Signals and Systems
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Lecture - 13
Representation of Continuous Time Convolution

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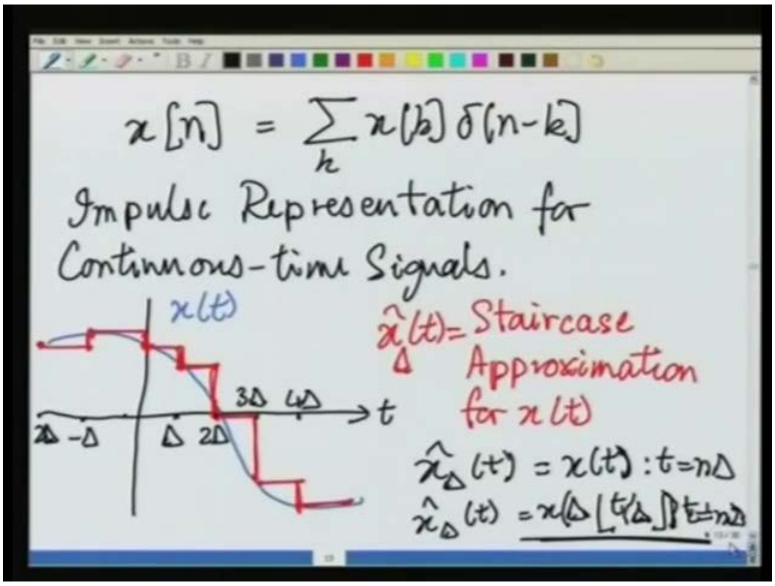

Our discussion of the continuous time case. So, in order to find an impulse representation for continuous signals; continuous time, we first to need to find out, how to go about it. Let us try a crude approach, the crude approach would be to get a very course representation, instead of an accurate representation very courses approximate representation for a continuous time signal to being with. So, in order to do this, let us first take some continuous time signal, a graph of a continuous time signal and see what we can do in it. So, here it goes this is the time axis and on this let sketch some continuous time signal x n, let this be x t.

Now, we want to representation for this, the first thing I will try to do is to form, what I call as stare case approximation for x t. Let us call this signal x hat t the idea of a stare case representation are a stare case approximation is that x hat t does not try be equal to x t every point t, instead at periodic intervals at periodic instance of time. Namely, t equal to 0, t equal to delta, t equal to 2 delta and so on. X hat t will be equal to x t at other points between this point of equality x hat will be constant at the most recent value. So, in order to do this, let me first mark out on the time axis. This is the time axis on the time axis let me just mark out first the points delta 2 delta and soon. We will also have of course, minus delta minus 2 delta and soon.

Now, let us start what does this x hat t do at minus 2 delta x hat t will try match x t exactly. So, at t equal to minus 2 delta, this will be a point on a x hat t, what is this values of minus delta? It is value at minus delta against to be equal to x t in between these two points x hat t should remain constant at the value, it found at x at t equal to minus 2 delta. So, it stays constant in between this two points.

Now, at t equal to 0 x hat t will have to take the value of x of 0, x hat of 0 has to be equal to x of 0, but between t equal to minus delta and t equal to 0, it will step out at the value x hat of minus delta or x of minus delta. So, it will remain constant like this. Next, at t equal to delta we have this value and from there up to this point it will have this value. Likewise if I go on doing this I would get the this is the stare case, just for psychological convinces I will have rather for the stare case as well I connect these parts of the stare case by vertical line knowing fully well that that is found upon my the purest. So, this red function is the stare case approximation x hat t of x t.

Now, because it has discretized the continuous time using a unit that we have called delta I will actually, call this x hat subscript delta of t and what can I say about the value of x hat delta of t. I can say the following x hat delta of t equals x of t at t equal to n delta at other points its value will be equal to the most recent value of x t, most recent sampled value of x t. So, thus x hat at delta by 2 will be equal to x of 0, x hat at 3.2 delta will be equal to x of till and soon.

So, we have this pause and at points not equal to n delta at point of time not equal to n delta we will get x hat delta of t equals x of delta times the floor of t by delta, as we have written over here. This completely describes the stare case approximation in term of the original signal, it is clearly an approximation. Now, the important things to recognize the about this approximation is that.

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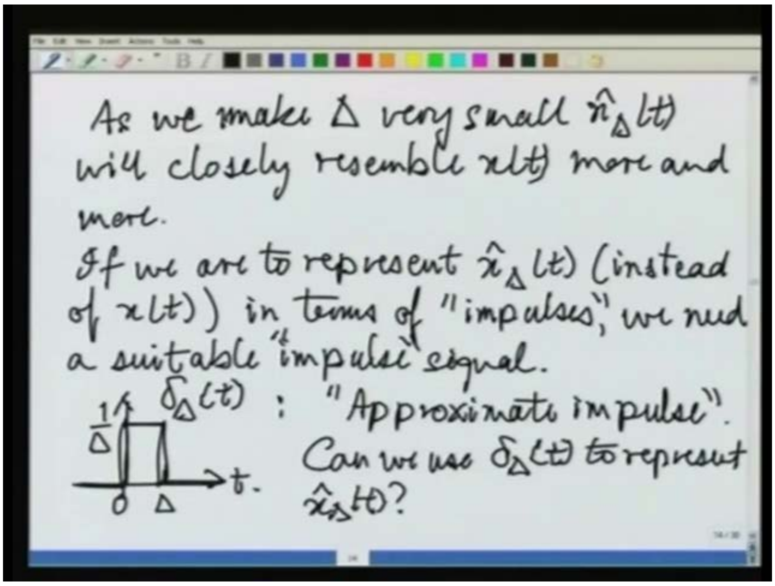

As we make delta very small x hat delta of t will closely resemble x t more and more. So, though at present and in the kind of sketches we have drawn x hat delta looks very crude, x hat delta of looks very crude compare to the original signal. This can be improve if you only make delta smaller and smaller, but that as separate issue right now. Suppose, we are prepared to except x hat delta in mu of x t x hat delta of t in mu, mu of x t as the signal that we want to study.

Then can be find something like an impulse representation for x hat delta of t the nice thing about x i delta of t is that it has only finitely, many different values between within a finite interval of the time axis. If you go back to the earlier figure, we find that in this interval of the time axis from minus 2 delta to approximately 4 delta, we have 1, 2, 3, 4, 5, 6 different values of x hat delta of t unlike the original x t, which had a infinite number of different values in this same interval.

So, in some sense we have reduced the problem of continuous time signal representation to something that resembles the discrete time signal representation. So, let us say if we are ask to represent x hat delta of t using impulses, represent x hat delta of t instead of x t in terms of impulses. We still do not know what we mean by impulse, but let us see. How would be go about it? We would try to defined a suitable impulses.

Impulse signal in this case it turn out to be convent to define that signal as a function sketched here, this is what I will call for lake of a better name I will call delta subscript capital delta of t, this is the time axis. This signal we will have a non zero support that is a support from 0 to delta, it is 0 outside this interval and within this interval it is constant and take on a value equal to 1 by delta. This is what I call delta delta of t, I will call this now for again lack of better name, I will call it the approximate impulse. Delta delta of t is called the approximate impulse can I use delta delta of t to represent x delta of t, x hat delta of t.

The answer is yes and we could do it by decomposing x hat delta of t into x hat delta k of t, as we did last time. Each of this x hat delta k of t would be non zero only in the interval k delta to k plus 1 delta and in this interval it would that the value of x t at t equal to k delta that is it would be equal x of k delta. It would be constant in that interval at all other end points of time it would be 0.

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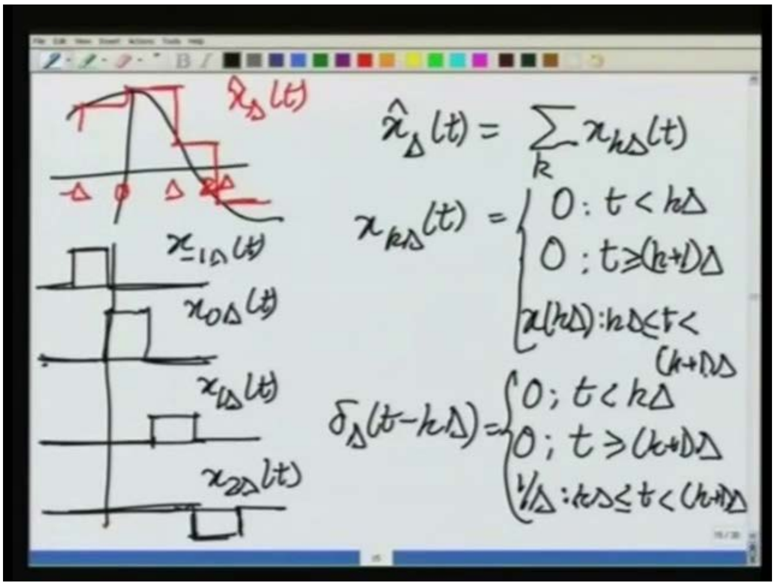

So, suppose we had X t as we are taken earlier then here is our x hat delta of t. Now, I will represent this as some of shifted and scaled approximate impulses, but before I do that I will first explicit as some of simpler function, these simpler functions would be like this these is minus delta 0, delta 2 delta so on. I have this, so I would express this as some of the following signals. In this place I would just have 1 pulse like this, this I will call x minus 1 delta of t. Then I would have the next, function x 0 delta of t. Then the next function x 2 delta of t sorry x 1 delta of t, then the next member would be 0 up to here and then being negative this would be x 2 delta of t.

So, if you added x 1 delta of t and x 2 delta of t plus x 3 delta of 3 plus so on, you would get x hat delta of t. So, we can write x hat delta of t equals summation x k delta of t for different case so far so good. We seen to have completely side track the original problem of finding a representation for x t, we have now become completely preoccupied with finding a representation for x hat delta of t. We have set our height a little lower we have recognize that handling x t directly, would be biting of more than we can choose. So, we have biting of only x hat delta of t for time being and we want to represent this.

That representation can be made in the following manner. Now, each x k delta of t can now the be represented in term function delta delta of t. How do we do it? Shift them and scale them and add them appropriately the game is very similar, what was done in the discrete time case. Let us try to characterized the x k delta of t, what can you say about x k delta of t x k delta of equals 0 for t less than k delta it is also equal to 0, for t greater than equal to k plus 1 delta. If non zero only between these two points and what value is it between these two points, it is equal to x the original signal x at k delta in the interval k delta less than t less than equal to t less than k plus 1 delta. This is our characterization of x k delta of t contrast, this with our characterization of delta delta of t shifted to the point k delta.

So, we want delta delta of t minus k delta delta delta of t minus k delta equals 0 t less than k delta 0 t greater than equal to k plus 1 delta. It is equal to 1 by delta in the interval k delta less than equal to t less than k plus 1 delta. So, this is delta delta t minus k delta that is the delta delta of x and the approximate impulse shifted to the point t equal to k delta. This is the component of the stare case approximation the k th component of this stare case approximation x hat delta of t, this k th component is what we have called x k delta of t.

So, you will see that both x k delta of t and delta delta of t minus k delta are constant in the interval from k delta to k plus 1 delta. One of them is equal to x of k delta the other is equal to 1 by delta. So, all we need to do is to find the appropriate scale factor and we would have to being with a means of representing the different component, simpler component signals x k delta in terms of the approximate impulse. So, let us do this now [FL] we have.

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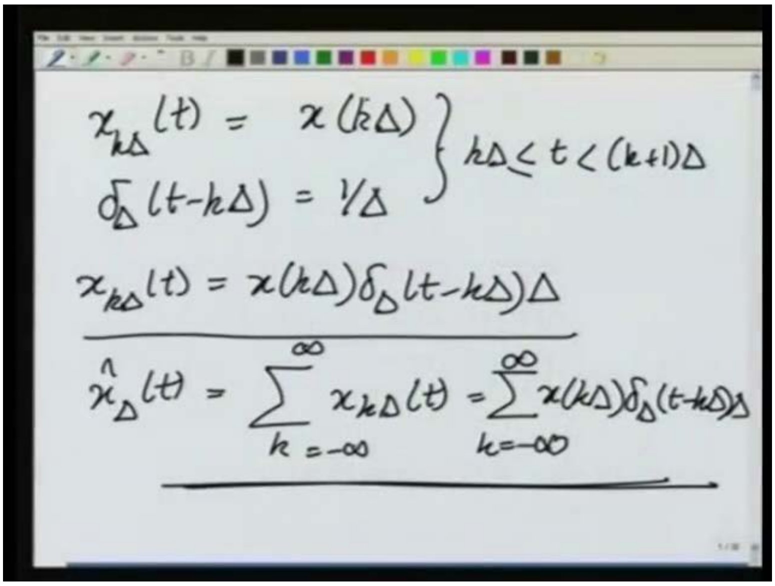

X k delta of t in the interest in the region of our interest, this is equal to x of k delta and we have delta delta of t minus k delta. Assuming a value equal to 1 by delta both this in the region k delta less than less than equal to t less than k plus 1 delta. So, clearly by comparing this two and both this functions are 0 outside this interval of k delta to k plus 1 delta. So, in order to make delta delta of t, in order to express x k delta of t in terms of delta delta of t minus k delta, all we have to do is scale delta delta of t by x k delta of t which is x of k delta.

We can therefore, write that x k delta of t equals x of k delta delta delta t minus k delta this is almost okay, except for this one deficiency that delta delta of t minus k delta has amplitude of 1 by delta not of unity. Therefore, we have to further multiply this by delta this gives us the expression for k delta of t, in terms of our approximate impulse delta delta of t minus k delta, the shifted scaled approximate impulse. This is the equivalent to our earlier expression where had x k of n in terms of delta n minus k times x of k. So, with this in hand, it is very simple to what we do next.

Remember that our original exercise is to represent x t, but our intermediate exercise is to represent x hat delta of t, which is the stare case approximation of x t. So, the stare case approximation is what we will now construct using shifted, and scale approximate impulses. For all k that is running from minus infinite to infinite, this is what we know is valid. Now, with our new information we can express this as k equals minus infinite to infinite, x k delta delta delta t minus k delta times delta.

So, this is the expression we were looking for. If our exercise was only to represent this stare case approximation then we have already achieved our objectives, we have now the means of representing x hat delta of t in terms of shifted and scale approximately impulses, but we have still got some distance should go from here because we actually, want to express x t in term x hat delta of t.

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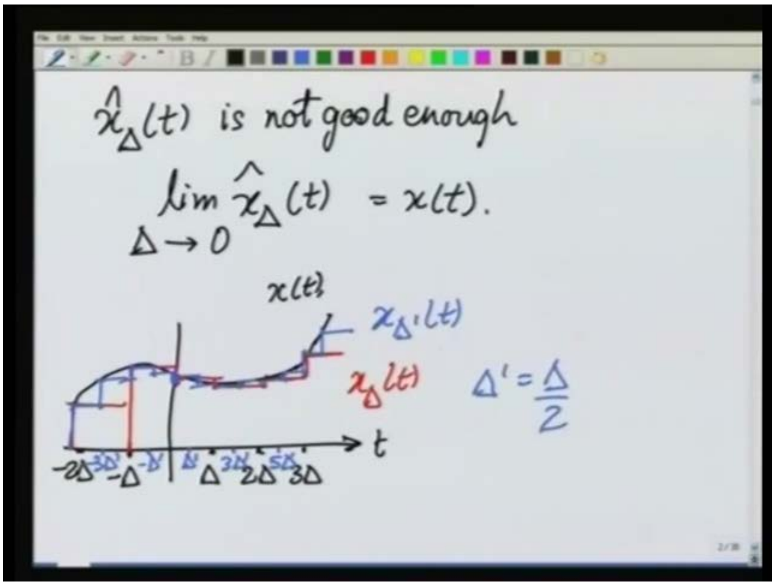

So, x hat delta of t is not good enough we have to see how to make x hat delta of t become close to or equal to x of t. That is reasonable easy to see its evident that as these we discretion size delta, discretion unit delta tense to 0 x hat delta of t will become more and more close to x of t. Thus we can write that the limit as delta tense to 0 of x hat delta of t equals x t. Now, when we do this we will have a larger and larger number of the component signal x k delta of t because in any given interval, we will have a larger number of units as delta tense to 0 because delta is the unit. Thus we will have more and more crowed representation plus try to make a diagrammatic exhibition of this fact.

Suppose, this is x and originally delta is this big delta 2 delta, 3 delta minus delta then the first stare case approximation would be like this, cut we have sketched here the same signal x t in black then there is a signal sketched in blue in red and that is x delta of t. There is a third signal sketched in blue which is x delta prime of t, we have chosen delta prime to be equal to delta by 2 half delta. We can see from the horizontal time axis what relationship delta prime bears to delta.

We can see that delta prime is half of delta we have mark 3 delta, 5 delta, 3 delta prime 5 delta prime here in this axis. We have constructed this two stare case approximation x delta prime of t and x delta of t, x delta of t is seen to be more distanced from x of t then x delta prime of t. This is achieved simply because x delta prime of t checks the values of x t every time k delta prime, which is as smaller time interval then k delta. Thus it evident that as delta tense to 0 x delta of t will tend x delta hat of t will tend x t. So, let see what this n tells, if you wish to make x hat.

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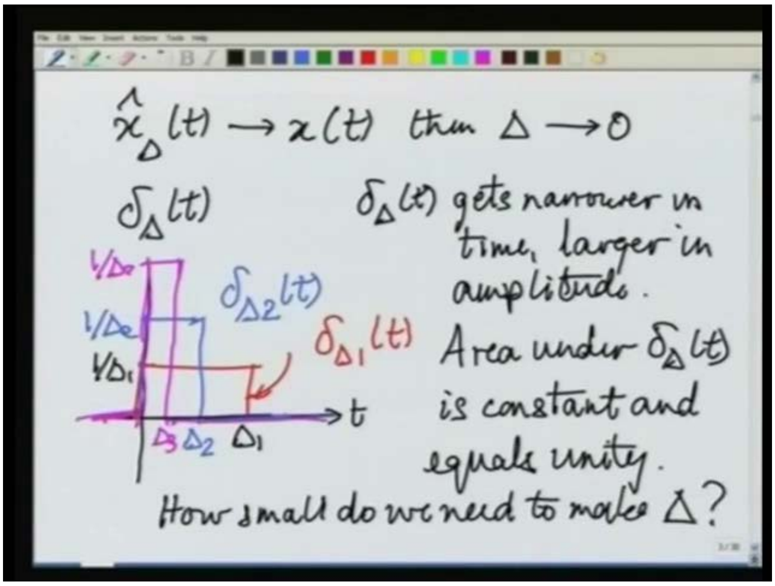

Delta of t tends to x of t then delta must tend to 0, let us see the consequence of the delta tending to 0. If we want every step of our derivation made until now to be valid then we should make appropriate definitions for the approximate impulse, the approximate impulse was earlier define as delta delta of t. Now, this approximate impulse is going to undergo modifications of a suitable kind, when capital delta the discretion step size tends to 0. Thus for example if you make a few versions of delta delta of t for varying values of delta you would get a diagram like this.

For large delta we would get something like this and we would say that this is let say delta 1. If this is delta 1 then this is 1 by delta 1 and we have this diagram. So, this is one choice of delta in which case, this function that we are just drawn will be what you can call delta delta 1 of t. If you make delta smaller let us make delta 2 that is half the size of delta 1, then the height of this new function in its non zero region in this support must be equal to 1 by delta 2 which is therefore, twice of 1 by delta 1.

So, we get a new function that has this shape this is delta 2 and what we just have drawn is delta delta 2 of t we could make a third one if you like which had an even smaller support the delta 3. Then we would have an even taller function, because 1 by delta 3 would be even smaller than 1 by delta 1 or 1 by would be even larger than 1 by delta 1 or 1 by delta 2 we would get this is not actually draw to scale, but this would be 1 by delta or to be 1 delta 3 and what we just drawn earlier is 1 by delta 2.

So, if we remain consist with the way we have defined our approximate impulse. Then it turns out that as the discretion step get smaller and smaller, these approximate impulses get narrower in time and taller in amplitude. It gets undergoes this two changes in a very systematic manner and that systematic fact is that if its width is delta. Then this amplitude is 1 by delta. So, that the area under delta delta of t is constant and equals unity. So, we have to maintain the area of delta delta of t to be 0 as the discretion step is made smaller and smaller.

So, finally the question is how small do you need to make delta, the answer unfortunately is that no finite size is small enough because we want to approximate a continuous time function x t and a continuous time function uses time as a real variable. Now, a real variable the real line is infinitely divisible. So, theoretically x of t can vary from instant to instant, you can take as smaller interval of x t as you like and still observe variation in it. So, if want x hat delta of t to be faithful to x of t at whatever level of magnification, we wish to observe a x delta of t in. Then it is imperative that delta must actually go all the way to 0.

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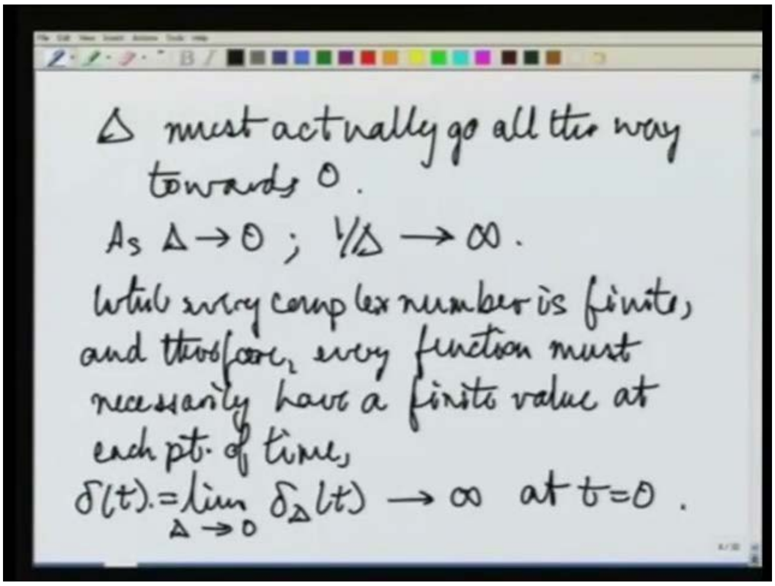

Only then can be hope that the stare case approximation x hat delta of t will be become actually, equal to x of t. So, this is what we have to aim for. Now, let see what happen to the approximate impulses as the unit as the unit of the discretion goes to 0. The area as we said must remain fixed at unity, the width of the support get smaller and smaller and tense to 0. If both these constraints had to made then necessarily, the amplitude 1 by delta should tend to infinite.

So, as delta tends to 0, 1 by delta will tend to infinite there is no help for us. So, what do we have in the limit what you have in the limit is acquired mathematical entity, it has in many ways it lacks the basic properties of a function, because a function as we have understood till now is a map from the real that is from the time axis into the range, which we expect to be the set of complex numbers. Every complex number in the set of complex numbers is finite. However, 1 by delta as we observed here is infinite.

So, while every complex number is finite and therefore, every function must necessarily have a finite value at each point of time delta delta of t will not satisfy these as t tense to infinite, as delta tense to 0. So, delta delta of the limit was as delta tense to 0 of delta delta of t will tend to infinite at t equal to 0. This is paradoxical situation and this is why delta delta of in the limit, which we call in fact we have a name for it we simply call it delta of t. We define the limit of this sequence of function of delta delta of t as capital delta tense to 0 as an object that we call delta of t. So, let us say what delta of t, what you can say about delta of t.

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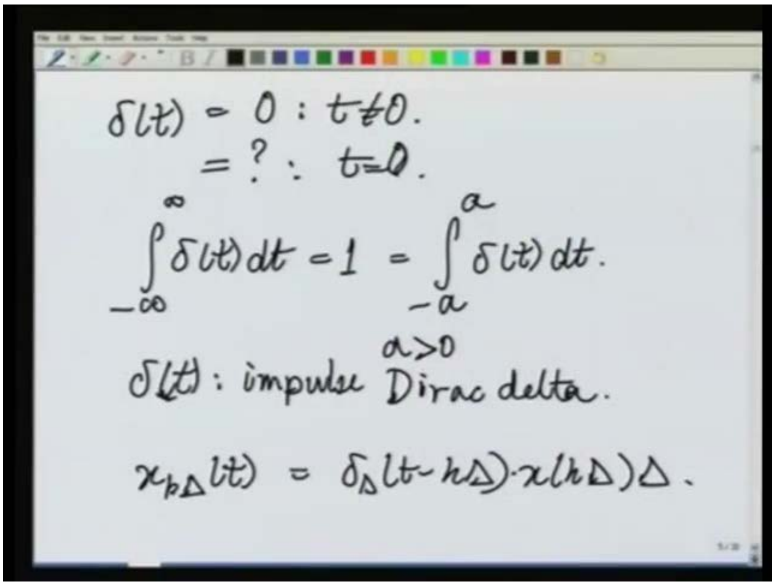

Delta of t is the limit of this sequence of delta delta of t and hence, delta of t equals 0 t naught equal to 0 this much we can say, because this is a common property of all the delta delta of t outside there are support and the support of delta of t which is the limit of that sequence is in this the 0, it has the width of the support of delta of t is actually 0. So, there is no problem it will be 0 all t naught equal to 0. Now, what value will it have at t equal to 0 we can't answer this question because the value it will take is not within the set of complex numbers, infinite is not within the set.

This value that is tense to take will be very, very large. However, we can this seize to discuss the property, this particular property of what value it takes at t equal to 0 and instant focus on more secure things, things we are more comfortable with the things that we are more confident about. One thing we are very confident about of delta of t is the area under the curve, the area under the delta of t must be same as the area under every number of the sequence because we have set that area constant and equal to unity. Thus we can say that the area elevated of delta of t d t equals 1.

Since, its value is 0 for all t naught equal to we could take any interval containing 0, and integrate delta t over that interval we would still get 0. In short this is equal to integral from any minus a to a where a is greater than 0, delta of t d t. This is the same it will have a area of unity. This delta this delta of t now called the impulse, simply called the impulse because it is no longer approximate it is the ultimate impulse, it is what we were hoping to construct.

Impulse for continuous time signal is just called the continuous time impulse it is also called the Dirac delta. So, all that we have said on the panel is the only if statement that we can make with certain t about the Dirac delta. Something we can say is that the Dirac delta is not a function because it is not possessed values within the set of complex number for all time, in particular at t equal to 0, where the value that the Dirac delta takes does not lay in the set of complex numbers.

So, it is not a function it is an entity that is all we will say it is right now, with this property that area under entity in the normal way we defined the area under any function evaluates to 1. It will evaluate one is that integral if carried out over any interval that contains t equal to 0. We can also say with certain t that delta of t is 0 for all points for all points of time not equal to 0. So, this much we have said there is one important property that delta of t will have.

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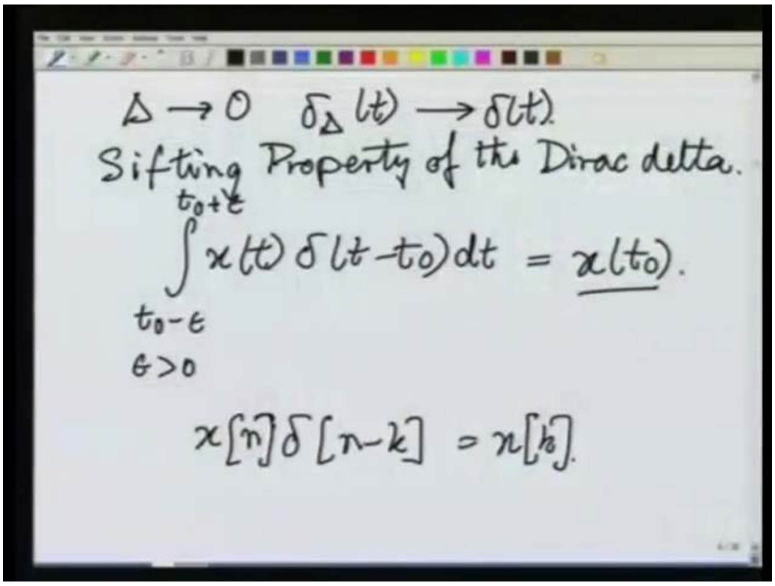

Remember that in our process in our exercise of coming to delta of t we have gone through the representation of x k delta of t, and we said that x k delta of t is just equal to delta delta of t minus k delta times x of k delta times delta. So, if we extend this property all the way until delta tense to 0, what we will get? We have essentially, here multiplied the value of x t at t equal to k delta, we have we have multiplied that with the approximate impulse and scaled it by delta and this gives the value of x t at t equal to k delta. So, as we make delta tense to 0.

Delta delta of t as now become delta of t the exact impulse or the ultimate impulse and delta of t will still, we used to sample the value of f c not in gross intervals of width delta, but at a particular point. In short there is something called the shifting property of the Dirac delta, the shifting property of the Dirac delta is simply this. If I try to take any function x t multiply it with a shifted Dirac delta located at some t naught that is delta of t minus t naught and evaluate this product.

The integral of this product over any interval of time that includes the point of occurrence of the impulse, which is now t naught say, t naught minus epsilon to t naught plus epsilon, where epsilon is of course, greater than 0, then this will be equal to x of t naught it is this property that was what, that was that that we saw at when we wanted an impulse representation for continuous time signals. In the discrete case we wrote that x k delta n minus k or rather x n into delta n minus k just one second, x n times delta n minus k was equal to x k that is the equivalent property we want here.

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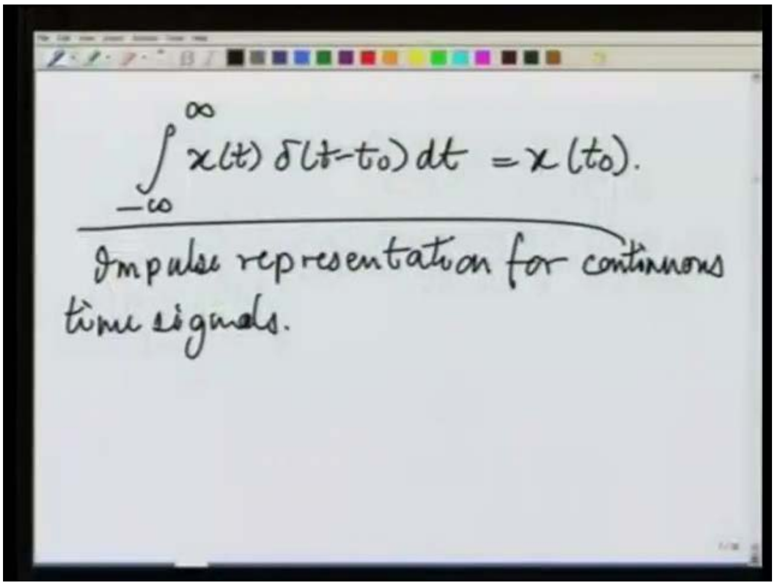

There we were able to sample the value of the function discrete function x n discrete time function x n at the point k by multiplying it x delta n minus k which is the shifted discrete chronicle impulse at n equal to k. Here we have the Dirac impulse located at t equal to t naught and we wish to evaluate x t naught, we wish to express the number x t naught using the shifted Dirac impulse to sample x t. So, we have this. So, in general we always write.

The shifting property as integral minus infinity to infinity, we can always use a larger interval because the Dirac impulse is anyway 0. For any point outside the point of its occurrence x t delta t minus t naught d t equals x t naught, this completes our impulse representation for continuous time signals. So, we have an impulse representation. Now, what we do next we have to carry out the same set of steps, that we used in order to obtain the convolution of discrete time signals.

We apply homogeneity, we apply time invariance then we apply the property of linearity of the system in question. Then we find that y t the output of a continuous time system, which as have been expressed, which has been impressed with an input x t may be computed using something that is also called the impulse response here of the continuous time system.