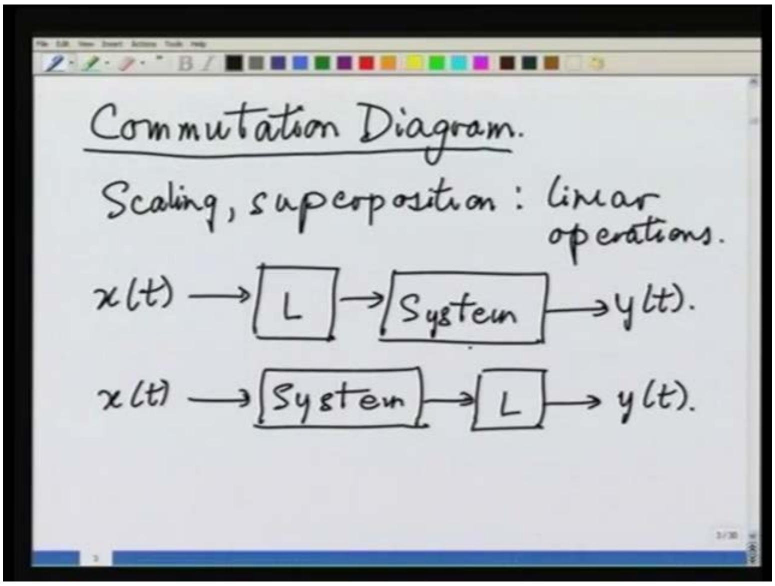
Signals and System
Prof. K.S. Venktesh
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 10
Communication Diagram as a Test for Linearity and Time Invariance

See let us now try to see if we can have rigorous mechanism for testing for linearity there is such a procedure and we call it the commutative commutation diagram.

(Refer Slide Time: 00:43)



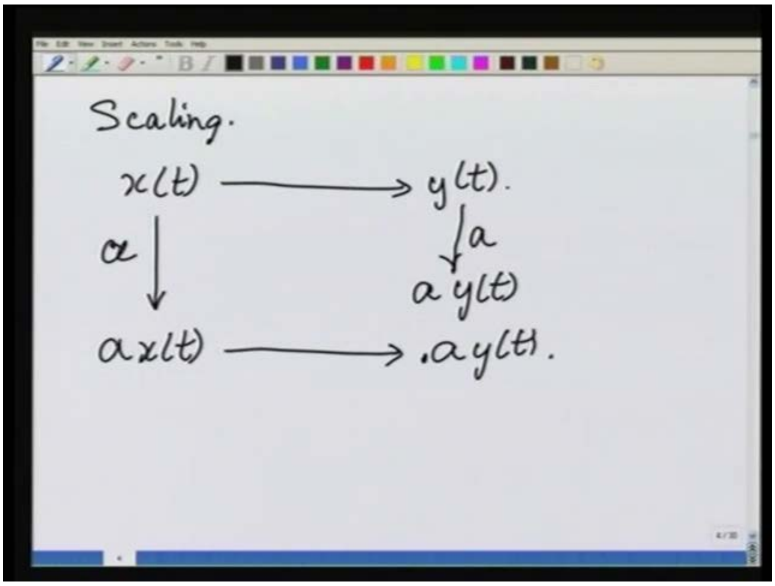
This commutation diagram approach is simply a formalization of what we have already been doing what we do is the following we have a system, and we wish to test whether this system is linear and all we need to do is to check whether scaling operations and super position operations when applied to the input actually or equivalent to the same scaling and super position operations applied to the operation output.

Let me explain this first by a block diagram let us first call both scaling and super position operations by one common name we call them linear operations . So, what I will now point out is that the property of linearity if possessed by a system is simply equivalent to commuting the linear operations with the system itself. So, suppose I start with x 1 of t or x of t I first apply a scaling operation to it a linear operation I will just represent by L and take its output to the system itself and I get some output over here which I will call say y t now if the system is linear then this output should be the same if

I first pass the signal through the system pass the input through the system and then apply the linear operation.

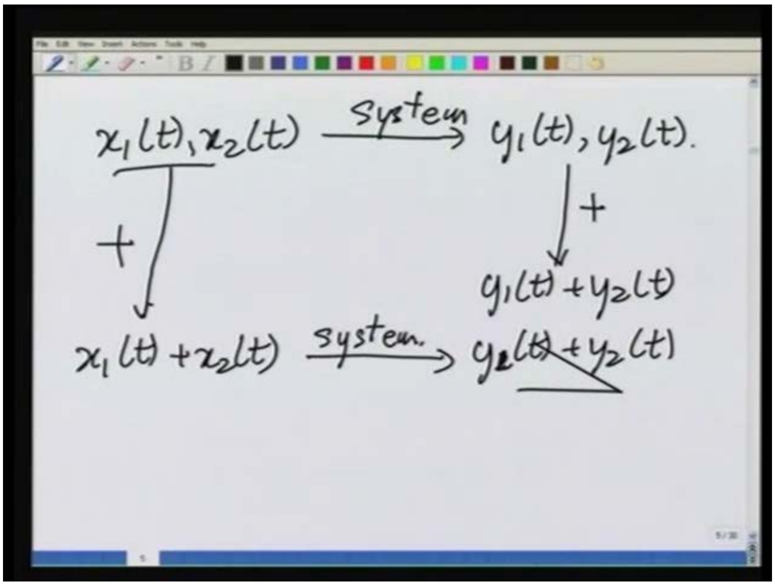
That is to say that if I did this instead of the above I should still get y t every system for which both these constructions that we have drawn here yield the same output or every system which satisfies this that the outputs are invariant to the order in which the linear operation and the system are applied is a linear system. So, we are essentially going to have a mechanism to rigorously test whether the system does in the remain invariant under an exchange of the linear operation namely scaling or super position and the system itself.

(Refer Slide Time: 04:37)



Let us now discuss the commutation diagram here we make the diagram as fallows to first test for scaling that is to test for homogeneity, let us apply x t and passing through system it produces some output y t instead we could also apply a scaling operation to x t scaling via factor a for example, to get a times x t, and we could see what output arises if a times x t is fed to the system if x t yields y t what does a x t give us and on the other side we have y t already here and we can apply the same scaling operation that is multiplication by a to y t. So, would get here something else we would get a times y t in this place now the question is what we will get in this place where a x t is applied to the input if here also we get a y t, then the system is homogenous a similar thing is done for super position what we will do is the following.

(Refer Slide Time: 06:04)

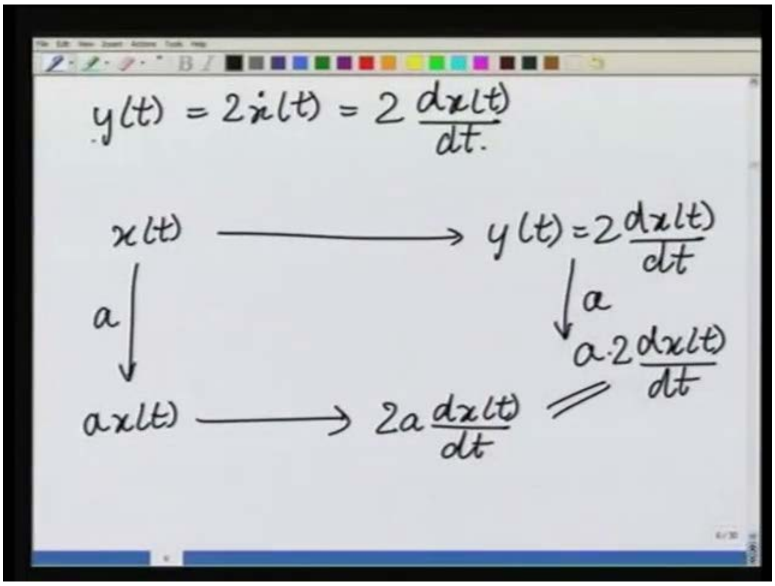


Apply consider two signals x 1 t x 2 t apply both of them to the system one after the other or whatever to get the corresponding outputs y 1 t and y 2 t now if these two signals applied at the input x 1 t and x 2 t are added to each other we would get x 1 t plus x 2 t what we will do of course, is as with the test for homogeneity, we will apply this combined signal x 1 t plus x 2 t to the same system and see what we get as the output here of course, we will on other side add y 1 t and y 2 t to get y 1 t plus y 2 t.

If x 1 t and x 2 t added together to give x 1 t plus x 2 t that signal is applied to the input then if the output is again equal to y 1 t plus y 2 t then this system is linear in short this system is additive in short let us put it this way the commutation diagram takes you from the input to the output in two different roots in one root you apply the system first and the linear operation later in this case addition. So, I apply the system first and then a linear operation and check the output compare that with the other path through the commutation diagram where you first apply the linear operation and then apply the system.

If the end result is the same irrespective of the two paths that is when you would say that the system commutes with that particular linear operation namely in namely addition in this case. So, if you test whether the system commutes with addition as well as test if the system commutes with scaling of the input then together we have succeeded in showing that the system is both homogeneous and additive and therefore, linear let us work out a few examples using the commutation diagram and that will help us understand how to use this diagram.

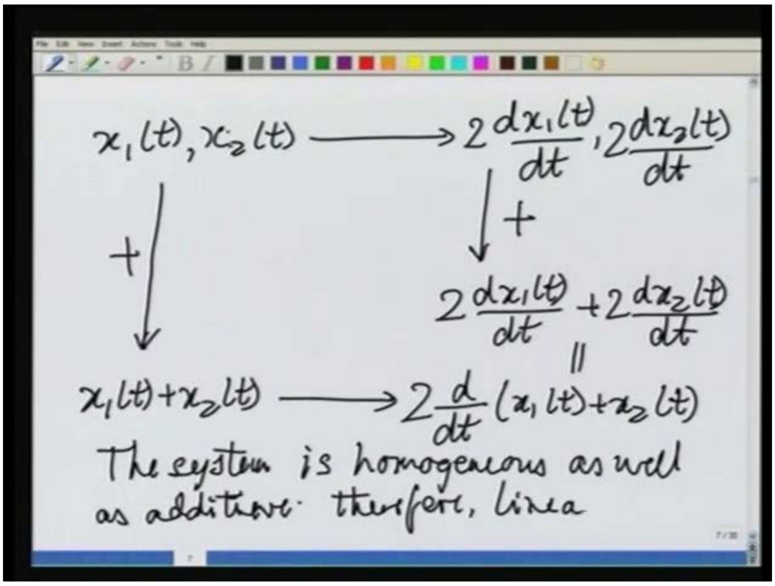
(Refer Slide Time: 08:56)



So, let us take an example like y t equals 2 x 2 x dot t let say which by which I mean 2 d x t by d t let say the system is differentiator and also scales the input by a factor of two. So, this is our system we wish to check whether the system is linear and we will do it by first testing for scaling to see if the system is homogeneous. So, apply a scaling operator to the input and see if the system commutes with the scaling operator.

Let us have x t over here and we make the commutation diagram the output for x t is y t equals 2 d x t by d t on this side we apply this scaling operator first to get a times x t and we wish to see what happens if a times x t is applied to system if a times x t is applied to the system and the system is in straight forward differentiator then we know that the output will be 2 a the d x t by d t this is what we get by coming through this path of applying the linear operation first, and then the system now through this other path we have already seen the output and we will apply the same linear operation on the output we have 2 x t 2 d x t by d t and if we scale this by a factor a this we will yield us a times 2 d x t by d t. Clearly these two are equal to each other and therefore, we have just seen that this system is commuting with the scaling operation by a. So, this system is homogeneous.

(Refer Slide Time: 11:15)

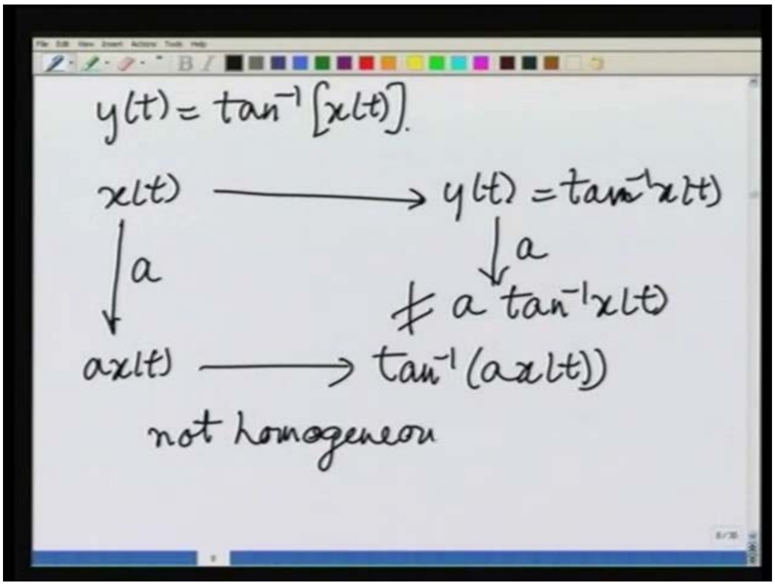


Now, let us see about super position let we have let us we have x 1 t plus x 2 t as the input the systems output would be y 1 t plus y 2 t which is two d [ah excuse me initially we haven't yet added the things. So, we just have x 1 t and x 2 t being applied to the input that would give us 2 d x 1 t as the output for the first signal 2 d x 1 t by d t and 2 d x 2 t by d t as the other output.

So, for this pair of inputs we get this pair of outputs separately now let see what happens when we add this pair of inputs we would of course, get the combined input x 1 t plus x 2 t is the combined input. Now there are two things to do first check the output when the input is x 1 t plus x 2 t that is very straight forward, because we know what the system does the system differentiates the input and scales this result by a factor of two. So, what we have now is two times the time derivative of x 1 t plus x 2 t which is d by d t of x 1 t plus x 2 t this is what we get going by this path of adding first and applying the system later.

Now, let see what we get by this path where we have already applied the system to get the two respective outputs y 1 t equal to 2 d x 1 t by d t and y 2 t equal to 2 d x 2 t by d t we just add the two now when we add the two we would just get 2 d x 1 t by d t plus 2 d x 2 t by d t these two are; obviously, equal to each other. So, this system has shown itself to be additive because addition is commuting with the system. So, collecting these two things together we can say the system is homogeneous as well as has additive therefore, linear that is it.

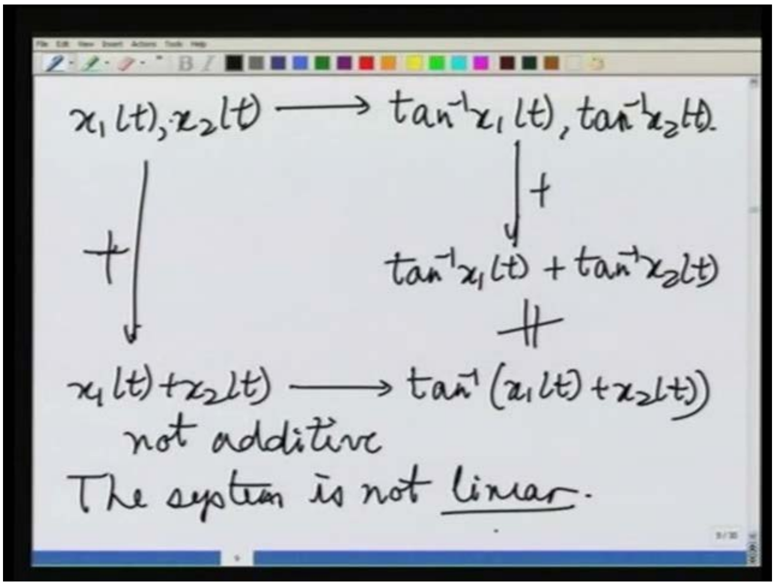
(Refer Slide Time: 14:40)



Now, one more example this time let us pick something which we suspected list to be non-linear. So, let us take y t equals say tan inverse of x t the tan inverse function is I think known to all of us to be non-linear, but let us see what happens when we apply the commutation diagram first for scaling we have x t you get y t equals tan inverse of x t here we scale x t by a factor a to get a times x t and apply the system to get tan inverse a x t from the other side we would get we have to scale the output by a factor a and. So, we would get a times tan inverse x t. Now, it is well known to all of us that tan inverse a times x t is not the same as a times tan inverse a x t. So, these two quantities are not equal to each other. So, the system is not homogeneous.

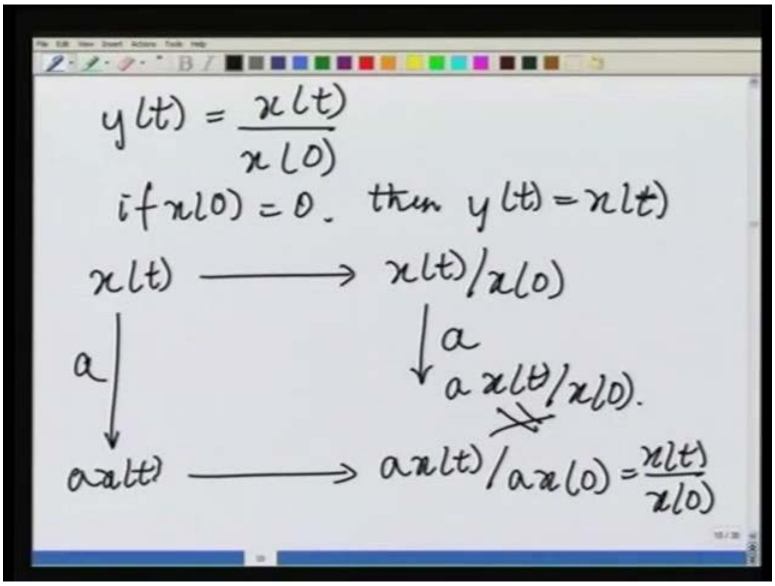
Let’s see about the additivity, you have x 1 t plus comma sorry x 1 t comma x 2 t as the two inputs and the corresponding outputs would be tan inverse x 1 t tan inverse x 2 t those would be the corresponding outputs now we have to add these two inputs to get x 1 t plus x 2 t on the one side apply the system to x 1 t plus x 2 t, you would get tan inverse x 1 t plus x 2 t together where as if you already take the two outputs tan inverse x 1 t and tan inverse x 2 t you simply add the two over here and get tan inverse x 1 t plus tan inverse x 2 t and certainly these two quantities are not equal to each other.

(Refer Slide Time: 16:30)



So, it is not additive the system is therefore, not linear we could work out any more any number examples on this principle, but let me just work out one last slightly tricky example.

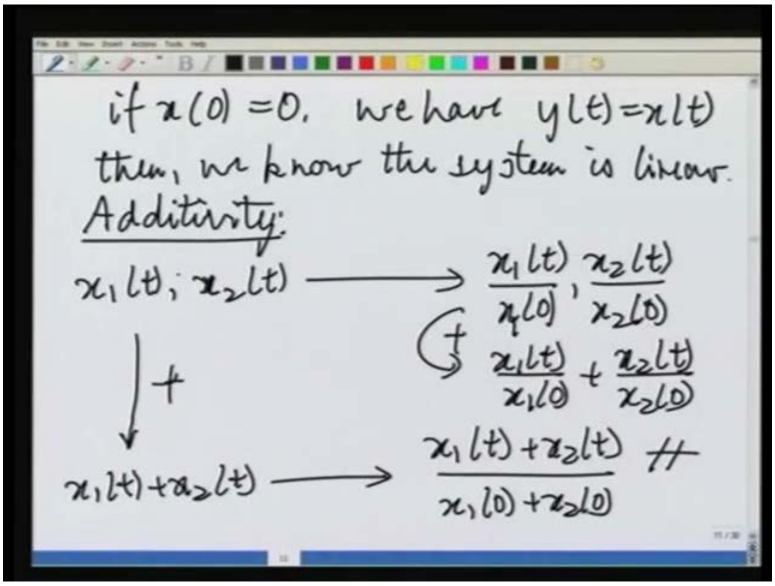
(Refer Slide Time: 18:30)



Suppose we say that y t equals x t divided by x zero is this system linear we want to know if the output is homogeneous such that the systems homogeneity and additivity can be demonstrated. Now, look at this expression carefully there can be a problem here because this definition of system itself is suspect the problem is that if x zero were equal zero then we will have to make a fresh stipulation, let us see if x 1 if x zero equals zero then y t equals say x t itself now let see what happens to this is the system homogeneous.

Let us just understand what I have written over here I have taken x t applied it to the system to get x t by x zero, if I scale x t by a factor a as I have done on this side then ax t is the input and when the ax t is the input the corresponding output by the definition of the system should be a x t by ax zero which is just equal to x t by x zero now let see what happens on the other hand if I scale the output of the system obtained directly before scaling that is x t by x zero by a factor a I would get a times x t by x zero. So, through one path I have got a times x t by x zero through the other path where this scale operation the linear operation preceded the system we have got x t by x zero now these two things are definitely not equal to each other there is no doubt that the system is not homogeneous for all signals where x zero is non-zero, because wherever x zero is non-zero this is what we would be writing now if x zero over zero then what happens.

(Refer Slide Time: 22:53)

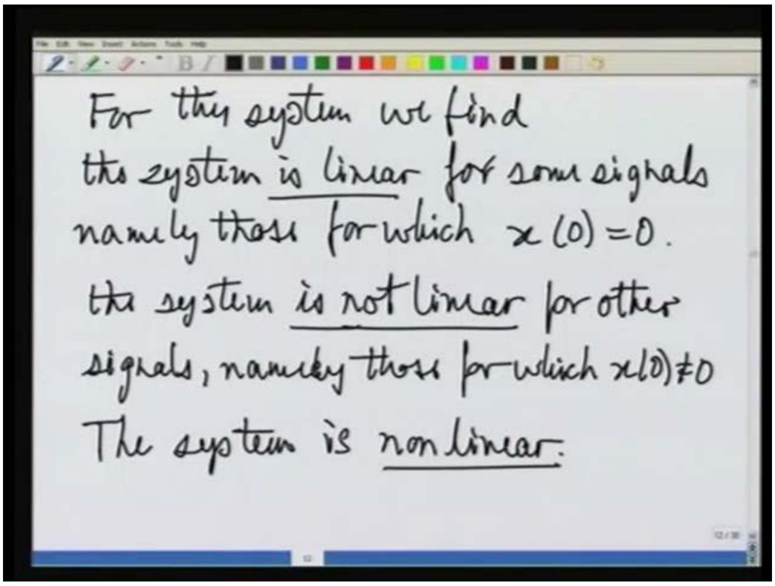


If x zero happens to be zero that is of the set of all signals in this world separate this set into two parts those signals which have x zero equal to zero and those signals for which x zero is non-zero we have just completed our exercise for only homogeneity for the case of systems where x zero is non-zero for that category of signals taken as inputs we have found that the system is not homogeneous. Now if we consider the other class of the signals the class where x zero is zero, then what do you think will happen then we will just make modifications in this vary diagram let us draw a fresh diagram.

If x zero is zero then we have y t equals x t and we know the system to be linear is. In fact, linear. So, it is also homogeneous. So, there is no problem in x zero is zero there is; however, problem has we just demonstrated earlier then the system is impressed with a signal for which x zero is non-zero because then this definition would come into play now it is really not necessary to test for homogeneity here, but let us do it anyway it takes only a movement to test for additivity you have x 1 t and x 2 t being applied simultaneously to the input or one after the other.

So, this feat in parallel to get the outputs x 1 t by x zero x 2 t by x 1 zero and this is x 2 of zero these are the two respective outputs now if you added the two inputs we would get x 1 t plus x 2 t and what would the output be under this circumference it would be x 1 t plus x 2 t divided by x 1 zero plus x 2 zero where as if we just added the two outputs we would get x 1 t by x 1 zero plus x 2 t by x 2 zero.

(Refer Slide Time: 26:29)



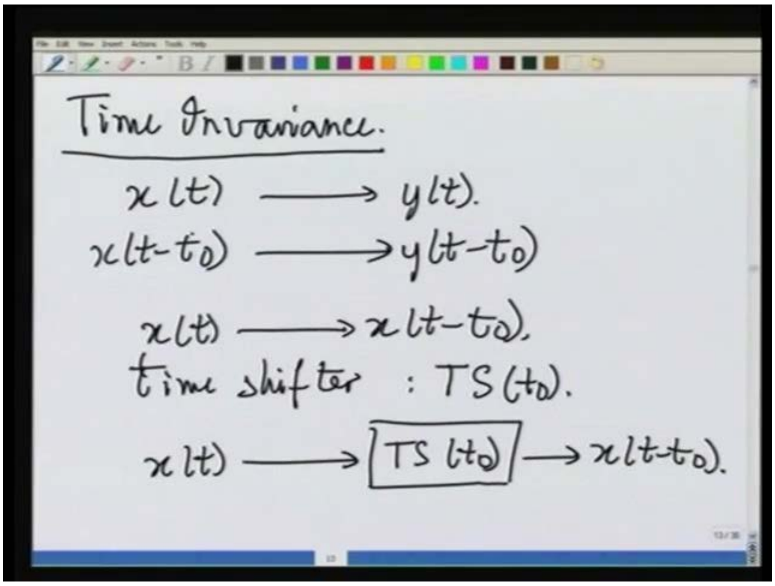
Now, these two are definitely not equal to each other. So, again for signals where x zero is non-zero the system has proven to be non-additive it is not additive just like it was also not homogeneous going back to the instance where we consider signals for having x zero equal to zero then the definition of the system changes to y t equals x t and that system is also additive just like it is also homogenous the whole system is linear. So, let summarize what we have learnt in this example for this example.

We find the system is linear for some signals namely those for which x zero equals zero for such input signals for which x zero equals zero the system is linear the system and the other hand is not linear. For other signals namely those for which x zero is non-zero. So, now, what happens how do we declare the system do we declare it to be linear or not linear this brings us back. In fact, to very, very important statement that we have been making every time we make any of this definitions of properties of systems a system is said to have a property like linearity or causality or stability whatever we have discussed. So, for only when that property is valid for every input signal using that principle over here and recognizing that this property of linearity is not available for all input signals only for certain input signals we will therefore, say that the system is non-linear or not linear.

Let me summarize what was very special about this particular example this is the first example that we have considered. So, for where the system behaves with different property exhibits very, very different properties depending upon which signal is applied for certain signals it shows linearity properties for certain signals it does not exhibit linear properties and for a situation like this as I have said we will have to go by the principle that unless that property that we are testing for such as linearity is available for every input signal we will not declare the entire system to be linear we will say. In fact, that the system has failed the linearity test that the system is not linear this example is therefore, a little unusual. Let us now move onto the next property of systems next important property of systems which we shall call time invariance.

Every time I introduce a new property I make some common somewhat whether it is a domain related property or a range related property thus causality memory were all domain related properties and stability linearity were range related properties now we come to this fifth property which we call time invariance and we would perhaps like to ask same question is this domain related property or a range related property and the fact is that this is also a domain related property what is time invariance.

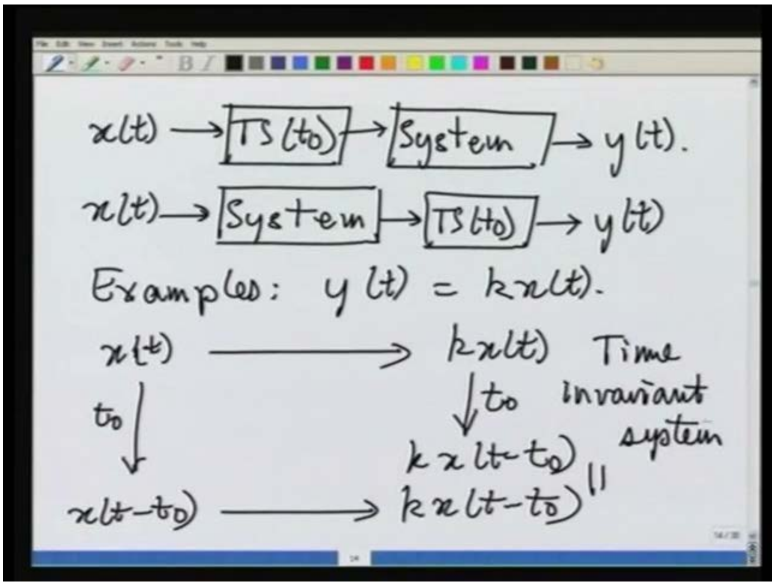
(Refer Slide Time: 30:19)



Time invariance is again a phenomenon that we would like to see in a lot of real world sound system sound is time invariance is one form of consistency of course, linearity was also a form of consistency in that we expected the system to be invariant to scaling and super position time invariance is a different kind of invariance it is invariance with respect to translation of the input signal on the time access in short what we expect is that if we time shift and input signal and applied to the system then the outputs that we get must be the corresponding time shifted version of the original output if x t yields certain output y t, and if the system is time invariant then x of t minus t naught which a time shifted version of x t we have already discuss what happens to the signal when it is given a time shift or time translation we know what how the signal changes what we want the system to do; however, is ensure that when x t minus t naught is applied to the system the output is y t minus t naught this we must understand is also a kind of commutation. If we consider this operation of time shifting to be another kind of transformation we apply on the signal that is we can say that time shift itself is a system in the following sense that we have a system which takes xt as the input and produces x t minus t naught as the output which I will call the time shifting system time shifter and I will denote it by say t s time shifter by the amount t naught.

Let this is a system now. So, I could alternately write this expression as x t applied to time shift of t naught use x of t minus t naught. So, I have made a system out of it just like my discussion of linearity I made a system out of it when I said that a linear operation be its scaling be it super position will be treated also has a system put it in a box now what we expect of tine invariance or what we understand from time invariance is the following.

(Refer Slide Time: 34:31)

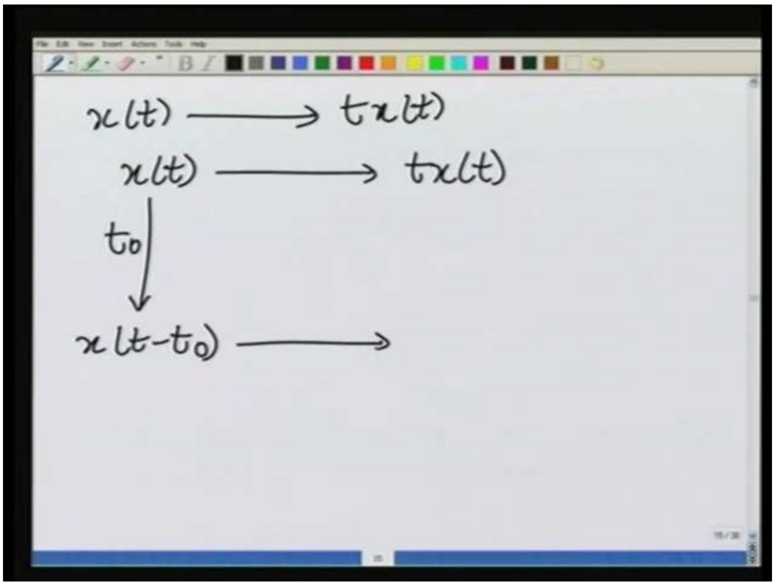


If I have x t and apply it to t s of t naught then applied to the output of this t s of t naught to the system I get an output which I will just call y t for now the alternative thing to do is to exchange these two systems in the order of application to get x t applied to the system first then applied to the output of the system t s of t naught this is the second manner in which we can apply the two systems on x t first the system then t s by t naught earlier we did t s of t naught, and then the system the output we get here must be the same as before y t in both cases must be the same irrespective of which of these two is applied first and that therefore, mix the approach to understanding time invariance where is similar to our approach of understanding linearity we could again. In fact, use the commutation diagram. Now unlike last time where I examined systems first examples of linear and non-linear systems try to find out if linearity was applied or not I will directly start evaluating examples using the commutation diagram; however, this commutation diagram will work with time invariance rather than with linearity. So, let us take examples suppose y t equals k times x t then x t is applied to the system as last time to get k times x t as output that is the first part of the commutation diagram the leg of the commutation diagram.

The second thing is to apply the time shift operated to x t if I did that I will just write this as t naught over here I will get x t minus now on x t minus t naught I will again apply the system what is the system do it just scales the signal by k and. So, it should do the same for x t minus t naught as well I would get k times x t minus t naught on the other hand

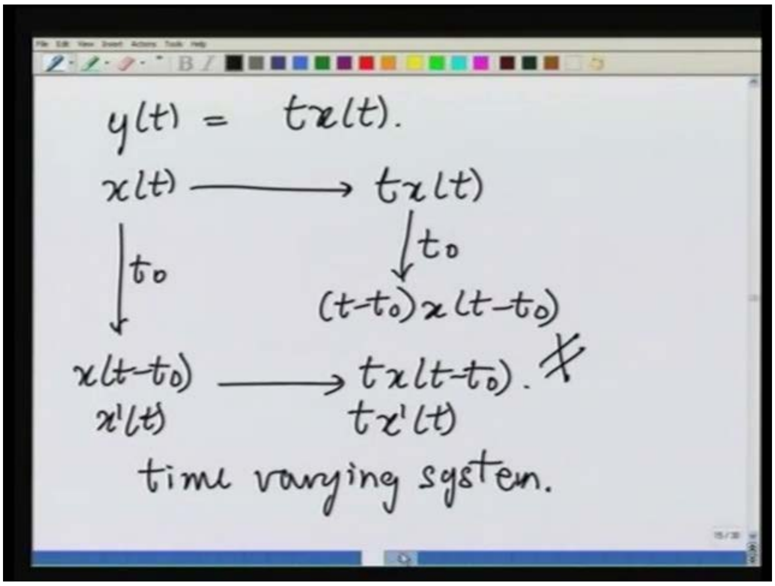
I already have the output of the original signal k x t and if I applied a time shift to k x t what would I get I would get k x t minus t naught once again in this approach I would replace all occurrences of t by t minus t naught that is what I would do whenever I time shift a signal wherever there is t replace it by t minus t naught that is what gives will the output now you see that both these outputs co inside they are the same and this demonstrates that this system is time invariant.

(Refer Slide Time: 38:52)



Let us take a slightly more interesting example suppose we have x t and the system produces t times x t that is the output now is this system invariant, we want to examine what happens when we apply time invariance the test of time invariance to this system we have here x t applied to the system to get t x t apply the t naught shift operator to it you would get x t minus t naught now apply the system to it what would we get.

(Refer Slide Time: 39:57)



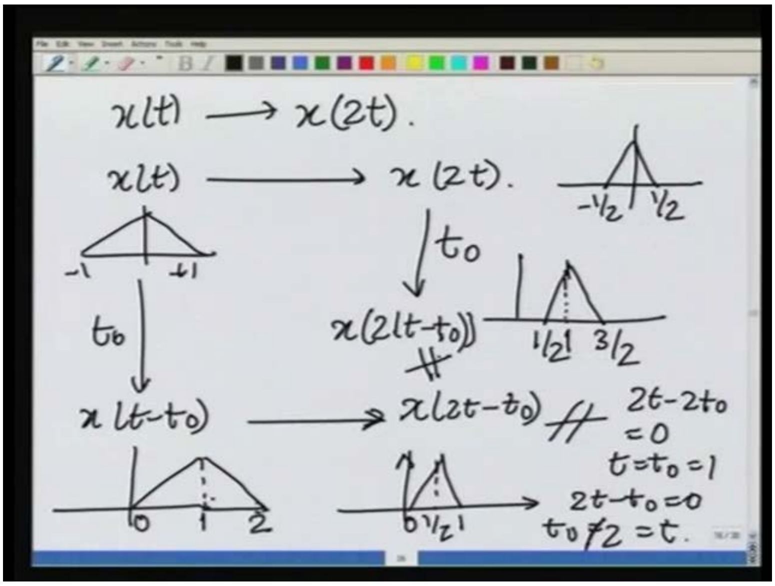
Let us take a second example let us say that y t is equal to t times x t. So, is this system time invariant it will turn out that is is not time invariant, but we can see that by using the commutation diagram we have x t and applying x t to the system gives us t xt according to the system now alternately if we first apply the time shift of t naught we would get x t minus t naught now let us see what happens we have to be a little care full here because it is very easy to go wrong I keep going very often we have here t x t as the output already produced when the input was x t now what happens when you time shift a signal t x t when you time shift a signal by t naught you have to replace every occurrence of t by t minus t naught and. So, if t x t is subject to a time shift of t naught the output would be t minus t naught x t minus t naught. So, that is what we would get if we time shifted the output signal the output of the original signal.

Original signal is x t applied through the system we get t x t time shifted we get t minus t naught x t minus t naught on the other hand if we apply the time shift to x t directly we get x t minus t naught this is just some other signal you could call it x dash t if you liked now if x dash t or x t minus t naught is subjected to the system we have to get t x dash t because that is what the system does the system multiplies the input by t. So, what is t x dash t that is equal to t times x t minus t naught.

So, through one path in a commutation diagram we get t x t minus t naught this happens when you time shift first and apply the system next through the other path you have the system applied first to give t x t and then the time shift applied to t x t to get t minus t naught x t minus t naught the expressions are completely different as you can see and these two systems are not equal to each other. So, this system is n time invariant we would simply say the system is time variant.

Let us take yet another example of a system which gives rise to a different transformation on the signal transformation, that is of the third kind that we discuss when we discussed domain transformations time dilution.

(Refer Slide Time: 43:34)



So, if x t is applied to the signal let the output be x of two t this means the signal has been stunk along time access you can go back the previous lectures and look at the diagrams you will see what is the difference. So, you have x t applied to the input and x 2 t coming out of the output now let us apply the commutation diagram the point is that whenever we apply the commutation diagram to time invariance it is very very common feature to make mistake. So, we have to be very care full x t applied to the system we get x 2 t let us make a diagram also here to help us understand it better suppose this x t say minus one to plus one let say this is x t then x 2 t will be of this form its speeded up. So, its compressing time.

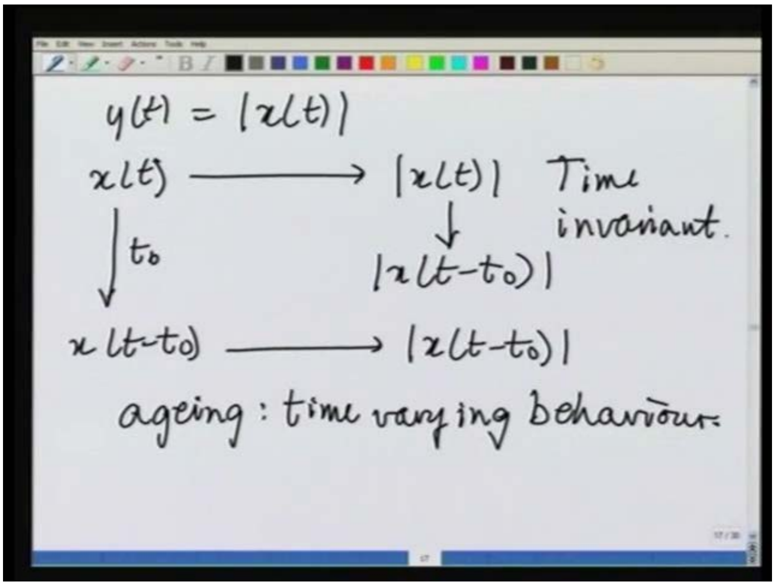
So, you just get from minus half to plus half this is what we have over here we time shift xt by t naught let say t naught is some amount to get x t minus t naught let us have a diagram for this let say t naught is positive let say t naught is equal to one. So, that the output is sketched as and it runs from zero to two with its peak at one this is the time shifted signal now what happens if we time compressed or what we have called is the apply the system to this x t minus t naught x t minus t naught when dilated will give us this signal is easier to not go wrong with diagrams. So, we would get a signal which is compressed in the time access we would get where the signal runs from the zero to one only and the peak is at half. So, this is the signal we would get on the other path we have this signal already given to us x 2 t and we just have to apply a time shift of t naught equal to one to this system into this output signal and when we did that you would just get this starts at minus half and ends at plus half the support starts at minus half and ends at plus half with a time shift of unity it would start at half and end at one point five. So, you would have this signal sorry that say little larger than life from half to three by two now clearly these signals are not equal to each other the signal that arrows from going through this path where we delayed first and applied this system later is disk signal which has a support from zero to on the signal we get by first applying the system and then applying the time delay runs with a support from plus half to plus three by two.

And so there is a difference between these two signals. So, what are the expressions for these two signals that is the next question disk signal now one which we get by applying the time delay to x of two t would be x of two t minus t naught you can verify this by checking for salient points on this graph for example, this original graph of x 2 t we have the peak occurring at zero and here we have the peak occurring at one let us just verify that is the right answer the peak is at zero here and the are carrying delay of two t minus two t naught.

So, the peak should occur at two t minus two t naught equal to zero that where the argument of x t is zero and; that means, that two t minus two t naught equals zero or the peak should occur at t equal to t naught which is equal to one that is what is happening over here we can see that the peak is indeed occurring at one on the other hand if we come through a this branch we have x t minus t naught already available here as time shifted version of the input now here we applied the system after the delay of t naught. So, we will get here is x of two t minus t naught and if this occurs at x of two t minus t naught you can again verify that this diagram is the correct one by looking for the position of the peak.

The peak should occur where the argument of x is zero. So, where two t minus t naught is zero or where t naught by two equals t that is at the point of time t naught being one that the point of time where t equals half one by two and that is what happening over here. So, this has as expression at x of two t minus t naught this has an expression of x of two times t minus t naught these two expressions also clearly or naught equal to each other this system is therefore, time varying has no place to write it, but you can note that down. So, this is another time varying this is a time varying system if we will find that since this is a domain related transformation time invariance is not usually to be faulted whenever the system only involves range operations of the signal.

(Refer Slide Time: 51:36)



For example take y t equals mod x t now taking the absolute value of a signal is a range operation on the signal. So, such a system should naturally be expected to be time invariant and we can verify this very easily using the time commutation diagram once again we have xt and x t when applied to the system gives us mod xt the time delayed version of x t is x of t minus t naught apply the system to it the system finds its absolute value of x t t minus t naught that is it now coming through other path we have mod xt and here applying delay is equivalent to replacing every occurrence t by t minus t naught. So, that you get mod x t minus t naught exactly the same as before. So, this is time invariant. So, these have been a few examples of systems which are time invariant and time varying now what is the fundamental significance of time invariance.

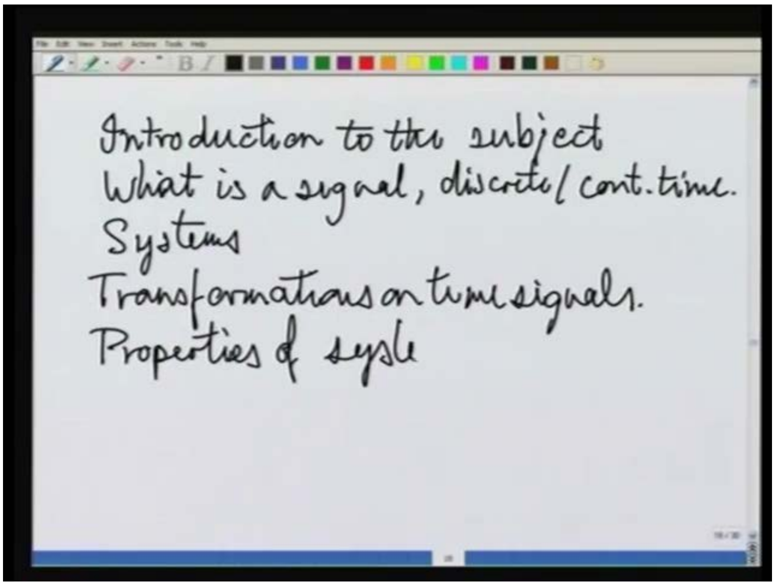
Time invariance as I said is another form of consistency and what kind of consistency is this it is the sort of consistency that physicist expect from physical loss for example, if we slid a cylinder down and inclined plan, if you roll a cylinder down and inclined plan you would get some particular acceleration and velocity for the cylinder at by the time it reaches the bottom of inclined plane this is some sort of a physical experiment now this is physicist would certainly expect that you get the same results for that particular experiment irrespective of which day of the weak, you carried out the experiment in short every physical experiment is carried out in the real world which is largely believed to be time invariant with respect to the physical loss that we understand the world in terms of and therefore, physical loss are said to apply not only at every place in the world, but also at every time that they are valid for all time now this is deeper question thus many professional physicists will tell you, but this is the laymen’s understanding of a physical law that the law should be verified irrespective of when you carry out the experiment to verify the law the result should be same.

If you carry out an experiment today you get a certain set of results today if you carry out that same experiment tomorrow you should get that same set of results not today, but tomorrow. So, whenever you carry out the experiment you get the corresponding results at the time that is fallows the experiment itself that is time invariance is the most straight forward thing most natural thing and something that you would expect from almost any physical system; however, in real life thing do change with time there are gradual changes which have to be accounted for when we consider a system at the engineering level for example, semiconductor devices gradually deteriorate with age.

So, if you have say an audio amplifier made out of solid state devices today its performance will not be the same if you used it daily for the next fifty years and try and tested its performance at the end of fifty years it would certainly be expected to be slightly inferior this happens because three semiconductor devices their behavior rather is not really tine invariant if thus change with time very, very slowly perhaps very, very gradually, but it does change with time a capacitor for example, most capacitors are known to have gradual deterioration in the dielectrics. So, as time passes we dielectric constant of the capacitor comes down. So, for same voltage the same capacitor will store less and less charges it gets older. So, after long time you probably fine that the circuit any electronic circuit be used small function because it is components age. So, this aging is one manifestation of time varying behavior thus two with only human beings as well if I gave the same lec forty years hence a probably not be able to even standup.

So, every ages with time and that sense if we look very closely perhaps now everything in this world is time invariant we do expect physical last to be time invariant because we aren't expecting the world to age the universe to age in the same sense that human beings age are electronic devices age would may be even the world is aging we do not know. So, this completes the list of five properties of systems with respects to which we try to understand our systems and in some sense this also completes one module the first module of this course where we have gone through the following steps we have had an introduction to this subject.

(Refer Slide Time: 57:29)



We have had discussion of what is a signal discrete continuous time examples etcetera systems introduction then of course, we went through lots of discussion on different kinds of signals we discussed exponential signals in detail unit step unit ramp etcetera in detail then we talked about various transformations on time signals and finally, we have spoken of properties of systems this discussion being preliminary was also extremely general we discussed all kinds of systems we discussed all kinds of signals because our objective was to make the study as general as possible from this very, very general setting we will now progress towards more constraints situation where not all kinds of systems are discussed, but only systems that poses certain very desirable properties.