order of the equation completely determine the form of the equation. The point we are making here is simply this that if whereas we said in the first point that if the input changes then the solution will change. In the second point, we have made here we assign that even if the input is unchanged, if the form of differential equation changes that is to say the either the order changes are even if the order does not change if any of the coefficient changes then the solution will change. Thus the solution depends both on the form of the equation and the form of the forcing function, the particular case of forcing function. Now it terms out that in order to get information about the solution that depend only on the form of the equation, we have to solve something called the homogeneous equation. Let me say that again I am saying that that aspect of the solution that depends only on the form of the differential equation is understood by solving the homogeneous equation.

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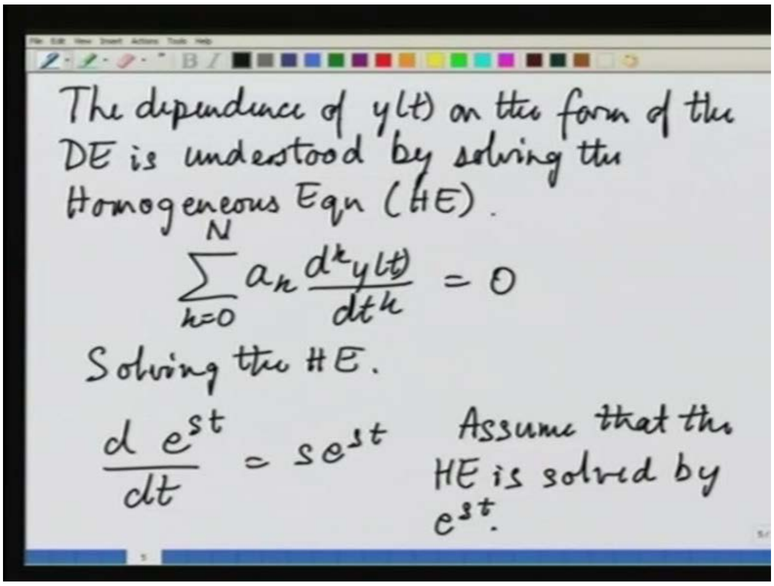

What is the homogeneous equation? The homogeneous equation resembles the original differential equation except that the right hand side where we have the forcing function, we replace with zero, the identically zero function of time zero. So, the homogeneous equation is simply, this is the homogeneous equation. Clearly this equation and whatever we learn about it solution cannot depend upon the input, because there is no input. Irrespective of what input we had applied in the original equation, we have stripped the input in these equations. So, irrespective of what input is applied there are certain thing about the solution that will depend upon the form of the equation alone that is on the left hand part of the equation alone and that is what we want learn by solving first this equation. So, solving the homogenous equation – HE for short.

Now what can we say about solving the homogeneous equation, how do we look for a solution. On the right side, we have zero, we somehow we want to find a y(t) such that when you evaluate the different derivatives of y(t) scale then by the respective scale factor a k and add them all up, the added up identical to zero. How do you go about this, in order to see how we go about this, we shall invoke one of the more one of the most powerful functions in mathematics the exponential function. So, in order to understand how we solve the homogenous equation, we have to recall so facts about the complex exponential function the general exponential function. We will recall from our earlier mention of the complex exponential that it is of the form e to the power s t, where s is generally a complex number. Now why is the complex exponential of interest here, what business thus it has with differentially equations, very very simple. This is the only function which posses the following property that when it is differentiated, it maintains it form. The result of differentiating an exponential function a general exponential function of these form is still the same function scaled by a certain complex number; no other change. Thus if you take the time derivative of this all you get is s e to the power s t.

Since s is a certain constant, which is the parameter of the exponential. We see that the result of differentiation is still a function of the same form. This fact about the exponential is going to be crucial for understanding how it place such an important role in solving the homogenous equation. Right now, we will take a jump, a jump which is arguably without justification and say that the solution for this differential equation is indeed a certain complex exponential e to the s t. Without yet knowing what that s is. So, we do not know what the s is, but we shall assume that the DE - the homogeneous DE is solved that is as a solution of the form e to the s t supposing it has. Then let see what happens if it is a solution for this differential equation. Then if I substitute e to the s t for y t and evaluate the left hand side, I should get the right hand side, which is zero. So, let us substituted and evaluated. Evaluate the left hand side.

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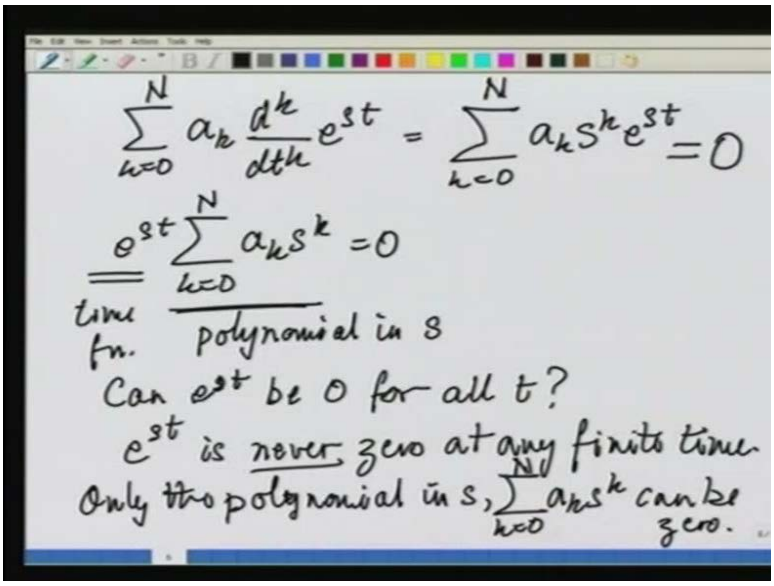

Left hand side is this you written it several time so ever this. And if you evaluate this what we get is a k times the k eth derivative of e to the s t is s to the k e to the s t, because every time we differentiate it once, s one more s come out of exponent and lands on the ground. So, we have a to the k, a k times s to the k e to the s t. So, you see that all the derivatives are gone now. We have already carried out the differentiation, we have this and since this whole thing was taken out of the left hand side of the homogeneous equation, you known that this entire expression this sum of capital N plus one terms must be equal to 0.

Now let us look at this carefully, we have e to the s t and index by k as one factor in every term of this expression. There are N plus 1 terms; each is of the form a k times s to the k e to the s t; e to the s t is a common factor in all this terms, and it has no index of k, so it can be take out of the summation. And we can write these as e to the s t summation k equals 0 to N a k s to the k, this equals 0. Now let us look at this carefully. This part the second of the two factors is a polynomial in s k, in polynomial in s, and this is a time function. So, e to the s t is function of time and that is getting multiplied by a polynomial in s, and this product of this polynomial times the time function e to the s t is identically zero. Identically zero, I mean by identically zero, I mean that it is zero for every point of time.

Now if this thing has to been identically zero, if it has to be zero for every point of time then let see which of the two factors can be responsible for it. Clearly when we have an expression that is the product of factors then the product can be zero only if one or more of the factors is zero. So, let us try to fix the blame on these factors and see which of them could have made the entire product zero. So, can e to the s t v 0 for all t, well from what you should have already learnt from what we already covered in this course a few lectures ago, where we want into considerable detail about the form of the exponential function for varies cases. S was a real number, s was the positive real number, s was the negative real number, s was a purely imaginary number, s was a complex number. For all this cases, we found one very very common and very very study fact about the complex exponential about the general exponential that e to the s t is not just not 0 at all time, it is not even 0 at a single point of time on the time axis.

In short we known that e to the s t is never 0 for any finite value of t e to the s t is never 0. Now why is this important for us, it is important for us because going back to our solution of the homogenous equation, where trying to find a solution to the homogenous equation, and we had two factors the time function and the polynomial in s. We were trying to see which of this could be responsible for the entire expression evaluate into zero. And this latest remark that we have put down clearly shows that if this product has to go to zero then e to the s t is not going to contribute in any manner; e to the s t is not going to help you to make this product of factors go to zero. Hence in order to make to solve the homogeneous equation in order for the left hand side of these expression to evaluate to 0 the burden fall on the polynomial in s. Only this can be zero. So, when can it be zero is the next question. In order to ask, when can this polynomial be zero. Let just look at the polynomial, it is an algebraic polynomial.

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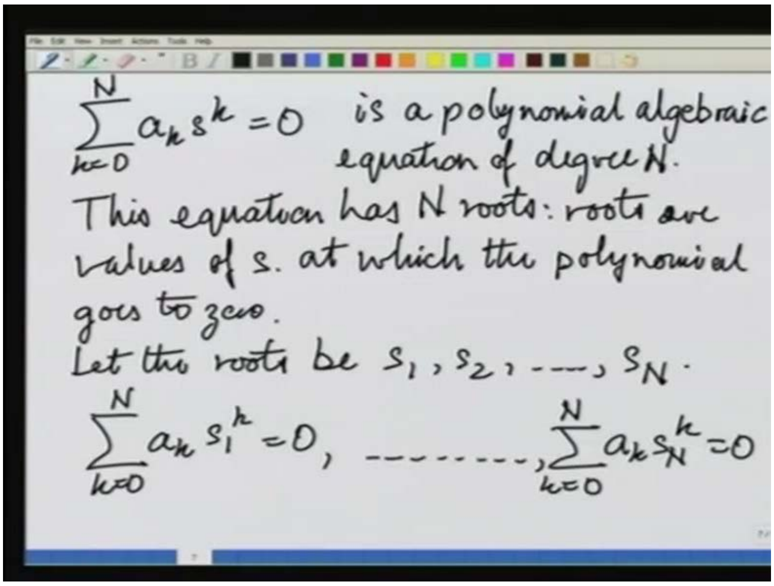

This is just an algebraic equation and it is a polynomial equation of degree N. So, let us dig out some of our result that we all know from high school about solving a polynomial algebraic equation. We known that this polynomial algebraic equation will have N roots, because it is of degree of N, it will have N roots. And what are roots? Roots are values of s at which the polynomial goes to zero. So, since there are N roots; lets name them s 1 to s N. These are the N specific numbers not necessarily real numbers, they could be complex numbers which if you substituted in this polynomial equation in place of s and computed evaluated the left side, it would evaluate to zero.

Now what is different from this homogeneous algebraic equation and earlier homogenous differential equation. We seem to be saying the same things about a very similar equation now. We had a homogeneous algebraic differential equation, we got read of the derivatives somehow by assuming that the solution was as exponential, a general exponential, and we have now got into a homogeneous algebraic equation. This as helped us in a very unique manner. Earlier our homogenous equation was an equation that related to function of time on the left side of the differential equation we had functions of time y(t) d y(t) by dt and so on. On the right side, we had 0 of t. So, it was real not one equation over there, the homogeneous differential equation you have infinite collection of equations one for each instant of time. All of them have the same form that why we can write it in one single expression, but theoretically or if you look at it carefully enough you recognize that one differential equation is actually I am infinite collection of equation applying for each instant of time. Here on the other hand, this polynomial equation which is also homogenous, because the right hand side is zero is an equation between numbers is only about number there is no time index any more.

We just have to solve this once whereas, conceptually to solve the differential equation, you have to solve it every time for every instant of time you have to solve it separately. Here you have do it on once, so that is the difference. We now have to look at how this roots are to be used to demonstrate that they are solution for this algebraic homogeneous equation. By saying that the roots are s 1 to s N, what we essentially mean is this. We mean for example, that k equal to 0 summation to m a k, s 1 k, s 1 to the k is 0 and so on for s 2 and other roots until we is also true that k equals 0 to m a k s N to the k equals 0.

So, every one of these is a root for the algebraic homogeneous equation that is fine the matter of solving the algebraic homogeneous equation ends here, but these was not our original problem. The original problem is to solve the differential equation, the general non-homogeneous differential equation containing the focusing function x(t) from there we digress to solving the homogenous differential equation, and from there, we have further digressed to solving the homogeneous algebraic equation. Now let us start working backwards and retracing our steps, beautiful [FL].