Signals and Systems
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Lecture - 14
Properties of Convolution

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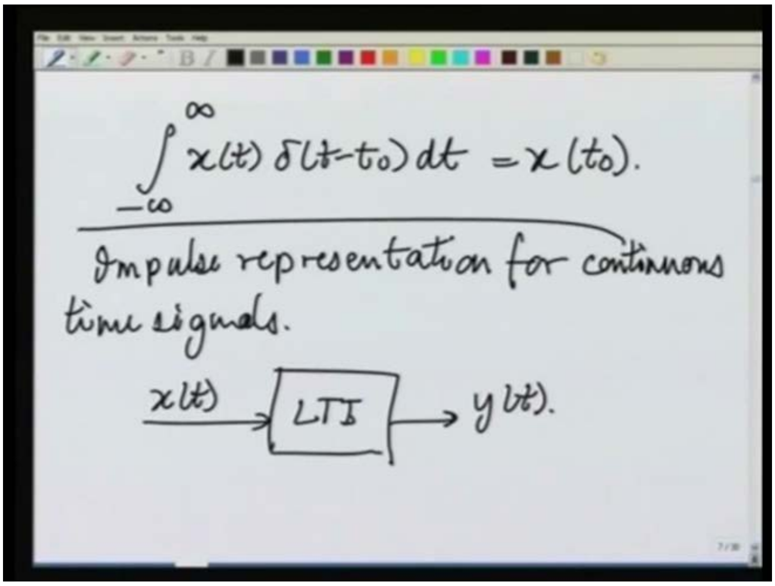

We have the continuous time system - L T I system, again L T I is very important linear and time invariant and on the left side we have x t being applied, we have y t emerging as the output. Can we have a means of computing y t from x t that is the question and as we said we will follow the same steps as we did last time. The first step is to see how to make use of the impulse representation, the impulse representation has to be made use of to obtain the same result that we got earlier for convolution stop. So, the impulse representation expression we already have, let us now examine the implications of linearity and time invariance in the continuous time case, when delta t is applied as the input to a system.

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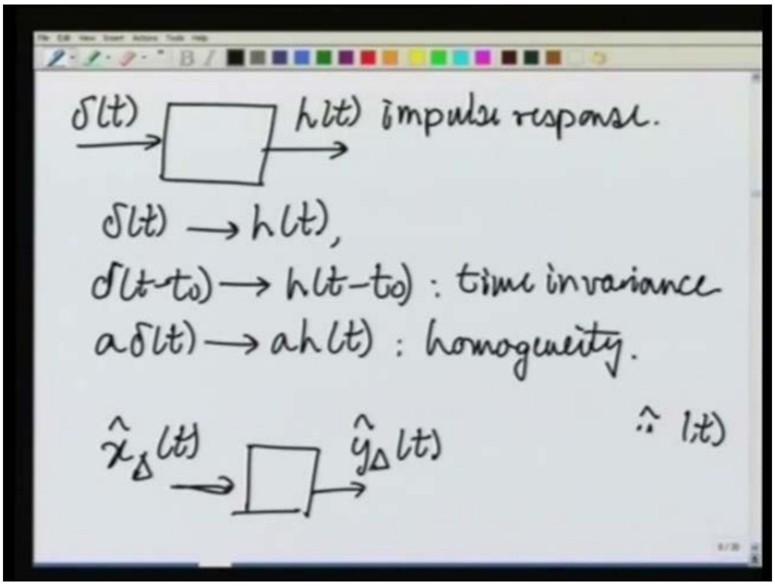

Though delta t is not really a function, we will allow ourselves to applied as the input to a system. When delta t is applied it has to produce some response from the system, and let us say that the response it produces is h t. Since, delta t is our impulse h t is our impulse response, so if delta t yields h t then by time invariance delta t minus t naught must yield h t minus t naught. This is as I said by time invariance then the next property we want to seek, we will arise out of homogeneity of the system and that is that if we apply a times delta t then we should get an output of a times h t, this is homogeneity.

Finally, we have linearity, but we will come to linearity in a little while. So, we have these two properties now the linearity will come into that the additivity will come into picture, when we apply the impulse representation. We now have a representation for any x t naught, for point we can always represent the value x t naught using the impulse representation. And then combining this x t naught’s together to get the entire function x t, we will see what happens. We now we will apply additivity as well and write out x t as the function integral minus infinity to infinity stop. [FL] point [FL] If we had a [FL] take if we go back to the stair case representation, then we had a stair case representation as follows.

If we go back to the stair-case representation that led to the exact representation, then if you recall we started with x delta of t, x hat delta of t. Let us say that we apply x hat delta of t to the system instead of x delta of t, instead of x t. Then when we apply x delta of t to the system, it is reasonable to assume that we will not get y t as the output, buy some modified signal appropriate to x hat delta of t rather than x t and that signal, we shall call y delta hat of t. Now, we will argue that as x delta of t tends to x t y delta of t will tend to y t. As x delta of t tends to x t x hat delta of t tends to x t y hat delta of t will tend to y t this is what we will try to arrange. Now, arguing in the same manner as we did in the discrete time case let us first apply the approximate impulse to our system.

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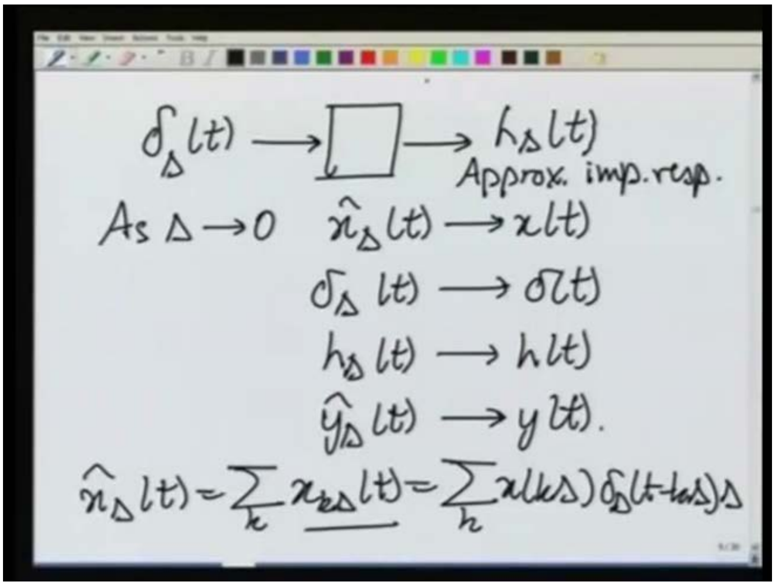

The approximate impulse was delta delta of t and when we apply delta delta of t to the system we do not expect to get the impulse response at the output. We will only get the approximate impulse response or the modified function the appropriate to delta delta of t we will call that h delta of t this is what we will get at the output. So, now our objective is to make delta go to 0. So, that x t goes to x, x delta hat of t goes to x t h delta of t goes to h t and because of these two changes x delta y delta hat of t will go to y t.

So, let us put all that down as delta tends to 0, x hat delta of t will become x t delta delta of t will become delta t. Hence, the approximate impulse response that we have over here will become the exact impulse response h delta of t, the approximate impulse response will become h t. Because of all these changes and improvements y hat delta of t will become y t. So, this is what we will work towards we already have an expression that we can construct for the approximate representation, the approximate representation said that x hat delta of t was the sum over all k of x k delta of t, which was equal to the summation over all k of x of k delta, delta delta t minus k delta times delta.

Now, let us see what happens as delta tends to 0 x k delta that we have over here x k delta of t will become a narrower and narrower function of time. Assuming the value x constant value x k delta within its support, 0 outside delta delta of t minus k delta will become a shifted dirac delta function occurring at some point. And x of k delta is the point at which x has been sampled at the points k delta, and these will become more and more numerous and we will populate the entire real axes as k becomes larger and larger for all values of k as delta gets smaller and smaller. Finally, this multiplication factor delta that we have to put in this approximate representation that will get smaller and smaller and will eventually, become what we call in calculus as d t. In the mean time using this approximate representation let us see whether, we can get the approximate output y hat delta of t, can we express.

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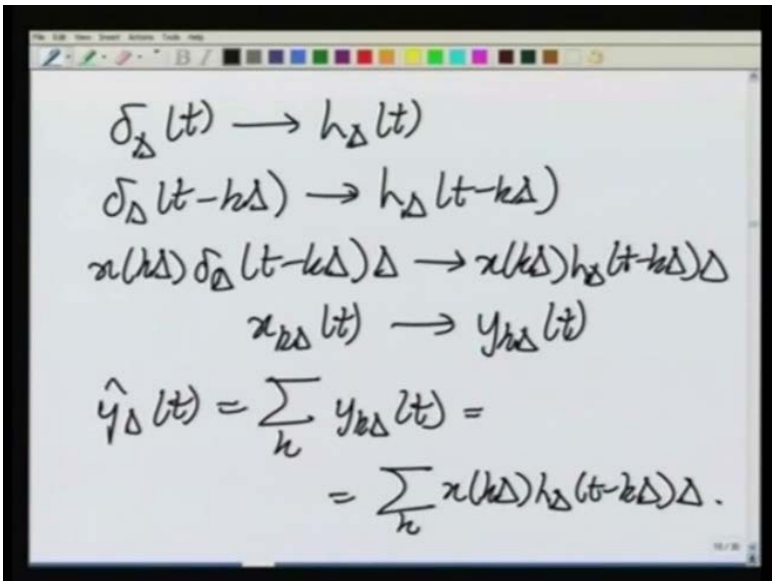

Can we find an expression for y hat delta of t indeed we can, y hat delta of t can be obtained using homogeneity time in variance and linearity as the expression. So, let us see if we can obtain an expression for y hat delta of t using the properties of linearity time invariance, and the approximate impulse response. This indeed can be done because we know that if you applied delta delta of t to the system, you would get an output h delta of t. If you therefore, applied delta delta of t minus k delta to the system you would get h delta of t minus k delta as the output. Finally if you applied x k delta, delta delta of t minus k delta times delta as the input, and this as we know is just nothing but x k delta of t.

Then we would have to get appropriately x k delta the scale factor, h delta t minus k delta times delta at the output. In short we could invent a new intermediate function, which is you could call the right as say y k delta of t as resulting from the left side, which we note to be equal to x k delta of t. So, you apply x k delta of t you get y k delta of t. Now, we have already used here time invariance in the first step, where we have shown that delta delta t minus k delta should give us h delta t minus k delta, we have next used homogeneity to say that if we applied x k delta, delta delta t minus k delta times delta. We have used scale scaling twice here because we are scaling by x k delta, as well as by delta.

Then the corresponding output by the homogeneity property of a linear time invariant system must be x k delta h delta t minus k delta times delta. So, the both the scale factors have appeared in the output namely, x k delta and delta this together has given us a expression for x k delta of t, as we have just written in the last line, thus if we say that y hat delta of t equals the sum for all k of y k delta of t. Then we can say that this is equal to the expression, summation overall k x k delta h delta t minus k delta delta.

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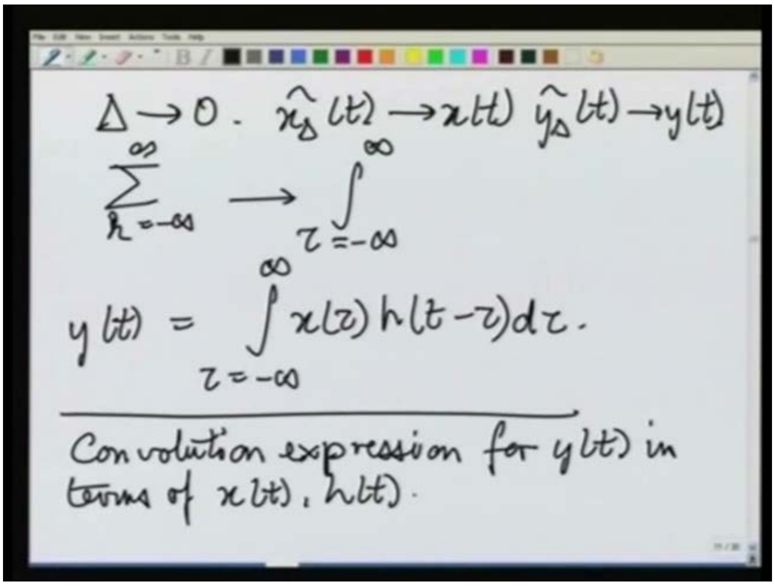

So, y hat delta of t is this expression and here we have used additivity in the last step prior to that we have used homogeneity and further prior to that we have used time invariance. So, because our system that we considered will have linearity time invariance both these properties, we can be sure that all that we have just written is valid, now the final step of taking delta to 0.

As we take delta to 0, let us take a various things which happen. The various things which happen will include x delta of hat of t becoming x t y delta hat of t becoming y t, as delta tends to 0. The summation overall k, k equals minus infinity to infinity that we have done will have to be replaced by an integral because the points, k delta will get extremely numerous and will populate the entire time axis. So, we will actually get some continuous variable which we can say is tau, tau going from minus infinity to infinity.

Thus the entire expression for the exact y t as delta tends to 0, will be y t equals integral tau equal to minus infinity to tau equal to infinity. That is the equivalent of the summation what has become of the summation x k delta is now, k delta is what has become tau. So, x k delta will be x tau h t minus k delta becomes h t minus tau and as we already said, delta will tend to be infinitesimal quantity d tau. This then is the exact expression for the output, when the input is applied and we use the impulse response the exact impulse response h t minus tau. This is the shifted version the actual thing is h t which has been shifted by h t minus tau. This is the convolution expression for the output of an L T I system in terms of the input and the impulse response, in terms of x t and h t.

So, we have through various very devious and complicated means obtained the same state of affairs, as we had for the discrete time case in the continuous time case as well. We can now construct the output of an LT I continuous time system, if you have a knowledge of its impulse response it’s response to the Dirac delta impulse, and if we know the input signal. So, this is the closing of the derivation of impulse response and it is use for continuous time systems, we can now again go through the properties of the continuous time convolution of the convolution formula. We can also take care of the notation if we have y t given by the formula just shown.

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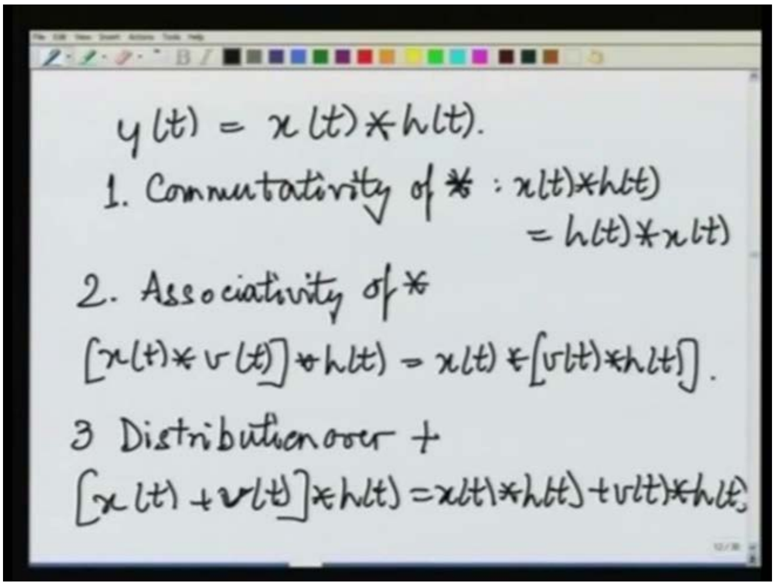

Then we will briefly write y t as equal to x t convolved this star with h t. Again, just like with the discrete time convolution as may be expected, we have commutativity of convolution, namely x t convolved with h t equals h t convolved with x t. We have associativity of convolution that is to say that x t convolved with say some v t, the whole thing convolved with say h t is the same as x t convolved with the convolution of v t and h t. Finally, we have distribution over addition of signals so, that x t plus v t convolved with h t is nothing but x t convolved with h t added to v t convolved with h t.

So, these properties are all given and they are very easy to prove the same thing, the same kind of steps we have to be followed that we followed in the previous case. There are many other properties of convolution that the student can try to prove as an exercise. For example, it can be shown that the area under the convolution of two signals is equal to the area to the product of the areas of the component signals. What I mean to say is if we define.

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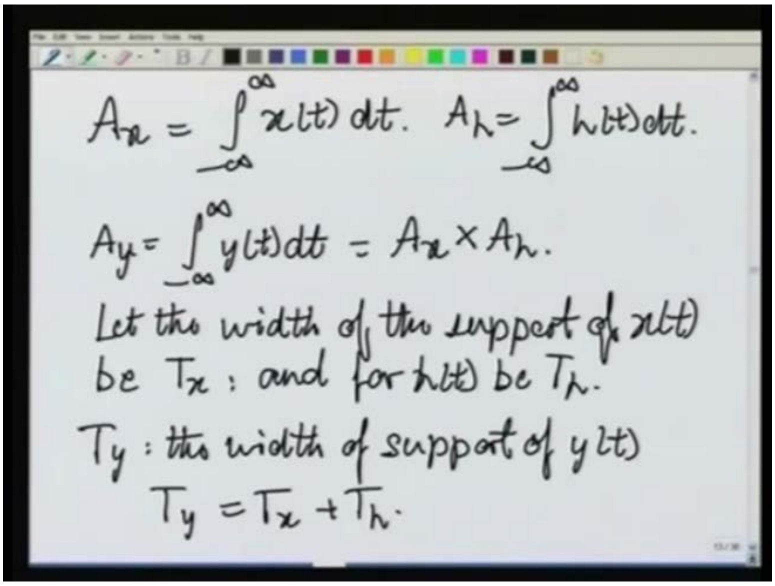

A x as the area under x t which is of course, given by this integral. Integral over all time of x t d t and A h as similarly, the area under h then it turns out that A y equal to the area under y is equal to A x times A h. So, the convolution of two signals gives us the area product of the component signal areas. So, there is this relationship about areas that you can investigate, and try to prove. Then there are other properties, the support of the convolution.

Suppose, we have two finite support signals let the width of the support of x t be say T x that is to say x t is non zero over the interval t x and 0 elsewhere. The same thing for h t be T h then the width of support of y t which we will call T y of y t is given by T y equals T x plus T h, again something that you can prove without too much difficulty. One more result this also pertains to the support, but this is a more strong result is this, let the support of x t be actually given in short we are not concerned merely with the width of the support, but with also the location of this support let x t.

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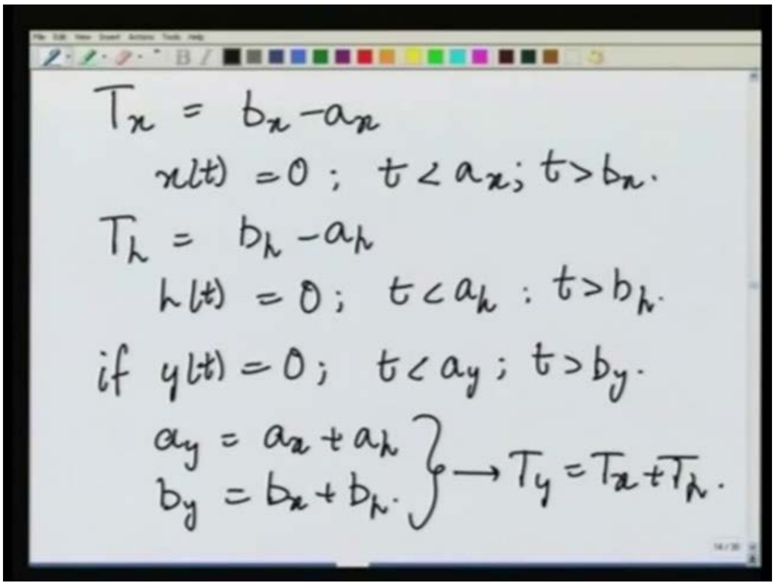

Let T x be equal to b x minus a x that is to say b x is the right side limit of the support of x t and a x is the support is the left side limit of the support of x t. In short x t equals 0 for t less than a x and t greater than b x. Similarly, let us say that T h equals b h minus a h which is to say that h t is 0 for t less than a h t greater than b h. Now, what about the limits of the support of y, we want a y and b y. If y t equals 0 for t less than a y t greater than b y, then can we find a y in terms of a x and a h b y in terms of b y and b h, b x and b h.

Yes, it can be done, it is simply that a y equals a x plus a h b y equals b x plus b h. In fact you will immediately see that this is what leads us to conclude that T y equals T x plus T h. So, without even doing the convolution you can say a few things about the output of a system. You can say for example, what is the starting point the point where the signal becomes non zero for the first time in the output, on the basis of your knowledge of h t and x t. You can say what is the final non zero point of the output from a knowledge of similar points on x t and h t, you can say what is the area under y t knowing the areas under h t and the areas under x t.

So, these are the interesting results about convolution. Now, there are other things that we can go on doing first of all remember that we had several properties of the L T I of systems. We had a property of not having memory that is the memory, the memory less property of a system, we had the causality property, we had the stability property, we had all these properties. Now, if it is really true that an L T I system is fully described by its impulse response, if all the information about the L T I system is covered or bourn by the impulse response.

Then it must also be true that we can determine whether, a system is causal or not whether a system is memory less or not whether a system is stable, or not by just looking at the impulse response by evaluating that fact from the impulse response. Thus we would like to know, how we can find out whether a system is stable or causal or memory less by looking at its impulse response?

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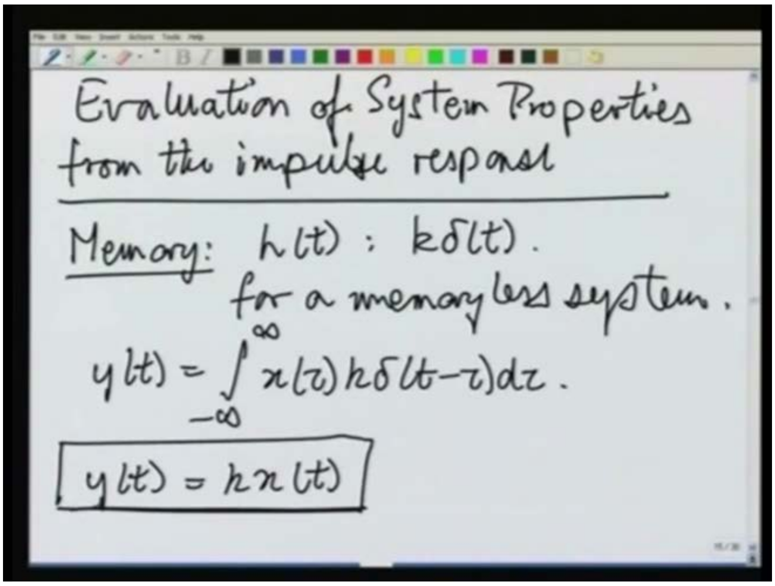

So, system properties evaluation of system properties from the impulse response, first memory, if a system is memory less then we know that the current value of the output only depends up on the current value of the input, right? Therefore, y t should not be dependant up on any value of x t apart from the point at which y t is being calculated, in short y t minus t naught should depend only on x t minus t naught. Now, in such a case what can we say about the impulse response or what do we find as a property in the impulse response?

We will find that the impulse response h t has to be of the form k times delta t for a memory less system, this is very easy to demonstrate we will use the convolution expression. And just write it out as y t equals integral minus infinity to t infinity to plus infinity x tau k delta t minus tau d tau fine. So, this is exactly the shifting property of the system, this is an expression of the shifting property which will we can use for the Dirac impulse that has been applied here, and say that y t equals k times x t.

The only type of linear time invariant system that is memory less is a system where y t and x t are related in this manner. If y t depended up on any other value of x t then it must appear in the right side. For example, if we said that y t was equal to two times x t minus t naught or if it was an integral of the values of x t over some interval of time. In all these cases, y t would not have this value would not have this form now, it has this clear form which says that y t depends on x t, this is the only form possible for an L T I memory less system. You can see that it is linear and time invariant if y t equals k times x t.

So, it is a it is linear time invariant it is also memory less as we can see and for such situations, we have h t given by x t k delta t x t h t equals k delta t. So, this you can immediately see if h t equals k delta t then it is a memory less L T I system. Now, what about causality if an L T I system is causal then what how does that manifest itself in the impulse response. This is an interesting question and it has a very, very straight forward answer in terms of some of the results that we have stated, though not proved. Remember that the support of the output can be obtained in terms of the support of the input, and the support of the impulse response. Now, suppose a system is causal and to this system we apply any x t which is 0 up to t naught and non zero only subsequently so let.

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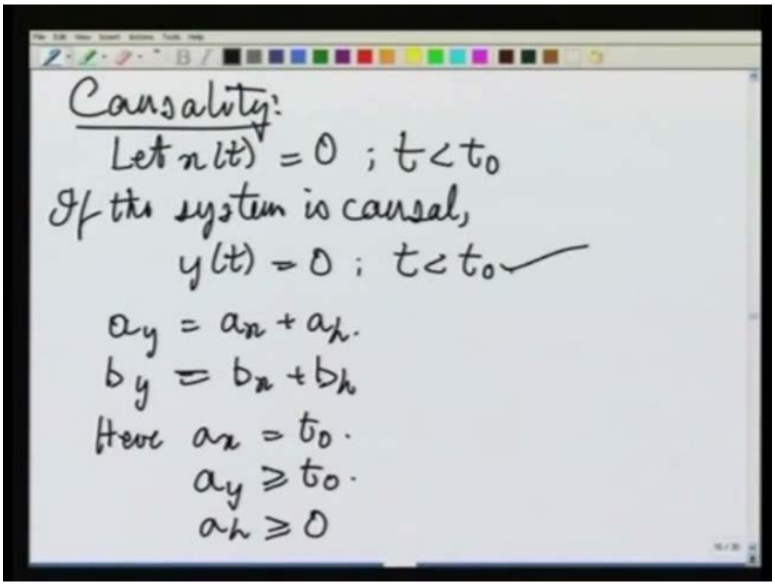

X t be equal to 0 for t less than t naught, and it is probably non zero for all t greater than t naught or at least for some of the t is greater than t naught, but it is definitely 0 for t less than t naught. Suppose, you have this then if the system is causal, what is one certain thing that we can say about the system one definite fact that we can state about the output. Irrespective of even whether the system is linear or time invariant we can say that if x t is equal to 0 up to t equal to t naught then y t must also be 0, for all times t less than t naught. In short for a causal system if the system is causal y t also equals 0 for t less than t naught, it could even be 0 for some interval of time after t naught, but that does not concern us we have to be sure that this much is certainly applied.

So, from this what can we say about the support of h t earlier we had a result that if a x and b x are the limits of support of x t a h, and b h the limits of support of h t a y b y the limits of support of y t. Then we had that a x I mean that a y equals a x plus a h, b y equals b x plus b h. Now, our concern now is with the left side limit of the support that is t naught, in this case a x equals t naught, t naught and we require since, the system is causal that a y must be greater than or equal to t naught. Therefore, if a y is greater than equal to t naught and a x is t naught then using this along with this result that a y is a x plus a h. We can immediately see that a h must be greater than equal to 0.

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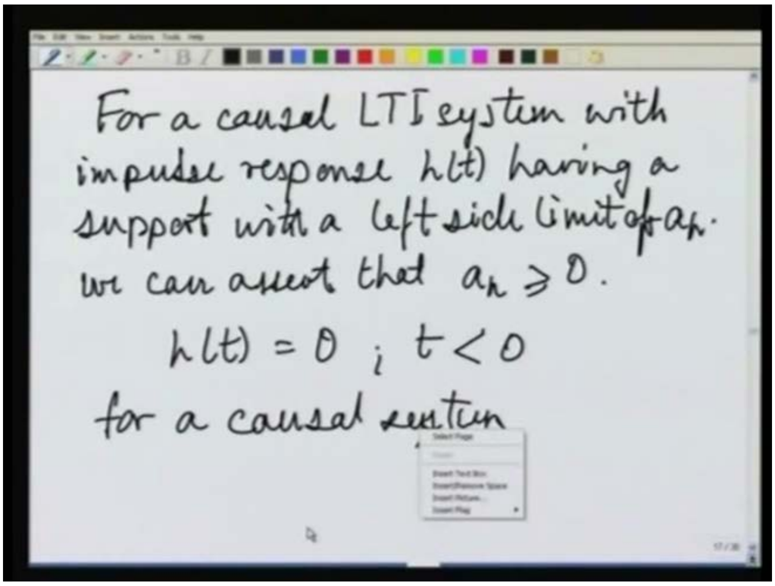

So, since the system is linear and time invariant, we assume that it have an impulse response, and if there is an impulse response, then the impulse response function, which gets convolved with x t to get y t, must have this property that its left side support limit, that the limit of the left side limit of its support must be greater than equal to 0, it cannot be less than 0. So, we will put this down for a causal system.

L T I system with impulse response h t having a support with a left side limit of a h, we can assert that a h is greater than equal to 0. Another way of just saying this is to say that h t is equal to 0 for t less than for t less than 0 not less than equal to 0, for t less than 0 for a causal system. So, that takes care of a causal system. So, we can by inspection say whether a system is causal or not by looking at its impulse response.

So, as we expected it is indeed the case that both causality, as well as memory. The presence or absence of memory can be discovered by just examining the impulse response, this as we said must be the case if the impulse response says, all that there is to be said about an L T I system. We should be able to reduce all the properties of the L T I systems from the impulse response and that takes us to the third property of stability.

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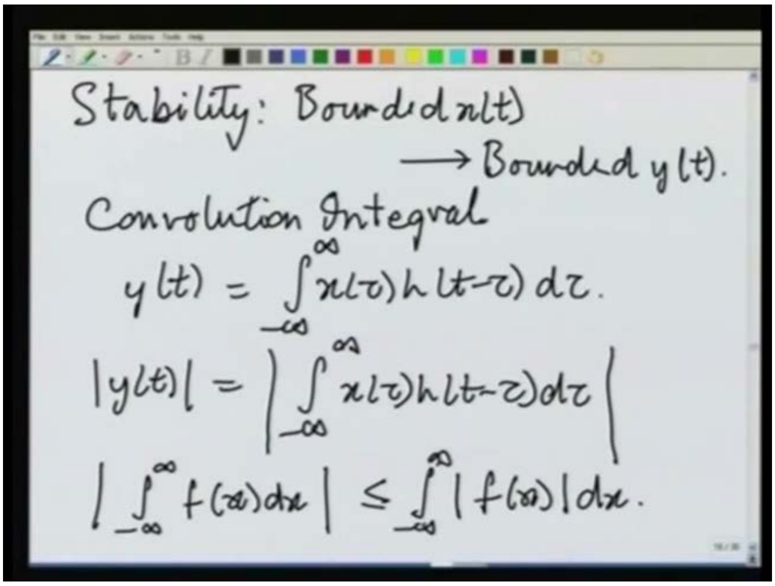

Remember that our definition of stability was that bounded inputs should yield bounded outputs, bounded x t must imply a bounded y t. So, let us see where this takes us, we again go back to our expression for the impulse response for the output, in terms of the impulse response that is the convolution integral. Y t equals the integral from for all time of x t rather x tau a different variable h t minus tau d tau this is what we have. Now, let us note that y t is a summation of functions like this is an integral of values of x t. so, we are now interested in a bound existing for y t, when a bound exists for x t. So, a bound as we know is a limit on the absolute value of the function. So, what is the absolute of y t? The absolute value of y t is this, we have to evaluate the absolute value of the integral.

Now, one very well known property about any integration is this, if you have an integral of some function and you take it is absolute value that number is always less than, or at just equal to the integral of the absolute value of the integrand itself. In short we can say that for any situation integral of the absolute value of the integral of say f of x d x is less than, or equal to integral of the absolute value of f. This property is called the triangle inequality, this is true and we can say these two things about the triangle inequality that well let me put that down.

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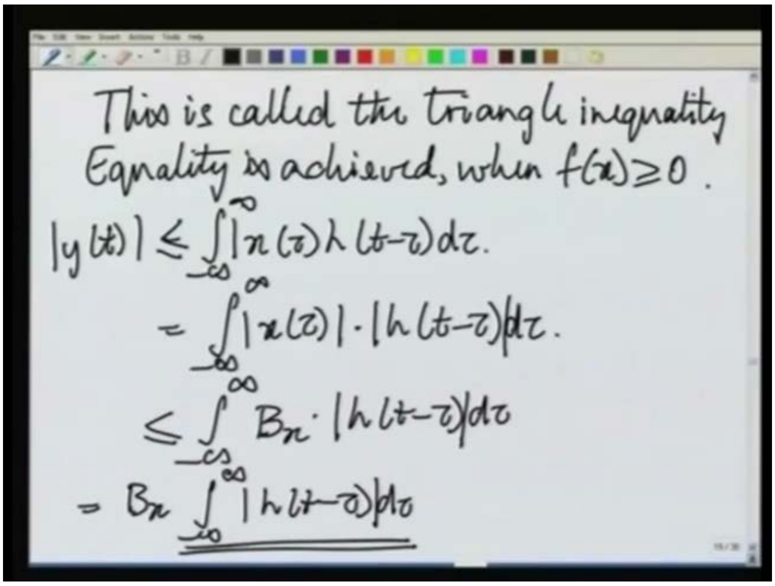

Equality is achieved, when f x is greater than or equal to 0. That is whenever f x is a non negative valued function then equality is achieved. Now, our integrand here is x tau h t minus tau which could be negative in some places. So, using that fact over here, we will write this down as mod y t less than equal to integral over all time of x tau h t minus tau d tau, which is equal to integral over all time of the absolute values of the components the factors of the integrand. So, we have brought it to this state.

Now, let us see how we can maximize the value of this integral, our intension is in some sense malicious. We want to do our best to see if y t is unbounded, or rather if the absolute value of y t is unbounded. In order to ensure that we will try to consider the worst case on the right side, the worst case on the right side will result for a signal x t which achieves its bound at all times. In short this integral that we have just written here is itself less than equal to the worst case situation where we have instead of absolute value of x t the actual bound x t.

The difference is that absolute value of x t is less than equal to b x at all times t, but here we have replacing it by a constant, which is again equal to the maximum value that x t takes or to the bound on x t. So, we have this h t minus tau d tau absolute value, this is what we have. Now, this is equal to b x times the integral of h t minus tau d tau b x could be taken out because we have replaced the function of tau that was x tau by the constant b x, b x is constant with respect to tau. So, it is been taken out.

Now, let us understand what is the significance of this integral, this integral is nothing but a time shifted version of h t. If you have h t and it has an area under it let me call a h then h t minus tau will also have the same area by the same logic, if the absolute value of h t has a certain area which we will call whatever we like. Then the same absolute value of the shifted version of h t that is h t shifted to h t minus tau will also have the same area. In short what we are looking at is the absolute area of h t, B x times the absolute of h t this means.

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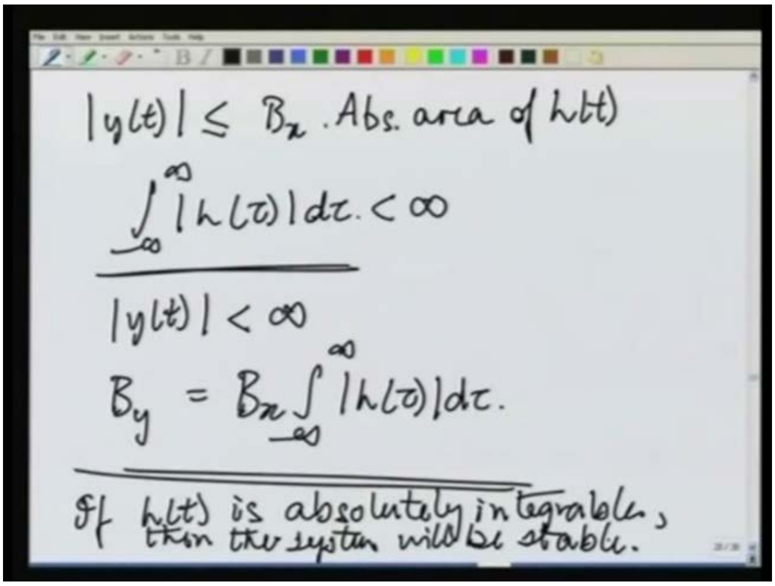

That we have mod y t summarizing as less than equal to B x times the absolute area of h t. Where the absolute area of h t is nothing but integral overall time of h tau d tau, if this quantity that we have called the absolute area of h t is finite. If this is finite, then B x times this is also finite, because any finite number multiplied by another finite number is always a finite number and thus mod y t will also be finite. Therefore, we will have a bound for y t. In fact we can find that B y is bounded B y a bound for y can be found in the form B x times the absolute integral of h t. Summarizing we can say that if h t is absolutely integrable, then the system will be stable.