Signals and System
Prof. K. S. Venktesh
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 11
LTI System

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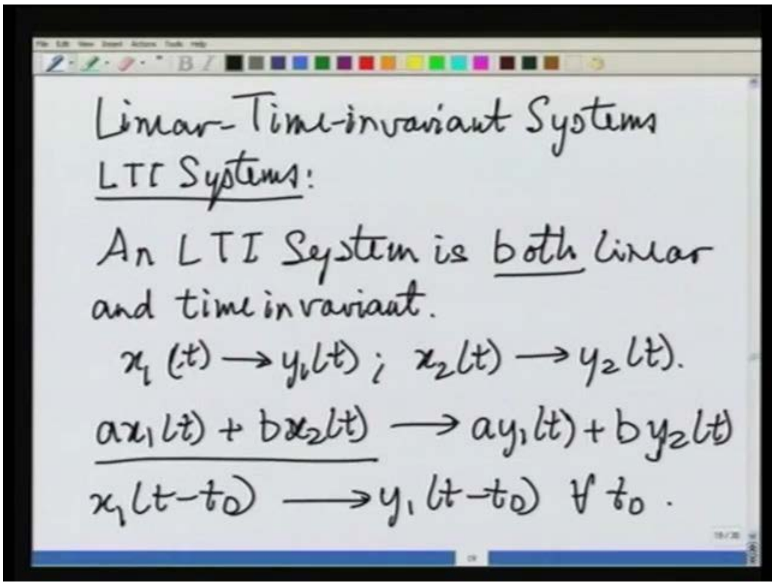

To move on to module 2, where we will speak of systems which are possessed of two very important properties and properties which are very particularly valuable to us namely linear time in-variant systems, briefly called LTI systems. LTI systems are practically all that we will ever be discussing in the rest of this course, the other systems are too complex to be discussed in an introductory course, but LTI systems with the simplifying property of being both linear and time in-variant allow us to get a first grasp of signal and system theory.

Now, what is an LTI system? An LTI system briefly is something that is both linear as well as time in-variant this should be understood clearly, we are not saying that it can be either linear or time in-variant, no it has to be both. So, the system, an LTI system is both linear and time invariant. Now, when a system is both linear, and time in-variant, in turns out that our description of the system becomes much more neat, much more compact, and much more capable of manipulation and construction.

So, we will now see what it is about linear time in-variant systems, that makes life, so interesting, recalling our definitions of linearity a linear system is 1 where if x 1 t gives you an output y 1 t, and if x 2 t gives you y 2 t, x 1 t yields y 1 t and x 2 t yields y 2 t. Then if the system is linear we know that x 1 t a x 1 t plus b x 2 t, if this signal which is this combination of the input signals is applied as a single signal to the input, then the output should be a times y 1 t plus b times y 2 of t, writing the single expression covers both the homogeneity and the additively property. So, a linear system is simply one which satisfies this single property that we have put down over here, time invariance says that if you apply x 1 of t minus t naught or x of t minus t naught whatever to the system.

Then the output should be y 1 of t minus t naught for all t naught positive or negative, it also goes without saying as we made particular example of in the last incidence we discussed with linearity, that any of these properties must be valid not only for all values of the parameters involved such as a or b in the case of linearity r as t naught in the case time invariance, but these properties must hold for every input signal on every set of input signals. Only then do we say that the system is linear and time invariant. So, a linear time invariant system satisfies this property we have put down here, for all values of a, all values of b, all values of t naught as well as all signals t 1 x t 1 x 1 t and x 2 t, this must be very, very clear at the outset. Now, once we have this property for a certain system we can see gradually what this yields in terms of our forming a new description for systems which are linear and time invariant. So, let us begin we introduce a property called convolution.

Now, in order to understand convolution, let us first begin with a discrete setting we will not consider continuous time signals, we will consider discrete time signals x n being applied to an LTI system to get y n, right. Now, we want to understand what makes linearity and time invariance so interesting to us, in order to understand that we will take this discrete signal x n and find a representation for it, a new representation for it which we will call an impulse representation for x n. Now, what is an impulse representation? Let me begin the discussion by introducing a new signal either to un presented either to unknown this signal is called delta n is called the Kronecker delta, and this signal is equal to 1 for n equal to 0, and its equal to 0 for n not equal to 0.

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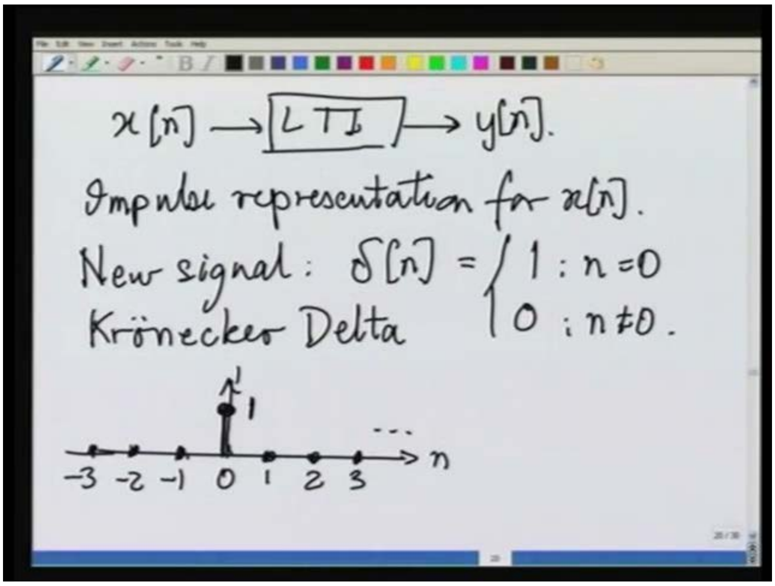

So, what does the signal look like what does its graph look like, very simple; you have the n axis over here, you have say this is the origin. This signal has a presence at n equal to 0 its value is 1 at all other points of time its value is 0. So, at n equal to 1, 2, 3, minus 1, minus 2, minus 3, etcetera; the signal is 0 except at n equal to 0 where this signal has a value of 1. Now this is our Kroneckaer delta and this signal is going to play a very, very important role, when we discuss signal and system theory. We will have a corresponding signal called dirac delta when we start talking of continuous time systems, but that is a little while later. Right now we have this Kronecker delta with us and let say we have an arbitrary signal x n, let me take a simple example first of an x n which has only finite support, let say that x n is this.

Let say this is x n, and let say that x n is 0 at all other points of time such as minus 1, minus 2, minus 3, and for values of n greater than 3 as well. So, let me just put down values of x n, x n equals 1 at n equal to 0, and n equal 1, that is what I have put down over here is equal to 0 at n equal to 2 is equal to minus 1 at n equal 3, and further x n is equal to 0 for n greater than 3, and n less than 0, this is our description of x n. Now, what can be do with this x n, the idea is to express x n as a sum of a shifted and scaled versions of the dirac delta of the Kronecker delta, you have the Kronecker delta given to us, and we know what is shifting a signal, we know what is scaling a signal.?

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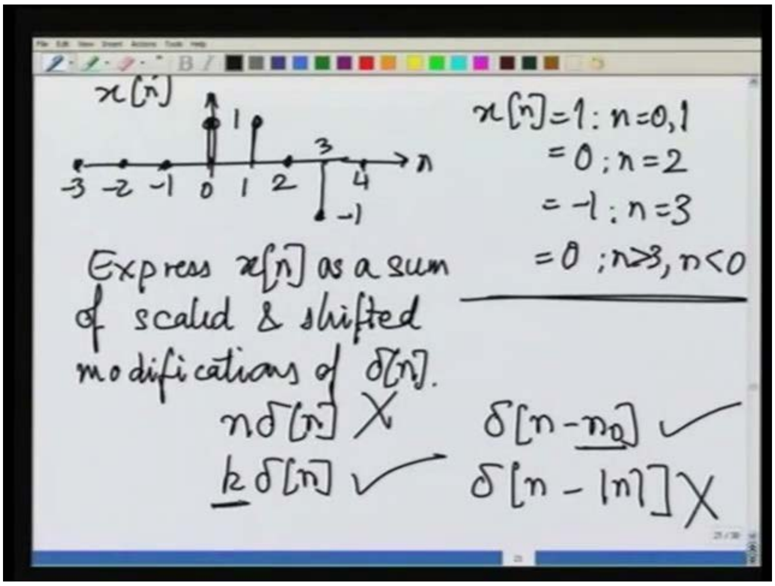

So, we want to express x n as a sum of scaled and shifted modifications of delta n, the first point you recognize here is that should be express as a sum. The final expression at the outer most level must be as a sum not as a product or any other kind of form, it should be as a sum. And each term of the sum can be scaled by some arbitrary constants, it can be shifted by an arbitrary amount, but only shifting and scaling by constant numbers are allowed the shifting and the scaling parameters cannot themselves be functions. For example, I would not consider, n delta n as a shifted version of delta n no, because n is a variable all I am allowed to do is to have a shift by a factor like k delta n.

So, k delta n is acceptable n delta n is not acceptable, because k is a constant likewise shifting I can shift by some factor n naught to get delta n minus n naught, this is acceptable. However, shifting by say by an amount like delta n minus mod n, say this is not what I would call a shifted version for our present purposes. So, this is not right.

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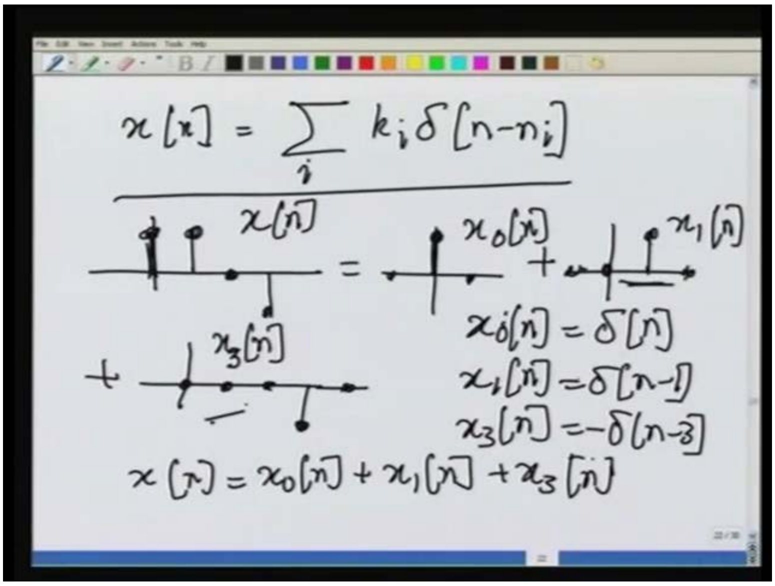

So, you can shift only by a constant by a fixed number or you can scale by a fixed number you can normally do both. So, we want x n to be expressed as a sum say over an index i of terms of the form k i delta n minus n i, this is our objective. Why are we going into this particular game, we are not in a position to explain right now. Let us wait a little we will understand the motivation for looking for such a representation for x n in a little while, but right now let us just see how this can be done, and whether this can done take the x n, we had earlier and in that x n we will find the following. x n is 1 for n equal to 0, and 1; x n is minus 1 for n equal to 3 and x n was 0 elsewhere. So, what I will do is to split x n into component signals, each component signal will be allowed to be non zero only at one point time at most and 0, elsewhere thus if I take the x n I had.

I am not pointing out the other points where the signal is anyways 0. So, this is our x n, I will express it as a sum of the following I will do everything is sketches. So, that things are more comfortable this equals the sum of the following, just this one point will zeros elsewhere fine, plus another signal where its 0 everywhere excepted n equal to 1, the first signal was 0 everywhere excepted n equal to 0, this is 0 everywhere except n equal to 1 then 2 more terms or just 1 more term. In fact, it is very easy to see that if I call these signals say x 1, x 0 of n x 1 of n and call this x 3 of n, where the subscript essentially indicates the point on the time axis where the non zero value of the original signal is substituted with all other values being kept at 0.

Then you can see that x n now equals x 0 n plus x 1 n plus x 3 n, you can verify this it is very straight forward. So, this is our first step. What do we do after this, we try to see if each of these component terms x n is already in the form of a sum, is in the form of a sum of much simpler signals than x n itself was; these signals are simpler because they are non zero at only at one point of time and they are 0 everywhere else, fine.

So, now let see if we can find an expression for x 0, x 1, and x 2; we can, we can write x 0 n as equal to simply delta n, Kronecker delta. x 1 n becomes equal to delta n minus 1, x 2 n sorry x 3 n becomes equal to delta n minus 3, sorry the minus sign there minus delta n minus 3, this is all we have very, very straight forward, you can verify each of these things minus delta n minus three is this signal, this signal, then delta n minus 1 is this signal, and delta 0 delta n is this signal. So, we have all these three things.

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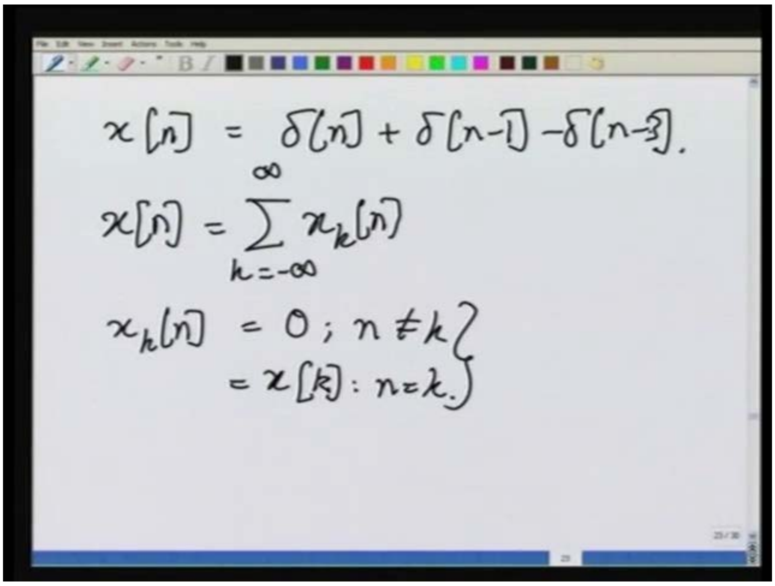

Now, we have achieved our expression of x n in the form we desired, and this form is delta n plus delta n minus 1 minus delta n minus 3. So, we have a representation, and the point your understand at this stage is that you can carry out such a representation for an arbitrary signal x n. If you have general x n not necessarily this 1 which had a finite support and these particular values, you would simply do the following. You had you take this x n split it up into components signals each component signal must be non zero at a particular point. So, x n is expressed as a summation of x k n, we can say let k run from minus infinity to infinity and x k n equals 0 for n not equal to k, and its equal to x k for n equal to k, that is all I expect of x k n.

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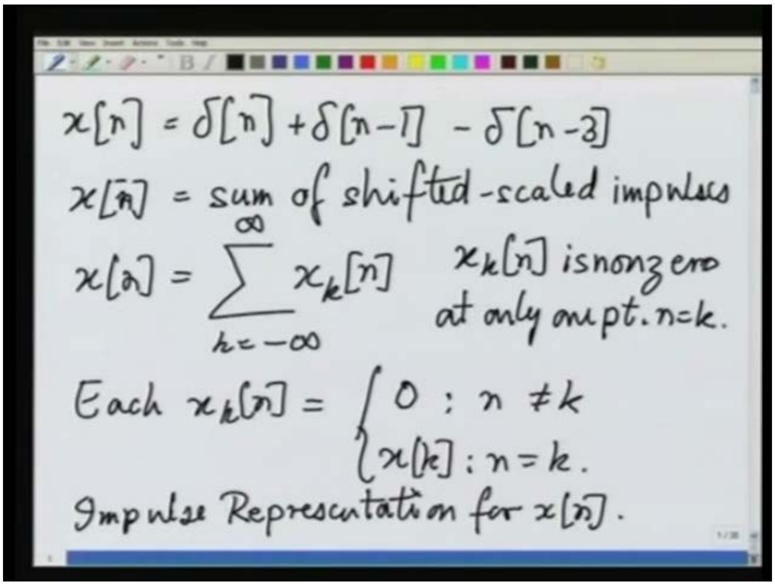

Thus we have expressed x n as a sum of shifted and scaled impulses, that is the essence of the argument. Now, if we had an arbitrary signal for x n not the highly simplified case that we just picked up to demonstrate the point, we would still do exactly the same thing, only difference is that our sum expression has a sum would probably bigger, and sum of this scale factors would be other than plus 1 and minus 1 as it has being in this case. In general therefore, if we had an x n with probably large values at some places probably infinite support, whatever x n, then the point to note is that x n is again expressed as a sum of simpler signals.

These simpler signals I will call x k of n x n i said is the sum of these signals. So, x n is actually the sum, in the general case k runs from minus infinity to infinity, and it is a sum of these x k n component signals. Now, each x k n itself being a very simple signal is non zero at only one point that is a very, very important feature of x k n and 0, everywhere else. So, x k n here is 0 when n is not equal to k, and at n equal to k its equal to x of k the original signal x, that we wish to represent evaluated at n equal to k. So, this makes up each x k n. So, when each x k n is given by this x n is given by the following sum or x k n k running from minus infinity to infinity, this gives us what we shall call an impulse representation for x n the significant point of course is that this is possible to do for any signal x n any discrete signal x n. Once we have the impulse representation in our hands the next point is to understand, how this impulse representation is useful.

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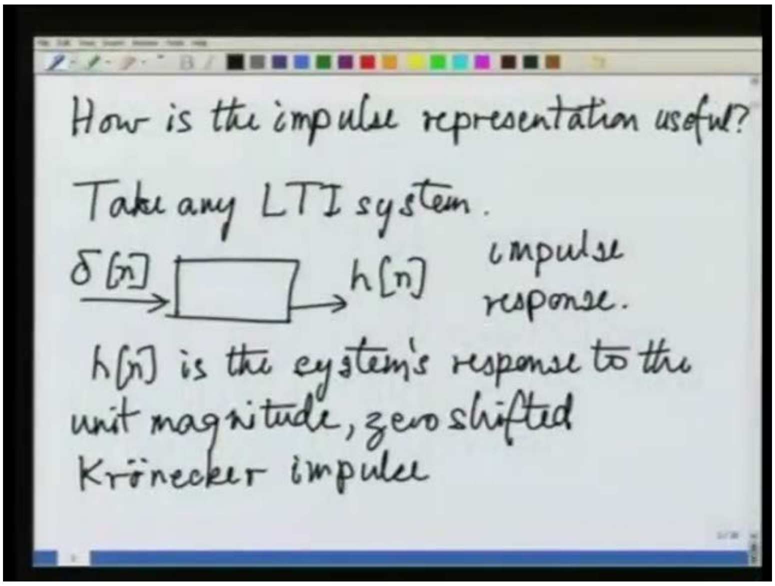

In order to understand the to get the answer to this question. Let us first see we have a system, let us say we have an LTI system, let let this system be generally unknown to us we do not know what output it produces for particular inputs. So, what we do is we take this system, and we apply a very special signal to this system is an ordinary signal just like any other signal, but it is is the simplest possible non zero signal.

This simplest possible non zero signal is of course, the Kronecker impulse, the Kronecker delta as I have called it. So, take the system take any arbitrary LTI system. Remember that it has to be linear time invariant that is very, very important. Now, we take the system then apply as the input, the signal delta n the Kronecker impulse, it is in terms of scaled and shifted Kronecker impulses that we have found a representation for arbitrary signals, but that is as far as the relationship of the Kronecker impulse goes with discrete time signals.

Now, we are bringing the same Kronecker impulse, which we have been discussing in the context of signals to the world of systems and applying this Kronecker impulse to the given LTI system. Now, this Kronecker impulse as I said is also a kind of a signal it is also particular signal, and so there must be some output that the system produces for this signal, let that output be some signal which we call h n is also a discrete time signal. So, you apply the Kronecker impulse to the input and observe the output call it h n, so h n is the another discrete time signal.

Now, we will see what happens with our impulse representation of discrete time signals on the one hand, and our knowledge of h n on the other hand with respect to a particular given system it will be shown in a few minutes that if you know h n which. In fact, is very appropriately called the impulse response it is the response to the unit magnitude 0 shifted Kronecker impulse. We will find that our knowledge of h n will help us to find the output of this particular system to any arbitrary input signal. Thus from a knowledge of only one distinguished input output pair namely delta n, h n; we able to construct every other input output pair in the lookup table that represents other system. So, how do we do it? Remember that is it is a linear time invariant system that we are concerned with and so a certain property is hold for the system its time invariant, so if instead of applying delta n to the system.

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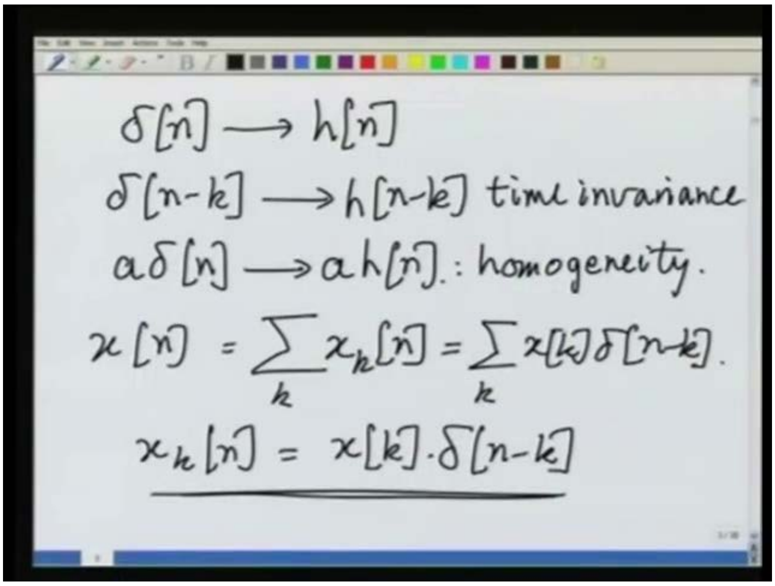

We know that delta n yields h n. So, if I shift delta n by time by some time k and apply it to the same system, because of time invariance the output cannot be something altogether view is has only got to be an correspondingly shifted version of h n. So, if I apply delta n minus k to the system, I should get correspondingly h n minus k, this is because of time invariance by virtue of time invariance the system has no choice, but to produce h n minus k when delta n minus k is applied. This will obviously be true for any value of k from minus infinity to plus infinity, any value of k you like this should happen. That makes the system more predictable in a way, you know what it is going to do when you apply delta n minus k by virtue of your knowledge of what it did when you apply delta n, when you apply delta n it gave you h n when you apply delta n minus k, it gives you h n minus k.

Another thing we know about the system is its linearity. So, we know that from the homogeneity property of linearity, from homogeneity aspect of linearity if instead of delta n, I apply some a times delta n as the input, the output again cannot be altogether new it has to be a times h n, it cannot be anything else. So, a delta n will produce an output a h n delta n minus k will produce h n minus k. So, this second part is by homogeneity. So, we have used time invariance, we have used homogeneity, all that is left unused is the linearity property is the aspect of linearity property that we call additivity. So, we are just going to use additivity also now.

Remember that we have a particular representation for an arbitrary signal x n. x n equals summation overall k x k of n, if you go back to the definition of x k of n you will that its non zero only at n equal k and at that point it takes the value of x k, and other places it is 0. So, x k of n may be written as x k which is just a number x k is a number the signal x n evaluated at n equal to k multiplied by the impulse function, this is a very simple and important fact to keep in mind. So, I will substitute that in this expression for x n, and get our impulse representation in the form x k delta n minus k, it takes this formula.

So, where do we go from here, remember that the system is additive if homogeneous is additive its time invariant, right now I am concerned with its being additive, because it is additive, instead of applying the entire signal x n to the input of the system, I can just apply each component part namely each h k of n separately to the system, and see the respective outputs. Once I have all the outputs I can add them out, and what I would still get is the response that I would get if there sum that is x n and we applied directly, thus instead of applying x n to the system I will apply sigma overall k summation overall k x k delta n minus k.

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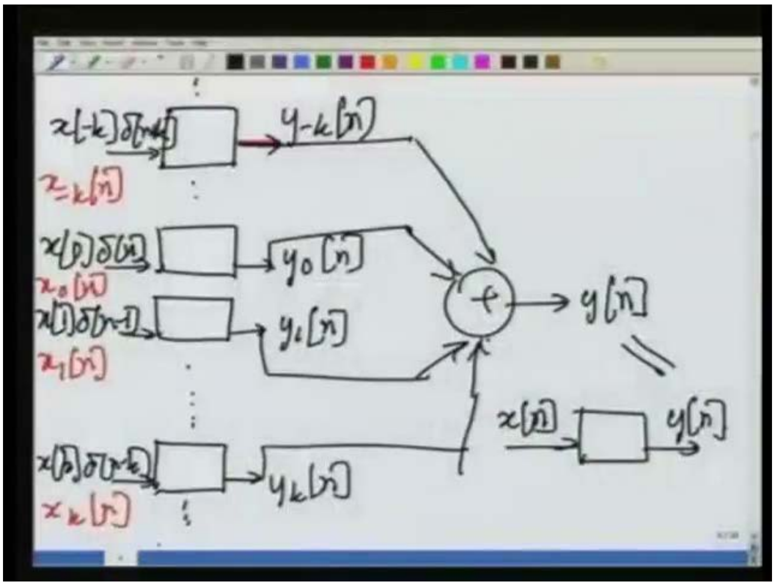

So, here is my system I will make a lot of copies of this system, identical copies carbon copies if you like. So, that they all have the same lookup table, and then I will start carrying out my I have made lots of copies of this system, and here I will apply x minus k delta n plus k here I will apply x 0 delta n here I will probably apply x of x 1 delta n minus 1 here I could apply something like x k delta n minus k.

So, I apply all these different signals each one this is x minus k of n, this signal is x minus k of n. Similarly this signal is x 0 of n, this signal is x 1 of n this signal is x k of n. So, I have applied each part separately to a copy of this system, since they are all copies of this system, they will produce the same output that the original system would have produced. So, here let me call the output, let me call this response to the application of x minus k of n, I will call this y minus k of n, this correspondingly will be y 0 of n, y 1 of n, y k of n. I have each respective output by the additivity property I can do the following, I can take each of these outputs, and add them all up, and when I add all these up I get a signal which should be the same that I would have got if I just got 1 copy of the system and I just applied to x n to it.

So, here I should still get y n, which is to say that is entire experiment with so many boxes and components and connection over here is amount to the same as just having one single system applying x and the original un decomposed signal, and obtaining y n. This y n is the same as this y n by the additivity property right, this y n is the same as this y n by the additivity property. So, I could manage with one block representing the system instead of making an infinite number of carbon copies as I have done over here. So, I suppose you would still be wandering what I am getting at I have tried as seen to a only made the whole thing more complicated rather than more simple. Now, we have so many blocks, so many component signals to deal with, and then we have the business of adding them all up only to get the same signal that one would have got directly by applying x, but their sub points to all this. The point is the following consider any one of the component blocks that we had just used in constructing the big block diagram.

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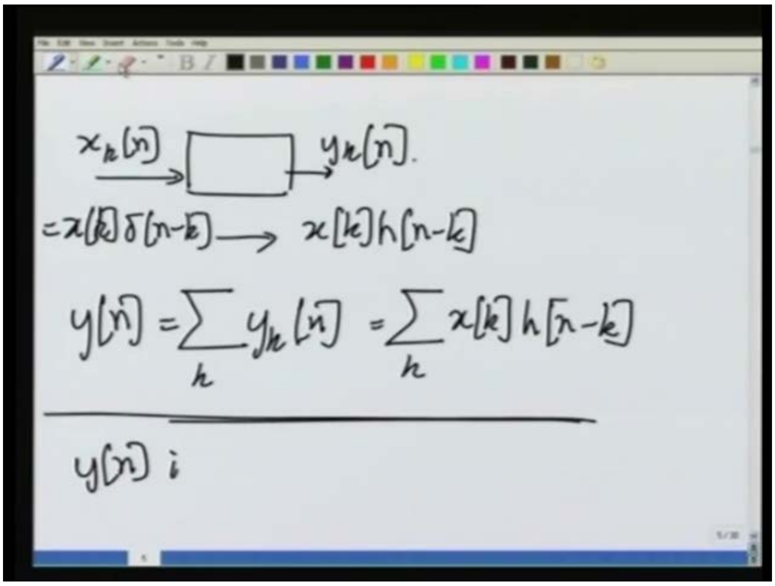

So, there is this component block to which I have applied x k of n equal x of k delta n minus k, this is what I have applied at the output, and I have got y k of n. The point is using homogeneity and the time invariance, I can actually find out what y k n is going to be instead of just leaving it as some kind of some unknown signal y k of n, I can exactly find out what it is in terms of surprisingly h n the impulse response. If we just go back to the 2 steps go back a couple of frames, then we will recognize that if we apply x k delta n minus k to the same system. Then using homogeneity and time invariant simultaneously, we find that the response to x k delta n minus k should simply be x k, which is the scale factor homogeneity use of homogeneity and by time invariance x k times, this should be h n minus k.

So, in this particular component block, we have succeeded in expressing the response to x n minus x k of n namely y k of n, in terms of the impulse response, in terms of a scale and shifted impulse response shifted by k scaled by x k. So, that is it we have now the response at each component block, we now have to add all these responses remember that y n, evaluates to the summation overall k of y k of n, and by just what we have said above this comes to summation overall k of x k h n minus k. x k is the value of x n evaluated at n equal to k delta n minus k is the Kronecker impulse shifted to the point k, time shifted to the point k and this gives us an expression for the output signal y n in terms of the input signal, and the impulse response.

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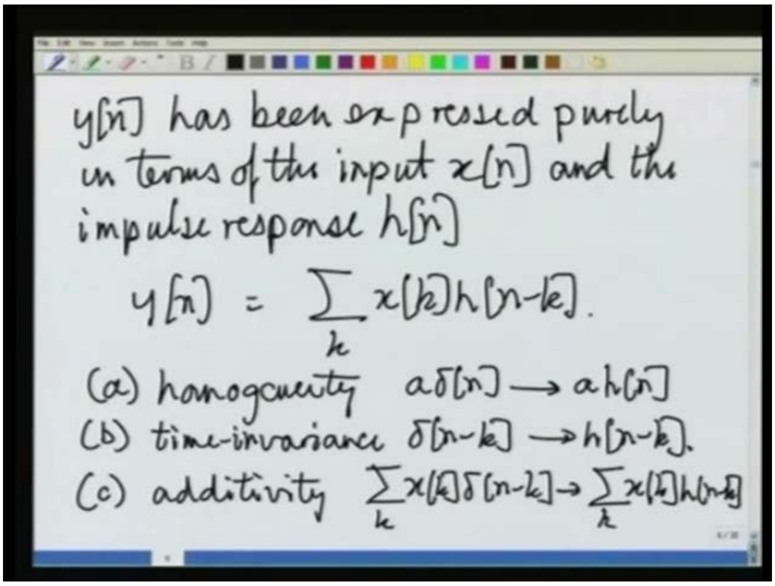

So, y n is let us go to the fresh page y n has been expressed, purely in terms of the input x n input signal x n. You want to know the input signal completely of course, and the impulse response. You just have the impulse response and the input, and we have evaluated the response to an arbitrary input, the response y n to this arbitrary input x n. Until now we required an entire lookup table to tell us what the output would be for each input, we had an infinite number of signals which we usually do, we have to have an infinitely long lookup table listing every input signal, which itself is generally a signal with infinite support, and against it you would have had to represent every corresponding output. So, we would have had a doubly infinite problem on our hands, instead now we have what can be called a formula a means of calculating the output without actually looking up in any table, this is a grass simplification.

A simplification that has reduced a huge table to just this grand formula namely y k, sorry, y n equals summation overall k x k h n minus k. So, this one function, this one discrete time function called h n taken along with the properties of our system namely linearity, and time invariance enables us to collapse all the information, that is present in an infinite lookup table to just a simple calculation a simple formula using, which you can calculate the output, you can compute the output without actually trying it out experimentally.

So, an experimental determination of y n has been replaced in some sense by a purely computational means of obtaining the output. So, as I said in order to get this result we have used all three properties that we have assumed of the system, in the very process of deriving this expression we have used a homogeneity, that is to say that a delta n must yield the output a h n. We have used time in-variance the property by which delta n minus k must give us h n minus k, and finally we have also used at the outer most level, we have used additivity or if you like you can just use the actual expression for the x k of n to get the output as the following sum of x k h n minus k.

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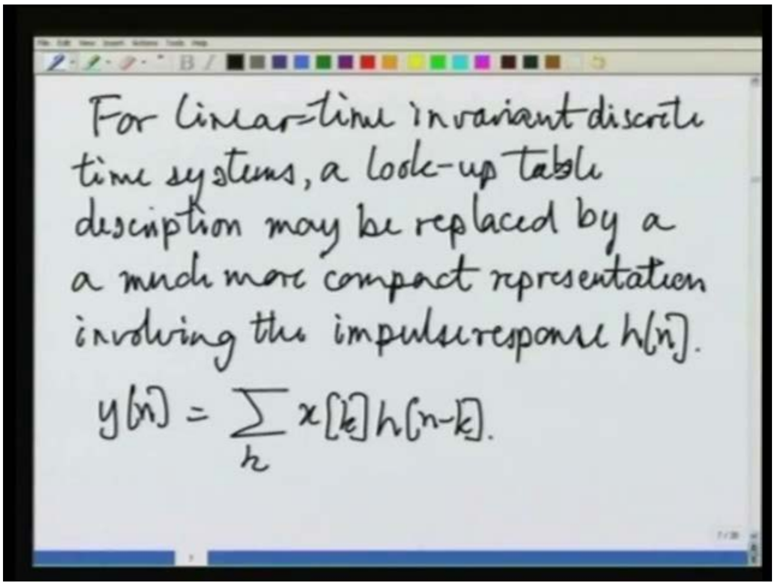

So, this is the major step forward that we have now taken for linear tine invariant systems, discrete time systems a lookup table description may be replaced by a much more compact representation involving the signal called the impulse response. So, if I have a knowledge of h n for a certain system I can potentially calculate the response to any arbitrary input signal. I need no more information than the impulse response to fully know the system, if I know the impulse response, I know the system completely, I know its entire input output relationship.