Signals and Systems
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Lecture - 12
Representation of Discrete Time Convolution

Recognize that the most important or the very important role that we have used, that has been played in this exercise of analysing linear timing variance systems has been played by the impulse representation.

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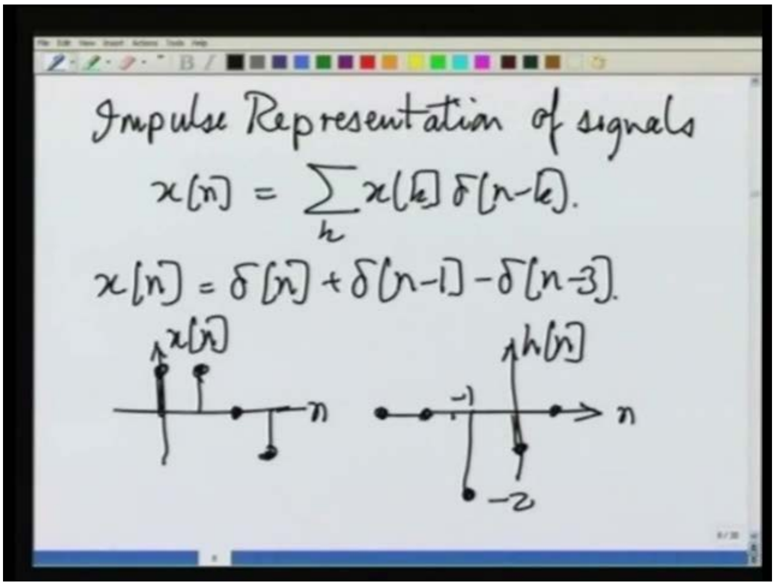

The impulse representation of signals in the form x n equals summation over k x k delta n minus k. This impulses from impulse representation of an arbitrary signal is crucial to our obtaining the impulse response approach to computing the output. If we do not have the impulse representation, we cannot have, we cannot go there; that is very, very important to understand. Once we know this, it is time to work out a few examples. Let us take the original signals we had, we had x n consisting of just three non-zero points delta n plus delta n minus 1 minus delta n minus 3. Here some signal we could sketch it over here. Let us similarly take a system now, which has a very simple impulse response, because we only want to demonstrate how we do this magic of obtaining the output.

So, let us take h n, let us say this value is minus 2 and, just 1 minute. So, h n will have a value at minus 2 at minus 1; let us say it has a value of minus 1 at n equal to 0; and let us say it has a value of 0 elsewhere. So, this is h n; it has values of minus 2 at n equal to minus 1, and minus 1 at n equal to 0; this is our system; this is the impulse response of our system.

Now let us apply our formula for obtaining the output; and try to see whether we can compute the output according to the formula. The procedure would be as follows. We would first decompose x n into component simple signals - elementary signals, each elementary signal is just once scaled and shifted impulse; we have already done this, we have x n equals delta n plus delta n minus 1 minus delta n minus 3.

We have sketched that signal over here. Then we have h n over here. Now that we have an impulse representation already laid out for x n; all we have to do is apply the additivity part of our derivation. Apply the three components of x n one after the other, and get the corresponding responses. So, I applied delta n. What happens, what will, what will be the output, if the input is delta n?

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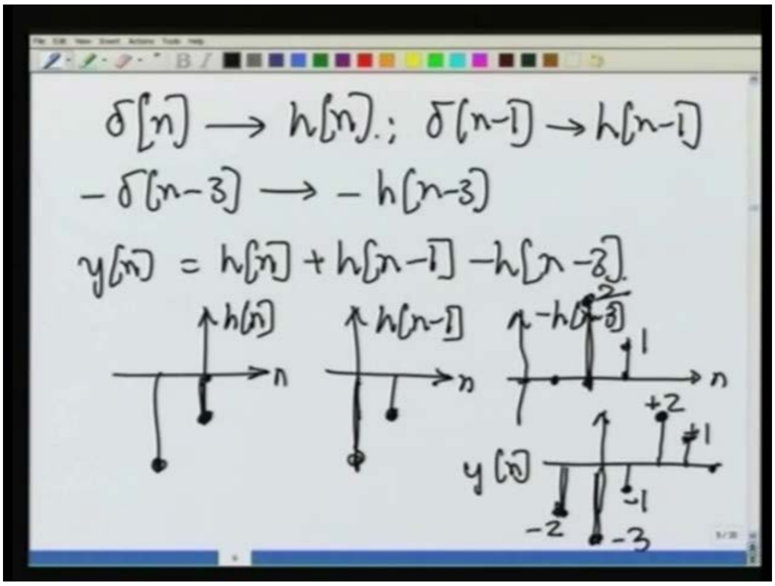

For delta n, by definition, the system will produce an output h n; for the other signal, delta n minus 1 the second member of the 3, you would get h n minus 1; and for the third member of the group minus delta n minus 3, you would get minus h n minus 3; that is it. So, by the additivity property, if these are the respective responses, for these respective input signals, then the response to the sum of these input signals which is of course x n, will simply be the sum of these respective responses. So, y n then equals h n plus h n minus 1 minus h n minus 3. If you want to do it with graphs, all we would do is first take h n, what is h n? Then we would take h n minus 1; h n minus 1 is nothing but h n shifted towards the right by 1 unit step, by 1 unit of time. So, we would have that is h n minus 1.

The next component of the response not of the input of the response is h n minus 3 or rather minus h n minus 3. Here is the time margin, you have minus h n minus 3 to be depicted over here, minus h n minus 3 is h n shifted to the right by 3 units of time. They have already shifted it by 1 unit of time to get h n minus 1, we have to shift it further by 2 units of time. And we have to invert the direction. So, it gets the value of plus 2 at n equal to 2, and plus 1 at n equal to 3. So, these are the three components of h n; if we add them, we would get y n, I will try to squeeze it into this screen.

So, y n would have a value 0 for n less than minus 1, because none of these three components is non-zero in these regions. At n equal to minus 1 we would have minus 2; at n equal to minus 1 we have y n equal to minus 2; at n equal to 0 we have y n is minus 1 contributed by h n, and minus 2 contributed by h n minus 1, so we would have minus 3 in this place. So, minus 2 at n equal to minus 1, minus 3 at n equal to 0, at n equal to 1 we have minus 1 contributed by h n minus 1, then at n equal to 2 we have plus 2 contributed by h n minus h n minus 3; and finally, at n equal to 3 we have plus 1 contributed by h n minus 3.

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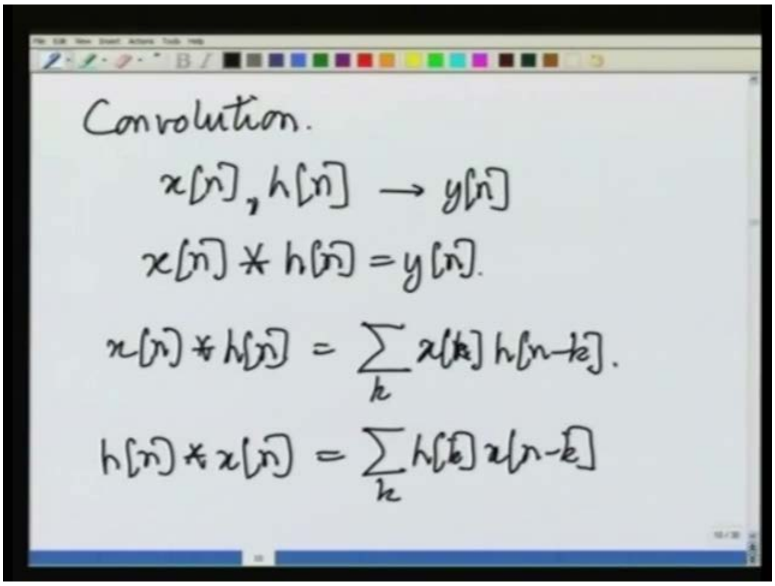

Let us just put down the values on the graph for each point of y n this is minus 2, this is minus 3, this was minus 1, this is plus 2 and this is plus 1; all other points of y n are 0. So, this has given us the output; from the input nearly by a knowledge of the impulses forms and by the computation. There is a name that has been given for this process, this process of computing the output from the input is called the convolution.

Convolution is the means of obtaining the output of an l T I system, given the input and given the impulses forms. Note that convolution essentially obtains a result by combining two input, two pieces of information; one is the input signal, the other is h n. So, we have x n and h n used together to give us the output y n. There is a notation for the convolution process; remember that just like addition and multiplication are operations that we can do on signals; convolution is also an operation that we can do on signals. That is why there is a symbol given for convolution, we would write normally that x n, convolution h n equals y n; and x n convolution h n is given by the formula that we had just obtained earlier x k h n minus k.

One important property of convolution is that it is commutative that is to say, it does not matter, if you exchange the positions of x n and h n you will still get the same by n. Putting this point down in the expression, we would get h n convolved with x n equals the summation overall k of h k x n minus k. Now is this true? It is true and it can be very easily shown as a I will just do now.

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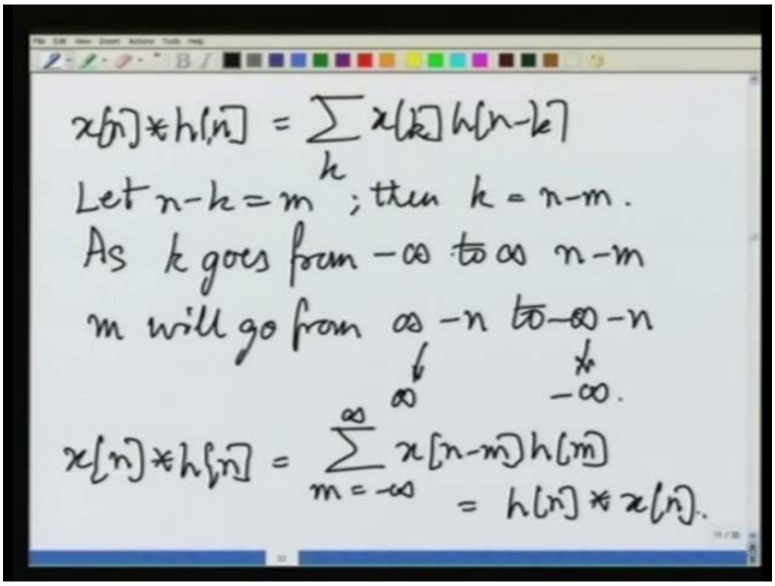

We have x k convolved with h sorry x n convolved with h n equals summation overall k x k h n minus k, we will carry out the change of variables in this infinite sum. Let n minus k be equal to say some m, now if n minus k is m, then n is sorry, then k is n minus m. And as k goes from minus infinity to infinity, infinity, what will happen? To n minus m n minus m will go from minus infinity to infinity because that is k; and so k will m will go from infinity minus n to minus infinity minus n; and both these this is nothing but infinity, and this is nothing but minus infinity.

And since summation is the same irrespective of what order whether you sum from minus infinity to infinity or infinity to minus infinity, this remains the same. So, rewriting in terms of these substituted variables, we get x k sorry x n convolved with h n equals summation m going from minus infinity to infinity x n minus m h m. This is exactly of the same form as this, except that the role of x and h have, the roles of x and h have been interchanged. Effectively therefore, this is nothing but h n convolved with x n that shows that convolution is commutative. That in fact, the order of the two functions been convolved does not matter, you could put either of them as the first and the other as the second; the result is the same.

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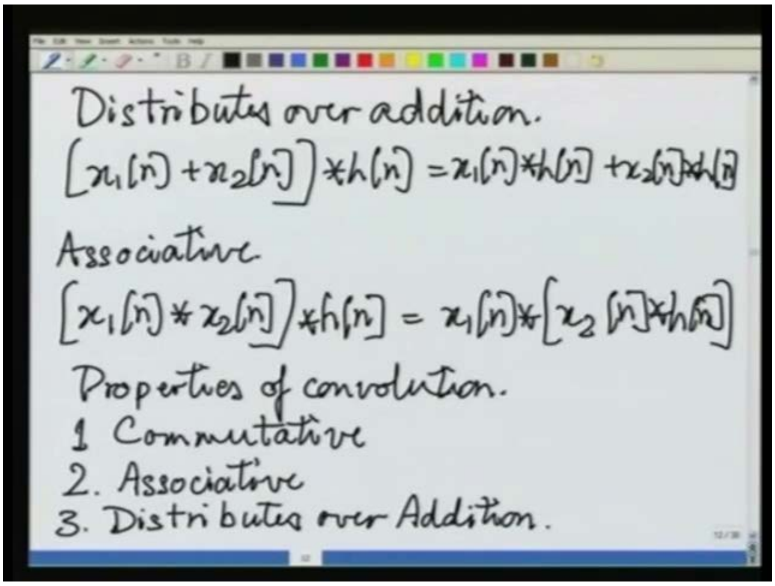

Convolution also has some other interesting properties convolution being a linear operation it distributes over addition, because if we have x 1 n plus x 2 n convolved with h n; this is nothing but x 1 n convolved with h n plus x 2 n, convolved with h n. So, it distributes over addition; this is very, very easy to show, you can just replace the convolution formula on the left side, and expand the single sum into 2 sums and you will get the expression on the right side.

Commutative is also associative, so that if you are convolving three functions simultaneously something that is perhaps immediately not relevant to us, but it is important to know the mathematical property so that we had say a convolution of x n x 1 n convolved with x 2 n, and this result of this convolution convolved with h n. We could equally rewrite this as x 1 n convolved with the result of convolving x 2 n with h n.

These two functions would amount to the, to be the same, and this can also be shown by writing the double summation for the left side, playing around with the variables, substituting some variables, replacing some variables, and we would get the expression on the right side; I do not go and had to do it, because it is available in most textbooks. Of course however, some textbooks will make you do this as an exercise, this really a various straight forward, just a little cumber some to do.

So, in short, convolution has the three following properties; first it is commutative, second it is associative and third it distributes over addition. These three properties characterize convolution; and what convolution does we know is provide us a means of computing the output of a system that is linear and timing variant. If the input is the output corresponding to the impulse as an input is known, in short if the impulse response is known.

Now what about continuous time signals? Can we carry out this very interesting exercise for the context of continuous time signals as well? Whether we can do it or not? Would depend upon whether we can replicate some of these steps we have done here. Among the steps we took, the first step was the impulse representation of a discrete time signal. So, the question will now arrive as to whether we can have an impulse representation for a continuous time signal. This is a problem of a much greater order of a much more indicate kind, for which a more thorough and study of real analysis is required. And what we will actually manage to do in this course is to take a few shortcuts wink at a few rules and get away with a rather superficial understanding of the real inter crisis of the problem.

We will however have enough of working understanding of the impulse representation for a continuous time signal. To do whatever we require in engineering, there is however much more to it; and that is important to know even if we actually have not worked out the details. The impulse representation for continuous time signals requires a knowledge of what an impulse would mean in continuous time, in the continuous time context.

So, let us try to see, what role the impulses played in the discrete time context? In the discrete time context, the impulse was an entity that could be placed anywhere along with time axis by shifting. And so what we did was to take the original signal, split it into components; each component was none zero only at one point, and zero everywhere else. Once we had this component, it could be expressed as a shifted and scaled impulse, the point is if you take a finitely long interval of the discrete time axis, then there are only a finite number of values of the discrete time signal between those two points.

So, if you take for example, x and between the points minus 1 and plus 3, then there are only the point minus 1, 0, plus 1, plus 2 and plus 3 - five different values on the axis; and at each of these points, the signal x n will take some value, so we have x minus 1, we have x 0, x 1, x 2 and x 3. Once we know this much, we know this signal entirely; this is the fundamental property of discrete time signal which is not present in the continuous time signal. Even if you take a finitely long interval of the continuous time axis, there are too many points and infinite number of points of x t on this interval. And at each of these points x t could potentially be different from its neighbouring point. In short therefore, we have a much larger number of points of the input signal to be represented, and that is why it becomes important. To understand how we handle that problem here. Secondly, there we had an option of an impulse; the impulse was also a discrete time signal like any other discrete time signal.

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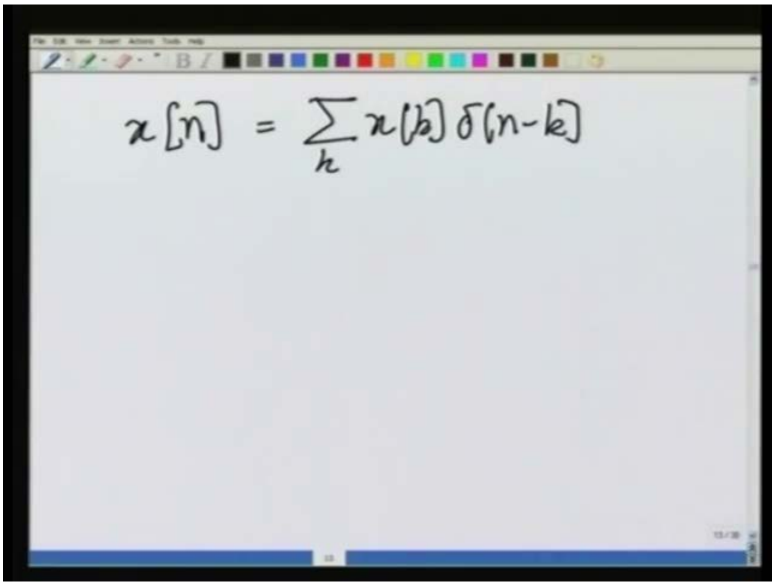

Here the impulse will turn out to be very different from the usual time kind of continuous time signals. There we could write x n equals summation overall k x k delta n minus k; this was our representation. We want a corresponding a similar representation for the continuous time case. How do we do it? Let...