Course Name- Nanophotonics, Plasmonics and Metamaterials

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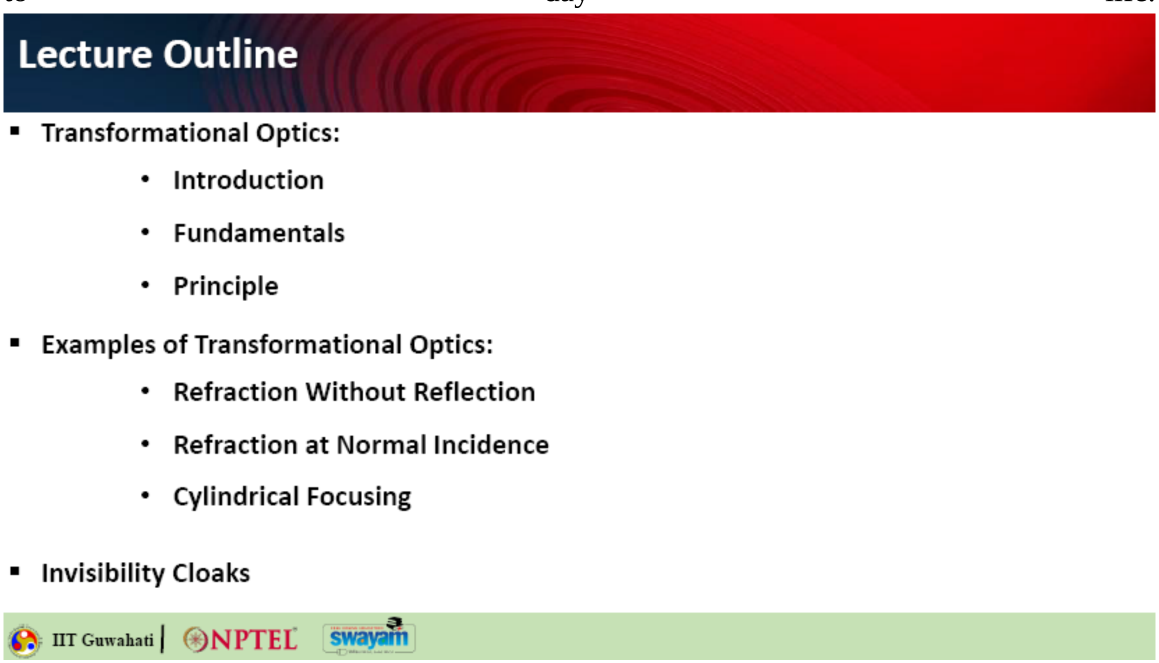
Week-11

Lecture -31

Hello students, welcome to lecture 31 of the online course on Nanophotonics, Plasmonics and Metamaterials. Today's lecture will be on Transformation Optics and Invisibility Clocks. So here is the lecture outline. So Transformation Optics, we will discuss about the basis of it, the fundamentals and what are the principles and mathematical forms in this particular field of science. And then we will take up some example of Transformational Optics and we will see how this helps us achieve some features which are not naturally seen. Something like you can actually make materials that give you refraction without reflection and then you can also think of refraction at normal incidence which is something not naturally occurring in the materials that we deal in day to day life.



You can also think about cylindrical focusing from a planar interface. So we will look into all of this and then finally we will discuss about Invisibility Clocks. So let's introduce Transformation Optics to you. So the first thing that should come to your mind is that this is coming from some sort of coordinate transformation.

So the requirement for Transformation Optics comes from the fact that you want to band light in a particular way depending on your will. So you may argue that there are graded index optics which allows you to do the same thing. So yes, graded index optics can be considered that determines how light can propagate in a medium with particular dielectric and magnetic properties. When you take it further you will be able to make light to do anything. So we will look into those examples today.

So let us first start with graded index material and see what happens there. In graded index materials or GRIN graded index green, they allow optical rays to follow some curved trajectories which are basically governed by the profile of the refractive index ny and that is considered to be constant along x and z but it only varies along y. So you can think of something like this. So here the refractive index basically varies only along the y direction. So, in this case there is a gradient along the y direction and as you can see a ray can get band from this interface at a different angle.

Transformational Optics: Introduction

- Graded-index optics can be considered to *determine how light propagates in a medium endowed with particular dielectric and magnetic properties.*
- Graded-index (GRIN) materials allow optical rays to follow curved trajectories governed by the profile of the refractive index $n(y)$ *i.e. considered constant along* x *&* z *but varies along* y.

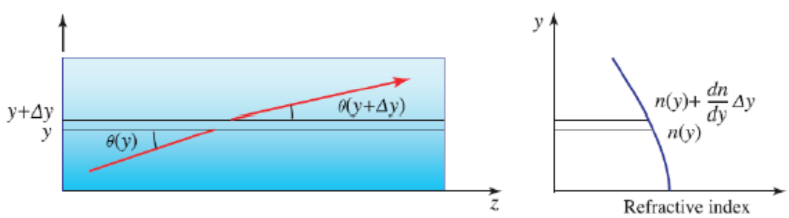


Figure: Refraction in a graded-index slab.

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019

So the incoming angle was θ which was function of the position y but then it changes. So this happens because this banding basically happens because the refractive index along y is also changing. So you can say that at a height of y it was ny and at a different height, here the refractive index changes to ny plus the slope of this refractive index profile. So that is dn over dy and then how much you have changed along y that is delta y. So, this gives you the new refractive index at this particular point.

Transformational Optics: Introduction

- For isotropic materials with graded electric permittivity $\varepsilon(r)$ and magnetic permeability $\mu(r)$, Maxwell's equations give rise to the generalized Helmholtz equations which were solved for layered and periodic structures such as photonic crystals.
- The synthesis problem of these graded-index materials is more challenging than the analysis problem in two respects:
  - Requires more advanced mathematical tools
  - Physical implementation of the graded medium often requires the use of metamaterials constructed from components that are available, configured in a particular spatial arrangement, and amenable to being fabricated with current technology.

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So this graded index materials allow you to direct light and you can actually make light to take any band path. Now for isotropic materials with graded electric permittivity εr and magnetic permeability ε_r, Maxwell's equation will give rise to generalized Helmholtz equations which were solved for layered and periodic structures such as photonic crystals. We have seen this already before. Now the synthesis problem of this graded index materials is more challenging than the analysis problem in two respects. First they will require much advanced mathematical tools as you can understand the refractive index continuously varies along one direction or the other.

Transformational Optics: Fundamentals

- **Transformation optics** is a mathematical tool that facilitates the design of optical materials that guide light along desired trajectories.
- The underlying concept relies on a geometrical transformation that converts simple trajectories into desired ones.
- In order that Maxwell's equations remain valid, the optical parameters associated with the transformed equivalent system must also be modified, and this establishes the character of the required optical material.
- As a simple example of such equivalence, **a local compression of the coordinate system by a scaling factor is equivalent to a local increase of the refractive index by the same factor**, so that the **optical path length** (product of length and refractive index) **remains unchanged**.

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

And also the physical implementation of this kind of graded medium will require the use of metamaterials constructed from the components that are available or which are basically configured in a particular spatial arrangement and they should be supported by the current fabrication technology. So these are the kind of requirements or you can say limitations getting imposed onto the graded index materials because of the current technology status. So both mathematical you require advanced mathematical tools so the analytical study is difficult as well as the fabrication of graded index materials will be challenging as well. Now to overcome these limitations you can think of transformational optics. Now transformational optics acts as a mathematical tool that facilitates the design of optical materials which can guide light along any desired trajectory.

So you are able to achieve the similar functionality of the graded index optics but here also you can do a lot of new stuff, okay, lot of new cool stuff I should say. The underlying concept here lies on a geometrical transformation that converts simple trajectories into desired ones, okay. So in order that Maxwell's equation will remain valid and the optical parameters associated with the transformed equivalent system must also be modified and this establishes the character of the required optical material. So you require new kind of material to do this trick for you, okay. So as a simple example of such equivalence you can think of a local compression of the coordinate system by a scaling factor which is equivalent to the local increase in the refractive index by the same factor, okay.

So it is like if you compress one coordinate system by a scaling factor, okay, in one case you can and another case you increase the local refractive index by the same factor, what will happen ? The optical path length in both the cases will be similar, okay. And how optical path length is defined? It is the physical length multiplied by the refractive index, okay. So in one case the physical length is coming down, refractive index is remaining same. In other case the physical length remains same but the refractive index is scaling up. So, their product remains same, okay.

Transformational Optics: Fundamentals

- A three-step design procedure provides a guide:
  - Begin with a *pilot* physical system for which the optical trajectories are known, such as a homogeneous and isotropic material.
  - Find a coordinate transformation that converts these trajectories to the desired ones.
  - Determine the transformed physical parameters of the equivalent material.
  - The new material will implement the desired optical trajectories in the original coordinate system.
- Since geometrical transformations generally involve changes of directions and introduce direction-dependent scaling, the transformed parameters are generally both anisotropic and spatially varying.

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So that is the optical path length. So a local compression of the coordinate system, so if you are compressing any material along z direction by a factor of a that can be also assumed to have similar effect as if you have the same dimensions along z but the material now have a refractive index NA, N was the previous refractive index now it will have NA refractive index. Now, Transformational Optics, let us see how you do the design. So there is a three-step design procedure which will provide the guidance. So you first begin with a pilot physical system for which the optical trajectories are known such as a homogeneous and isotropic material and then you find a coordinate transformation that convert these trajectories to the desired ones, okay.

And then you determine the transformed physical parameters of the equivalent material. So this is the third step and with this new material you will be able to obtain the desired optical trajectories in the original coordinate system. So this is what you are looking for, okay. So you want the features to work in the original coordinate system. So using this kind of transformational optics you transfer the desired configuration you can say onto the material properties, okay.

And then you put that new material into the previous coordinate system. We will take examples, it will become more clear in the subsequent slides. So since geometrical transformations generally involve changes of directions and introduce direction dependent scaling, the transform parameters are generally both anisotropic and spatially varying, okay. So this is how you will get the transform parameters. So, let us consider, ε_{ij} as a tensor, this is the permittivity tensor and μ_{ij} as a permeability tensor of the original material in the original coordinate system.

Transformation Principle

- Let $\{\varepsilon_{ij}\}$ and $\{\mu_{ij}\}$ be the elements of the permittivity and permeability tensors of the original material in the original coordinate system (x_1, x_2, x_3).
- The elements of the permittivity and permeability tensors of the equivalent material (denoted by the superscript "e") in the transformed coordinate system (u_1, u_2, u_3) are then related to the original elements by the matrix equations:

$$\varepsilon^e = \frac{A^T \varepsilon A}{|\det A|} \quad (L31.1)$$

$$\mu^e = \frac{A^T \mu A}{|\det A|} \quad (L31.2)$$

- Here ***A*** is the 3 × 3 Jacobian transformation matrix, whose elements are the partial derivatives:

$$A_{ij} = \frac{\partial u_i}{\partial x_j} \qquad i, j = 1,2\,3\,\ldots \qquad (L31.3)$$

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So original coordinate system has the three axis x_1, x_2 and x_3, okay. Now the elements of the permittivity and permeability tensors of the equivalent material, okay. So that can have a superscript of e that is the equivalent material that will bring the desired functionality, okay. And this will be in the transform coordinate system u_1, u_2 and u_3, okay. And then related to the original elements of this one by these equations.

Transformation Principle

- For Jacobian transformation matrix:

$$A_{ij} = \frac{\partial u_i}{\partial x_j} \qquad i, j = 1,2\,3\,\ldots \qquad (L31.3)$$

- The quantity $\mathbf{A}^T$ is the transpose of **A**, and **ε** and **μ** are 3 × 3 matrices whose elements are $\{\varepsilon_{ij}\}$ and $\{\mu_{ij}\}$, respectively.
- Since **A** is generally dependent on (x_1, x_2, x_3), the equivalent material is generally inhomogeneous, even if the original material is homogeneous.

IIT Guwahati | NPTEL swayam — Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So you can obtain what is the equivalent materials, permittivity and permeability tensor. So you can see here that you require a matrix A and this matrix A is a 3 by 3 Jacobian transformation matrix, okay. So and what are the elements of this matrix? So, the matrix elements A_{ij} is nothing but doh u_i over doh x_j where i and j are basically 1, 2, 3. So the matrix elements are basically partial derivatives, u is basically the transformed coordinate system and x_j is basically representing the original coordinate system. So once this, this matrix A which is a Jacobian matrix, you can find out what is the equivalent materials permittivity that is A transpose, then you have this epsilon A divided by the determinant of this matrix A.

Transformation Principle

- In the **special case** for which the original material is both homogeneous and isotropic, say free space, then $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are diagonal with equal diagonal elements ε_0 and μ_0, respectively, whereupon:

$$\varepsilon_0^{-1}\varepsilon^e = \mu_0^{-1}\mu^e = |\det \mathrm{A}|^{-1}\mathrm{A}^{\mathrm{T}}\mathrm{A} \qquad \text{(L31.4)}$$

- The tensors ε^{e} and μ^{e} are then identical except for a scaling factor.
- Under these conditions the impedance, which depends on their ratio, remains unchanged for all polarizations, which in turn implies that the equivalent medium introduces no reflection at any boundary with free space.
- Let's see a number of examples to understand the transformation principle.

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

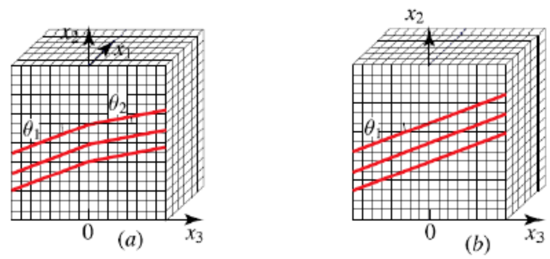
Similar equation is used also for permeability, okay. Now this you have seen the Jacobian transformation matrix, these are the elements. Now when you have this A transpose, okay, this actually is required to calculate this equivalent material properties and epsilon and mu they are also 3 by 3 matrices whose elements are this one, okay. So, you can represent the elements as ε_{ij} and μ_{ij}, fine. Now since the Jacobian matrix A is generally dependent on x_1, x_2 and x_3, the equivalent material that you will see will be generally inhomogeneous even if the original material that you started with was homogeneous, okay.

Now in a special case when the original material is both homogeneous and isotropic something like free space, okay, in that case you will see that epsilon and mu, these tensors are basically or these matrices are basically diagonal and they have equal diagonal elements mu naught and epsilon naught, okay. So in those case you can simplify the equation and you can write epsilon naught inverse epsilon e that will be equal to mu naught inverse mu e equals this, okay. This is the determinant of A inverse A transpose A, okay. That is basically coming from those previous equations, there is nothing new here. Just that here things got simplified because you have only the diagonal elements and another important thing is here mu and epsilon are similar, okay.

Example 1: Refraction Without Reflection

- An optical material implementing ray trajectories that refract without reflection at a planar surface (Fig. (a)).
  - Begin with an initial homogeneous medium, say free space, with rays that follow parallel straight trajectories at an angle θ_1 (Fig. (b)).

Figure: Geometrical transformation implementing refraction without reflection.



Desired optical trajectories

Free space with straight-line optical trajectories

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So the tensors ε_e and μ_e are then identical, okay, except for a scaling factor, okay. So only the scaling factors are different, other than them they are similar. Now under these conditions the impedance, okay, so what happens to the impedance? The impedance depends on the ratio of the permittivity and permeability, okay and they will remain unchanged, okay because both of them got a factor, okay and so the ratio will remain same and that actually helps you because when the ratio remains unchanged for all the polarization you can consider this equivalent medium will have no reflection at any boundary with the free space. So it becomes a reflection free boundary, okay. So let us see a number of examples now to understand the transformation principle.

So this is the first example that we will be discussing today. Let us try to create refraction without reflection. So usually if you have seen that refraction and reflection both come together but you are able to make some material here which can actually give you only refraction and no reflection. So the optical material shown here, okay, shows the ray trajectory where it only refracts. So these are the rays incoming angle is theta 1 and the refracted angle is theta 2, okay.

So this is the geometrical transformation that implements the refraction without reflection kind of feature. Now how do you begin designing this kind of a system, okay? So let us first start with an initial homogeneous medium, say free space, okay where the lines will follow the straight parallel trajectories at angle theta 1, something like this, okay. So this we need to design the material for this one that does this kind of bending, right? So this is our desired optical trajectory. Now we are starting with by assuming that this part is basically free space. So if it is free space what will happen? There will be straight line optical trajectories, right? There will be no bending or anything.

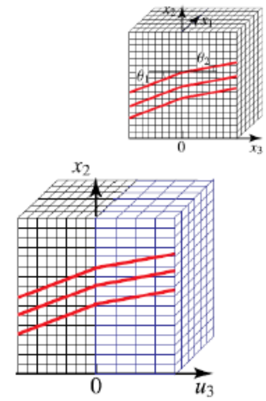
Refraction Without Reflection

- Apply a geometrical transformation that stretches the coordinate system by a scale factor s along the x_3 direction in the region $x_3 > 0$.
- Stretching the coordinate system by the factor s for $x_3 > 0$ causes the rays to change slope and follow the desired trajectories.
- The desired refraction is achieved by choosing s as the ratio of the initial and desired slopes:

$$s = \tan\theta_1 \,/\, \tan\theta_2 \qquad (L31.5)$$

- This transformation is implemented by the relations:

$$u_1 = x_1, \qquad u_2 = x_2, \qquad u_3 = s^{-1}x_3$$



Stretching the coordinate system by the factor s

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019

Now it is our time to apply the geometric transformation. Now if you apply geometric transformation in the form of stretching the coordinate system by a scale factor of s along x_3, x_3 direction for the part where x_3 is greater than 0. So only this positive half, so this part is x_3 less than 0. We will do nothing to it, okay because this is the part that we want. We only want things to change here on the right side, okay of this particular line.

So we want to stretch this portion which is where x_3 is greater than 0. And when you do the stretching you actually get something like this. So you see you just stretched the entire coordinates by a factor of s. So all these lines they got stretched, okay. So, stretching the coordinate system by a factor of s for x_3 greater than 0, okay this one will cause the rays to change their slope.

And now they will be able to follow the desired trajectory, understood? So we started with free space where the lines were simply going straight. Then we stretched this axis so that you acquire the slope that you want to be in your desired optical response, okay. So the desired refraction is achieved by choosing s as a ratio of the initial and the desired slope. So, you can choose $s = \tan\theta_1/\tan\theta_2$. This is how you can find out the stretching factor.

And once you have that you can find out the relationship between the transformation. So this is where the transformation is coming. So this is the transformed axis. So, as you can see in other two, so this is the plane is basically the x_1. So x_1 axis is coming out of, so sorry the plane is x_1 equals 0, right.

Refraction Without Reflection

$$u_1 = x_1, \quad u_2 = x_2, \quad u_3 = s^{-1}x_3$$

- This type of scaling of the Cartesian coordinate system, in which the directions of the axes do not change, converts a cube into a cuboid.

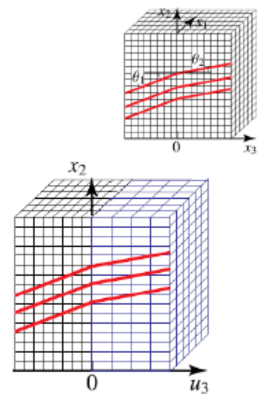
- Based on (L31.3):

$$A_{ij} = \frac{\partial u_i}{\partial x_j} \qquad i,j = 1,2\,3\ldots \qquad \text{(L31.3)}$$

$$A = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \frac{\partial x_1}{\partial x_3} \\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \frac{\partial x_2}{\partial x_3} \\ \frac{\partial s^{-1}x_3}{\partial x_1} & \frac{\partial s^{-1}x_3}{\partial x_2} & \frac{\partial s^{-1}x_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^{-1} \end{bmatrix}$$

the Jacobian matrix $\boldsymbol{A}$ is diagonal with diagonal elements $(1, 1, s^{-1})$ and determinant:

$$\det \boldsymbol{A} = s^{-1}$$



Stretching the coordinate system by the factor *s*

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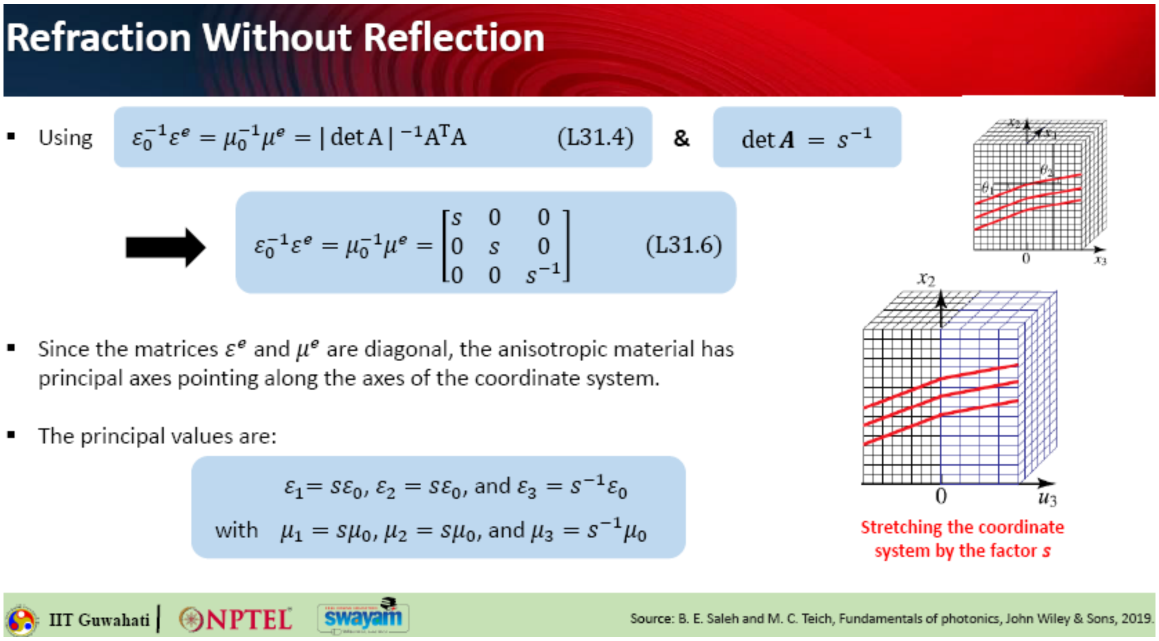
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So x_1 is coming out of this screen. So u_1, the new coordinate system u_1 is same as x_1. There is no change in this one, this direction. Similarly, you are not stretching along x_2 so that also remains same. So y_2 equals x_2. Only thing has changed is u, this is sorry u_2 equals x_2 and only thing that is changing is u_3.

So $u_3 = s^{-1}x_3$. So that is how you are stretching it, okay. So this type of scaling of the coordinate, Cartesian coordinate system in which the direction of the axis does not change. So what they will do, they will basically convert a cube into a cuboid, right. So now you have known the transformation. Let us find out what is the Jacobian matrix for this.

So Jacobian matrix a_{ij} can be calculated as doh ui over doh x_j. So that way you can fill up all these partial derivatives and using this relationships shown here you can calculate all this and they boil down to a very simple equation, simple matrix that is this one, okay. So, 1 0 0 0 1 0 0 0 s^{-1}. So this is the Jacobian matrix. So here you can see the Jacobian matrix is basically a diagonal one where the diagonal elements are 1 1 s^{-1}, okay.



And you can also calculate the determinant of this matrix A that will come out to be s inverse. Now once you know all these parameters you can easily find out that what will be the effective permeability of the new medium, okay. So you can go back to this particular equation, okay and you can now see what happens because this all factors are known, okay. You can do the computation and you will see that $\varepsilon_0^{-1}\varepsilon^e = \mu_0^{-1}\mu^e = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s^{-1} \end{bmatrix}$.

So the elements are s s and 1/s, okay. Now since the matrices that permittivity and permeability of the equivalent material they are diagonal, okay, the anisotropic material has principal axis pointing along the axis of the coordinate systems, okay. That is clear. And the values in this case are $\varepsilon_1 = s\varepsilon_0, \varepsilon_2 = s\varepsilon_0$, and $\varepsilon_3 = s^{-1}\varepsilon_0$ with $\mu_1 = s\mu_0, \mu_2 = s\mu_0$, and $\mu_3 = s^{-1}\mu_0$

So this is basically the requirement of the new material which has to be replaced on this right side, okay. So we understood that the parameters of the equivalent material may be obtained by batching this, okay. So what are the requirements we are fulfilling here? The phase shift that is encountered when a plane wave crosses the stretched free space segment with the phase shift encountered when the wave is transmitted through a unstretched segment filled with this new material. And this new material will have all this property, okay. So the whole idea is the phase shift that will be there for the light should be equivalent in both the cases.

Refraction Without Reflection

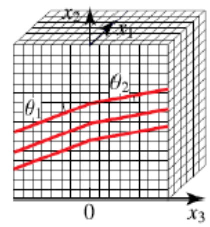
- The parameters of the equivalent material may also be obtained by matching :
  - the phase shift encountered when a plane wave crosses the stretched free-space segment with
  - the phase shift encountered when the wave is transmitted through an unstretched segment filled with the new material.

- To determine the parameters, we consider three waves in turn, each with the electric field along one of the coordinates:
  - Wave 1 is a plane wave traveling along the x_3 direction with electric and magnetic fields in the x_1 and x_2 directions, respectively.
    - The appropriate permittivity and permeability are thus ε_1 and μ_2 so that:

      $k = \omega\sqrt{\varepsilon_1\mu_2} = \omega\sqrt{s\varepsilon_0 s\mu_0} = sk_0$, corresponding to a refractive index $n_1 = s$

      The phase shift accumulated over the distance d is therefore sk_0d

      The impedance $\eta_1 = \sqrt{\mu_2/\varepsilon_1} = \eta_0$



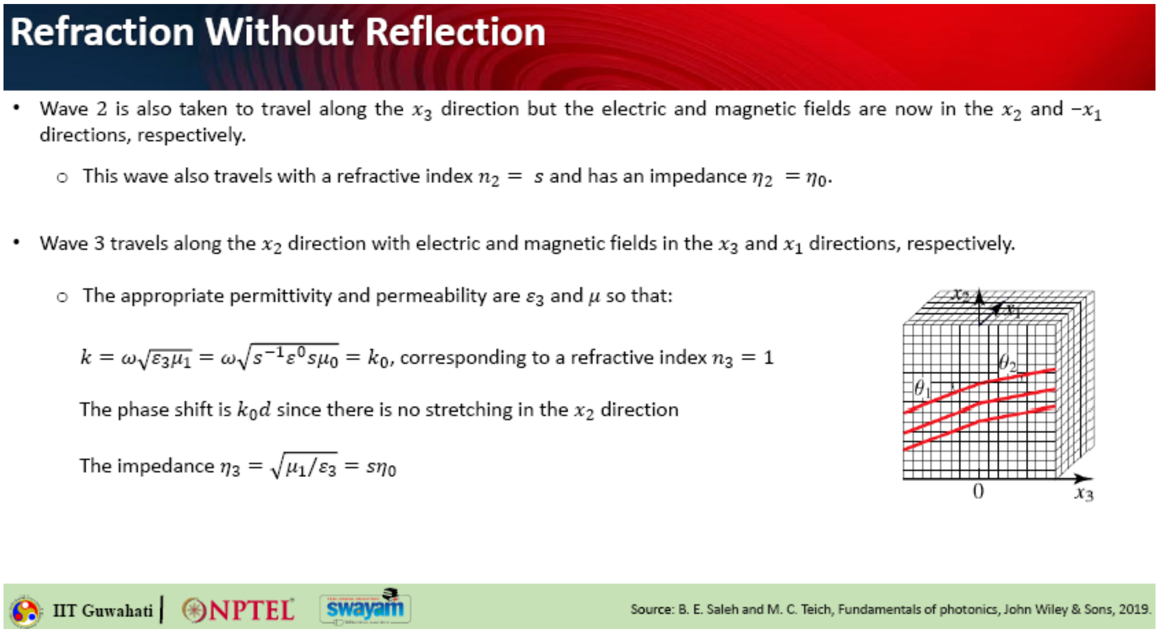
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

And then only you will be able to, fill the unstretched segment with the new material and that is how you will be able to bend the light without any reflection, So to determine the parameters we considered three waves in turn, each with the electric field along the, along one of the coordinates, okay. So let us consider one wave that is wave 1, okay. So, wave 1 is a plane wave traveling along the x_3 direction, okay. So, this is x_3 direction with electric and magnetic fields along x_1 and x_2 directions respectively. So, in that case the appropriate permittivity and permeability are basically ε_1 and μ_2.

And what is the relationship? So $k = \omega\sqrt{\varepsilon_1\mu_2} = \omega\sqrt{s\varepsilon_0 s\mu_0} = sk_0$. So, you can take out s and this thing actually gives you sk_0, okay. So, the wave vector is basically s times the k_0 in this new material, okay. Now this actually implies that the refractive index is basically s in n_1 direction. In the original material, stretched material, the refractive index was 1.

So here it is s. So, the phase shift that will be accumulated over the distance d can then be written as sk_0 into d. And you can find out what is the impedance. Impedance will be $\eta_1 = \sqrt{\mu_2/\varepsilon_1} = \eta_0$. Now this is what was mentioned before that there will be a scaling factor, okay. So if you see that the scaling factor appears both places, so they will basically cancel out and you will get the impedance to be same as the free space.

So in that case what happens? There is no reflection. So that is for one particular wave. Now if you consider another wave, wave 2 that is also traveling along x_3 direction, but the electric and magnetic field directions are now changed. So now electric field is considered to be along x_2 direction, okay, and magnetic field is along minus x_1 direction. So, this is x_1, so you can understand which one is minus x_1, okay. So, in this case the wave again travels with a refractive index n_2 = s and you can find out the impedance eta 2 will be same as eta 0.



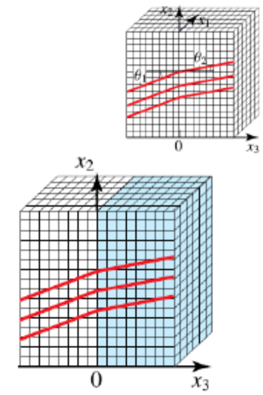
It is the same calculation as the previous one, so we are not repeating it. Then you can also think of the third wave, so this is wave 3. Now wave 3 travels along this direction, x_2 direction with the electric field along x_3 and magnetic field along x_1 direction. So in that case what happens? You can see that the approximate permittivity and permeability are basically ε_3 and mu, okay. So, it should be actually mu 1 because it is along x_1 direction, so consider this as mu 1 and you can then calculate what is the k.

So k will be omega square root of ε_3 mu 1 and ε_3 can be written as s inverse epsilon 0 and mu 1 can be written as s mu 0. So s inverse and s they cancel out each other you get the same wave vector, okay. In that case the refractive index along n_3 direction is also 1, okay. So we have figured out all the cases, okay. So, you can calculate the phase shift along this direction that will be simply k0d and this is similar in the unstretched case because there is actually no stretching along the x_2 direction.

So let us calculate what is the impedance here. The impedance will be eta 3 equals square root of mu 1 over ε_3 and that will come out to be s eta 0, okay. So you can see that there is a change in impedance in this particular direction. Now using these results you can conclude the design that the final design is basically a piecewise homogeneous. So this material is one material, this is another material that is why they are shown in different colors, okay. And the left half of the free space is basically, sorry left half is basically free space and right half is basically an anisotropic material and this is also if you see this is basically a_2 axis are same and third one is different so you say uniaxial material, okay.

Refraction Without Reflection

- Using these results, we conclude that the final design is a piecewise homogeneous medium with free space in the left half plane and an anisotropic uniaxial material in the right half plane (Fig).
- The anisotropic material is birefringent with $n_1 = s, n_2 = s$, and $n_3 = 1$, but it introduces no reflection at the boundary with free space since the impedances are the same as that of free space: $\eta_1 = \eta_2 = \eta_0$.
- Two factors distinguish refraction at the boundary of the synthesized anisotropic medium from conventional refraction at the boundary of a homogeneous and isotropic medium:
  - Refraction is not accompanied by reflection
  - The relationship between the angle of refraction and the angle of incidence, $s \tan\theta_2 = \tan\theta_1$, differs from Snell's law.



Equivalent anisotropic, homogeneous material that causes the rays to change slope in an identical way

IIT Guwahati | NPTEL swayam

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

And this anisotropic material as I mentioned it is birefringent because along two direction it has got the same refractive index but along the third direction it has got a different refractive index, okay. Now the boundary in the boundary along the free space, okay the interesting thing what is the boundary along free space that is along x1 and x2 these are the boundary along the free space. So you see the impedance in those direction eta 1 and eta 2 is same as eta 0 it means there is no impedance mismatch with the boundary with the free space. So there will be no reflection so you can only have transmission, okay. So, the two factors that distinguish refraction at the boundary of this synthesized anisotropic medium from the conventional refraction at a boundary of a homogeneous anisotropic medium is that here the refraction that you are getting does not come with a reflection, okay.

And second thing is that the relationship between the angle of incidence that is theta 1 and the angle of refraction theta 2 is different here. Here it is s tan theta 2 equals tan theta 1. So this is different than Snell's law. So you can actually define your own law if you have your own material, okay. So this is how you can get your desired refraction without any reflection if you are able to realize this anisotropic uniaxial material, okay.

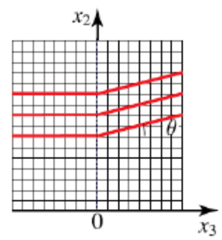
So now let us move on to the next example that is refraction at normal incidence. Now normally there is no refraction at the normal incidence, okay. So here we are trying to design that as well. So, consider the design of an optical material that implements refraction at an angle theta at normal planar surface.

Example 2: Refraction at Normal Incidence

- Consider the design of an optical material that implements refraction by an angle θ at a normal planar surface.

- This type of refraction cannot occur at the boundary between two isotropic dielectric materials, but can occur at the boundary between an isotropic and an anisotropic material.

Figure: Geometrical transformation implementing refraction at normal incidence.



Desired optical trajectories

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So this is what is the desired optical trajectory. So this is the interface, okay and light is falling normally but then still you want a angle theta. So this type of refraction cannot occur at the boundary between two isotropic dielectric material but they can occur at the boundary between one isotropic and another anisotropic material. So this possibility is there, okay. So we will start in the same process with a pilot system of free space where the rate rejected is along the horizontal parallel straight lines something like this, okay and then we will try to implement the coordinate transformation. So, first thing is so this is where the right side is basically a free space and that is where you are able to see straight line, okay.

Refraction at Normal Incidence

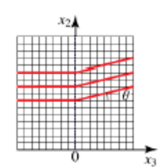
- We begin with a pilot system of free space with ray trajectories along horizontal parallel straight lines (Fig (a)), and implement the coordinate transformation:

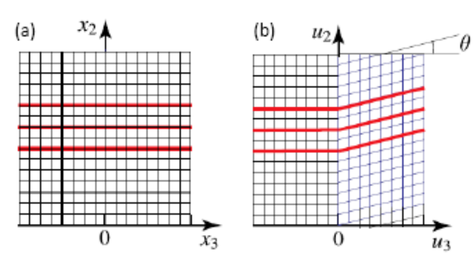
$$u_1 = x_1, \quad u_2 = x_2 + s x_3, \quad u_3 = x_3$$

for $x_3 > 0$, with $s = \tan\theta$.

- This deflects the trajectories, as desired, by shearing along the x_2 direction.

- Shearing the coordinate system along the x_2 direction for $x_3 > 0$ refracts the trajectories as desired (Fig (b)).





Free space with straight-line optical trajectories

Shearing along the x_2 direction

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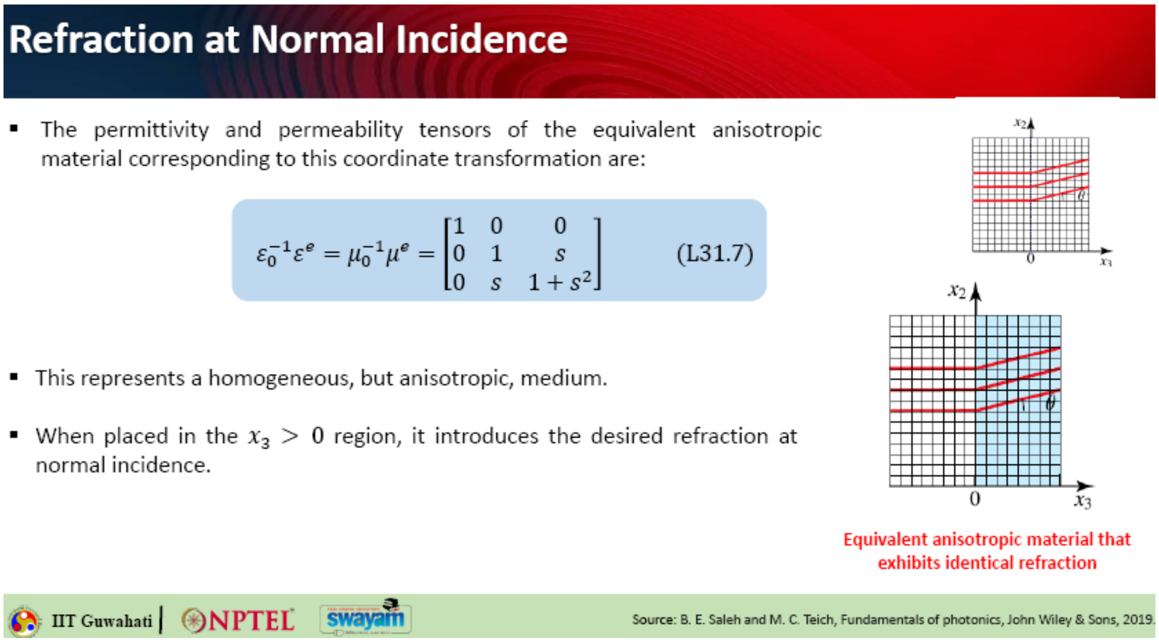
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

These are parallel straight lines but then what you want? You want this to bend at an angle of theta. So, you can actually think of this kind of transformation that u1 the new transformed coordinate $u_1 = x_1$ but x_2 there has to be some change, okay. So, the new coordinate system $u_2 = x_2 + sx_3$ and $u_3 = x_3$. So, you can use this kind of a transformation and you are only applying this transformation for the positive x_3 and what is s? s is basically tanθ.



So, this is the angle theta at which you want this refracted rays to go. So you can use this over here. So once you know that this is this will be obtained when you apply this particular transformation, okay. So, this transformation will deflect the trajectories as desired and this is done by shearing along the x_2 direction. So, you are actually changing the coordinates along the x_2 direction. So, shearing the coordinate system along x_2 direction for the positive half will reflect the trajectories as shown here, okay.

So this is what you want. So it will become like this because you are shearing it. So along x_2 things will change and that change is basically bringing this angle theta, okay. So how do you do now? So now we have to find a material that does it for us, okay. So, we have to again go back to find out the permittivity and permeability tensor of the equivalent anisotropic material and you can use the same formula that is $\varepsilon_0^{-1}\varepsilon^e = \mu_0^{-1}\mu^e$ should be equal to that determinant of A inverse then A transpose and you have A, okay.

So that is the equation. Once you do it you will get only the diagonal elements they are 1, 1, okay. Now here you get also some off-diagonal elements, okay. So, you get 1, 0, 0, 0, 1, s, 0, s, $1 + s^2$. So this represents a homogeneous but anisotropic medium, okay. So, once you are able to replace this new material which is in blue color, okay with this which will have this kind of property then you can go back to the previous coordinate system, okay and you will get the same effect.

Example 3: Cylindrical Focusing

- Parallel straight-line trajectories are to be refracted at a planar boundary such that they all meet at a common focal point at a distance *f* from the boundary (Fig (a)).

- Begin with straight trajectories (Fig (b)) in a Cartesian coordinate system
  and
  apply the coordinate transformation:

$$u_1 = x_1,\ \ u_2 = (f - x_3)\sin(x_2/f),\ u_3 = f - (f - x_3)\cos(x_2/f)$$

for $x_3 > 0$

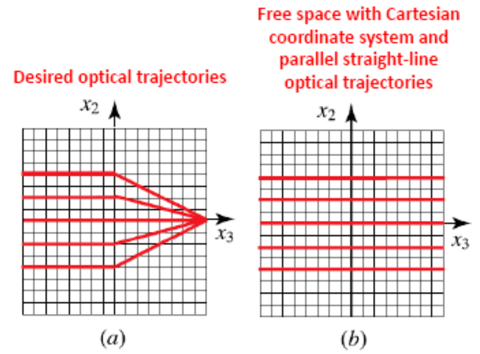


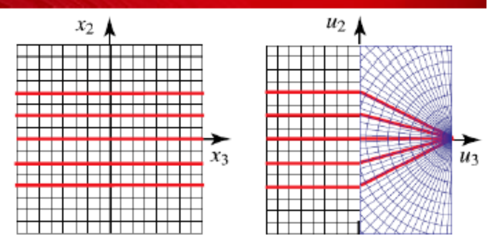
Figure: Geometrical transformation implementing cylindrical focusing.

The third example shows cylindrical focusing. So again here you have a planar interface but this is the desired optical trajectory. So you want to achieve a cylindrical focusing functionality. So what you have? You have parallel straight line trajectories on the left side that needs to be refracted from a planar boundary so that they meet at a focal point say capital F at a distance of small f from the boundary, okay. So first what you will do? You consider this as a free space, okay. So, you start, so when it is a free space there will be all parallel straight line optical trajectory and then you have to apply the coordinate transformation here so that this kind of feature is obtained.

Cylindrical Focusing

$$u_1 = x_1,\ \ u_2 = (f - x_3)\sin(x_2/f),\ u_3 = f - (f - x_3)\cos(x_2/f)$$

for $x_3 > 0$



Conversion to a cylindrical coordinate system centered at $x_3 = f$ for $x_3 > 0$ produces the desired trajectories

- The result is a cylindrical coordinate system centered at $(u_2 = 0, u_3 = f)$.

- This transformation converts
  a line $x_2 = a$ in the plane $x_1 = 0$ in the original coordinate system
  into
  a line $u_2 = (f - u_3)\tan(a/f)$ in the new coordinate system.

- It also converts a line $x_3 = b$ in the plane $x_1 = 0$ **into** a circle $u_2^2 + (f - u_3)^2 = (f - b)^2$ of radius $(f - b)$ centered at the point $(u_2, u_3) = (0, f)$.

So in this case we will be using this particular feature that will focus at a point. So, again there is no changes along x_1 direction so $u_1 = x_1$ and u_2 and u_3 will be changed as this equation. So, one will be sine function of x_2 over f, another one is a cosine function.

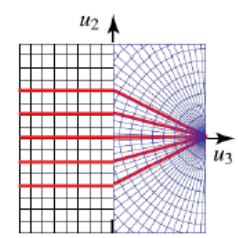
So, this will actually change the right side into a cylindrical coordinate system, okay and this centers at u_2 equals 0 and u_3 equals f, okay. So this particular transformation converts the right side into a cylindrical coordinate system so that you are able to get this focusing effect. So what is happening? This particular transformation basically converts a line that is x_2 equals a, okay in the plane x_1 equals 0 that is this particular screen you can think this screen is x_1 equals 0 in the original coordinate system into a line that is $u_2 = (f - u_3)\tan(a/f)$ in the new coordinate system.

Cylindrical Focusing

- Using: $\varepsilon^e = |\det A|^{-1} A^T \varepsilon A$ (L31.1) $\mu^e = |\det A|^{-1} A^T \mu A$ (L31.2) & $A_{ij} = \frac{\partial u_i}{\partial x_j}$ $i, j = 1,2\ 3 \ldots$ (L31.3)

The transformation yields the diagonal matrix:

$$\varepsilon_0^{-1}\varepsilon^e = \mu_0^{-1}\mu^e = \begin{bmatrix} s & 0 & 0 \\ 0 & s^{-1} & 0 \\ 0 & 0 & s \end{bmatrix} \qquad s = \frac{f}{|x_3 - f|} \qquad \text{(L31.8)}$$



Conversion to a cylindrical coordinate system centered at $x_3 = f$ for $x_3 > 0$ produces the desired trajectories

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

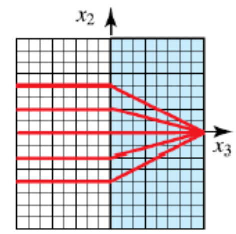
So this is how they are getting focused or they will meet somewhere and you will get this, okay. And if you consider x_3 equals b that is this particular lines that you see here because of this coordinate transformation all these lines will be now converted into circles and I will not go into the mathematics of it but I will just tell you that they will get converted into circles and these are the equations of the circles $u_2^2 + (f - u_3)^2 = (f - b)^2$. So,, this tells you that the radius will be f minus b and the circle will be centered at 0f that is basically u_2 and u_3 coordinates. So with that you are able to do this particular focusing. Now once the coordinate transformation is fixed now how to obtain your values of epsilon e and epsilon mu e they can be done using this formula, okay.

So you do the calculation again in the same manner you will be able to find out what is this matrix they are again diagonal matrix. So epsilon naught inverse epsilon e will be equal to mu naught inverse mu e that is s 00, 0 s inverse 0 and 00 s. What is s here? s is basically f over modulus of x_3 minus f. So, with that you can understand that the permittivity and permeability tensors of the equivalent medium they have the same principal axis along x_1, x_2 and x_3 axis, okay. But the principal values of the permittivity and permeability tensors are dependent on the position x_3.

Cylindrical Focusing

$$\varepsilon_0^{-1}\varepsilon^e = \mu_0^{-1}\mu^e = \begin{bmatrix} s & 0 & 0 \\ 0 & s^{-1} & 0 \\ 0 & 0 & s \end{bmatrix} \qquad s = \frac{f}{|x_3 - f|} \qquad \text{(L31.8)}$$

- The permittivity and permeability tensors of the equivalent medium therefore have principal axes along the (x_1, x_2, x_3) axes.
- The principal values of permittivity and permeability tensors are dependent on the position x_3, *i.e.*, the equivalent material is graded along the x_3 direction with larger anisotropy near the focal line.



Equivalent anisotropic material with identical trajectories

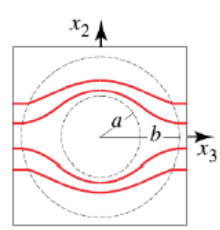
So this is important because s is basically all dependent on this particular position, okay. That is the equivalent material is basically graded along the x_3 direction with larger anisotropy near the focal line. So when this goes to very close to the focal line this value will be very large, okay. So this is a kind of larger anisotropy that is what they are discussing, okay. So here these parameters are basically function of x_3 and that is why they are all graded, okay.

So you can understand that creating such material could be challenging but this particular tool of transformation optics can allow you to think of this kind of materials which can do all this unusual optical transformations.

Invisibility Cloaks

- An invisibility cloak is a device that guides light around an object such that the object appears transparent, and therefore invisible.
- For example, the trajectories shown below avoid a sphere of radius *a*, emerging as if they had followed straight lines and passed right through it.

Figure: Geometrical transformation implementing cloaking of a sphere.



Desired optical trajectories

With that let us move on to the next topic that is invisibility clocks which is basically an application of this transformation optics. So an invisibility clock is a device that can guide light around an object such that the object appears transparent and therefore it is invisible. So here is an example say you have an object here, okay. But the trajectory shows that you are basically avoiding this sphere A and you are allowing the light to go through in this shell and they will exit this at the same point where this was so all these four points will be in the straight line with these four points.

So it will look like as if there is nothing in between this. So this is how clocking works. So anything any object that you can put in between will not be seen, okay. So this is that desired optical trajectory in this case. So again how do you design this kind of a system? You can start with a free space Cartesian coordinate system which has got all parallel straight line kind of optical trajectories, okay. And then you have to bring in the coordinate transformation. So, you can convert to a coordinate system which has got u_1, u_2 and u_3 such that the points of the sphere with radius r, okay.

Invisibility Cloaks

- Following the design steps for transformation optics, we begin with the straight-line trajectories shown in Fig. (b) in a Cartesian-coordinate system (x_1, x_2, x_3).

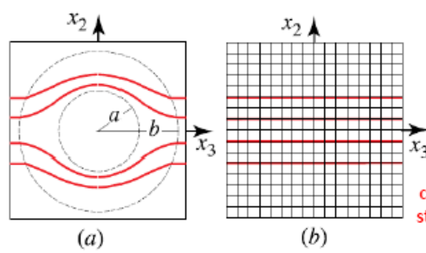


Figure: Geometrical transformation implementing cloaking of a sphere.

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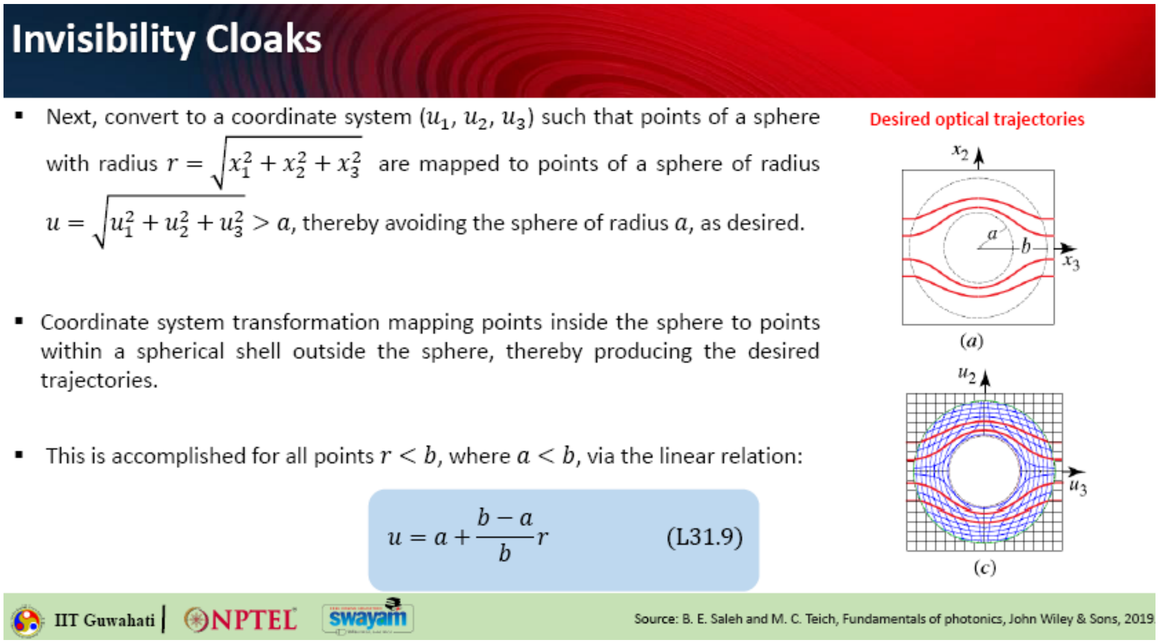
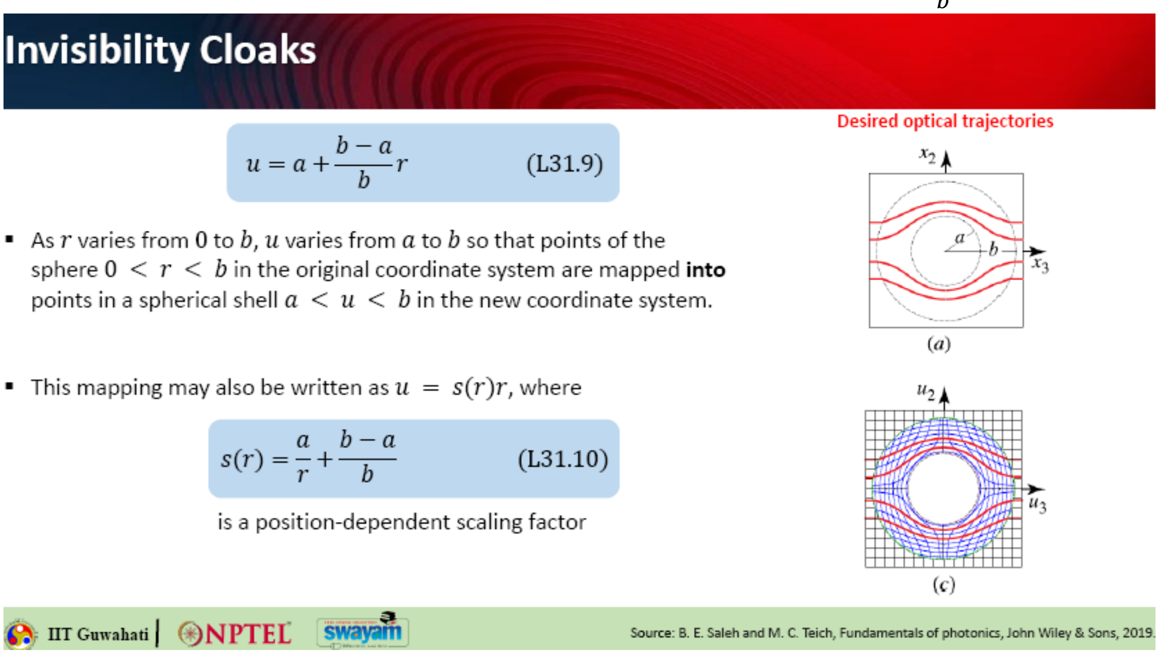
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So you can think of radius r, r is having $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. They are basically mapped onto a sphere of radius u which is greater than a. So that you know this all these points are basically mapped outside. So coordinate transformation mapping points inside the sphere to points within a spherical shell outside the sphere will allow you to get the desired response. So, what you are doing basically all the points that is inside this sphere of radius a has to be moved onto this shell, okay.



That way there will be no light inside, okay. So this is accomplished for all the points where r the radius is less than b, okay and b is basically greater than a. So this is what you want to do. All these points has to be mapped here, okay. So, you can actually use a transformation like this $u = a + \frac{b-a}{b} r$.



So you can see what happens to the point over here, okay. You can put r equal 0, you can put r equals a, okay and you can find out what is happening to this transformation. So this is the transformation that allows you to do that and when r varies to b, okay you can see that as r varies from 0 to b, u will vary from a to b, okay. So the entire thing from 0 to b is the entire space. So, the entire space in the real coordinate system is basically now mapped onto the new coordinate system which has got u and in that case everything falls within a to b. So, what you are doing? The entire system including this thing is mapped onto this shell now, okay. So this mapping can also be written as u equals sr times r where sr is basically, okay like this a over r plus b minus a over b. So you are just bringing r in the denominator. So this becomes a position dependent scaling factor. So, if you try to think of this in terms of scaling, so you can think that this is position dependent scaling factor, okay.

Invisibility Cloaks

$$s(r) = \frac{a}{r} + \frac{b-a}{b} \qquad \text{(L31.10)}$$

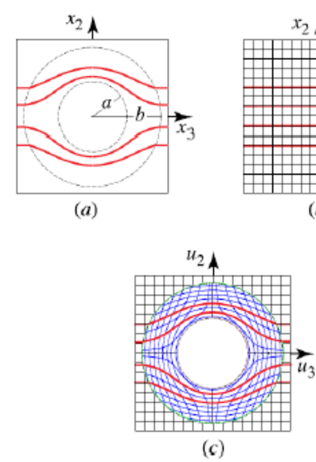
- When applied isotropically, this scaling produces the coordinate transformation:

$$u_1 = s(r)x_1, \qquad u_2 = s(r)x_2, \qquad u_3 = s(r)x_3$$

- This results in
  points on the straight line $x_2 = f$ in the plane $x_1 = 0$ of (b)
  getting mapped **into**
  the curved trajectory in the u_2–u_3 plane of (c):

$$u_2^2 + u_3^2 = a^2\left[\frac{u_2}{u_2-(b-a)f/b}\right]^2 \qquad \text{(L31.11)}$$

Desired optical trajectories



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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So now it is your task to apply this. So when applied isotropically the scaling factor will bring this particular coordinate transformation. So, all the coordinates x_1, x_2 and x_3 will get transformed, okay by this coordinate by this scaling factor and what will be the result? The points on the straight line say x_2 equals f in the plane x equals 0, okay of this one. So x_2 equals f is say this one, okay. Will now get mapped into the curve trajectory, okay one of these trajectories in u_2, u_3 plane.

So this is how all of them will now go around. So all the straight lines will now take this particular curved trajectories, okay. So this one you can think of like this going around. The top one you can think of like this because of this transformation. So, you can also write $u_2^2 + u_3^2 = a^2\left[\frac{u_2}{u_2-(b-a)f/b}\right]^2$, okay.

So this is the trajectory that you are seeing here. So the red curves are computed using this one and these are the four values. You can also see these are two and two. So these are the four values. So, this is how the top two goes to this two curved red lines, the bottom two and the bottom two red curved lines, okay.

Invisibility Cloaks

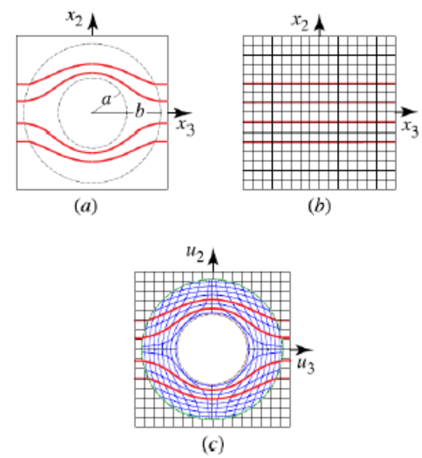
$$u_2^2 + u_3^2 = a^2\left[\frac{u_2}{u_2-(b-a)f/b}\right]^2 \qquad \text{(L31.11)}$$

- The red curved trajectories shown in Fig.(c) are computed from (L31.11) for four values of f.
- The grid shown in Fig.(c) within the shell $a < u < b$ is computed by use of (L31.11)

  and

- a similar equation determined by mapping the straight lines $x_3 = f$ in the plane $x_1 = 0$ into the u_2–u_3 plane.

Desired optical trajectories



So that is how you are able to do it. The grid that is shown in this figure within the shell that where u is like in between a b and you can actually obtain this using this particular relationship. So there is nothing here. So the entire space of radius 0 to b has been moved on to this shell. So now you can put anything here, doesn't matter. That will never enter this, it will go around and it will exit from this side, but these two points will look there as if coming straight without any deviation.

Invisibility Cloaks

- The parameters of the equivalent material to be placed in the $a < r < b$ spherical shell shown in Fig. (d) may be determined by use of following equations together.

$$\varepsilon^e = |\det A|^{-1} A^T \varepsilon A \qquad \text{(L31.1)}$$

$$\mu^e = |\det A|^{-1} A^T \mu A \qquad \text{(L31.2)}$$

$$A_{ij} = \frac{\partial u_i}{\partial x_j} \qquad i,j = 1,2\,3\,... \qquad \text{(L31.3)}$$

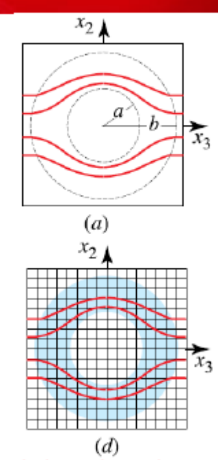
$$u_1 = s(r)x_1, \qquad u_2 = s(r)x_2, \qquad u_3 = s(r)x_3 \qquad s(r) = \frac{a}{r} + \frac{b-a}{b}$$

&

$$u_2^2 + u_3^2 = a^2\left[\frac{u_2}{u_2-(b-a)f/b}\right]^2 \qquad \text{(L31.11)}$$

- The result is:

$$\varepsilon_0^{-1}\varepsilon^e = \mu_0^{-1}\mu^e = \frac{b}{b-a}\frac{1}{v^2}\begin{bmatrix} v^2-u_1^2 & -u_1u_2 & -u_1u_3 \\ -u_2u_1 & v^2-u_2^2 & -u_2u_3 \\ -u_3u_1 & -u_3u_2 & v^2-u_3^2 \end{bmatrix}, v^2 = \frac{u^4}{2au-a^2} \qquad \text{(L31.12)} \qquad a < u < b$$



Equivalent anisotropic material with identical trajectories

So that is how your cloaking will work, okay. So, you can also think of similar kind of mapping for x_3. So x_3 equals f is this one, okay and you can think of putting this on this plane. So that is how you are getting this curved grids. Now once you know the parameters, the transformation, the parameters of the equivalent material to be placed in this shell, spherical shell will be now determined by these equations, okay.

So these are the equations we started with. This is the Jacobian matrix, this is the transformation, okay. This is the relationship of the transformation scaling factor, okay. This is how the transformation has finally come up from straight lines to this kind of curved lines. Once you know all these things, you can find out that you require the material equivalent permittivity and permeability to have these values, okay, where v square is basically u to the power 4 over 2au minus a square where u is basically ranging from 0 to, sorry, a to b. So, this is how this shell has to be replaced by this kind of material that will do this trick for you.

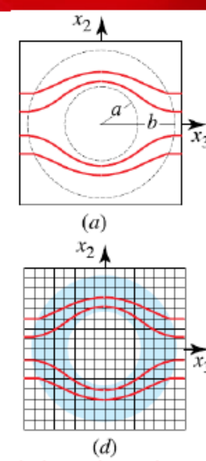
Invisibility Cloaks

$$\varepsilon_0^{-1}\varepsilon^e = \mu_0^{-1}\mu^e = \frac{b}{b-a}\frac{1}{v^2}\begin{bmatrix} v^2-u_1^2 & -u_1u_2 & -u_1u_3 \\ -u_2u_1 & v^2-u_2^2 & -u_2u_3 \\ -u_3u_1 & -u_3u_2 & v^2-u_3^2 \end{bmatrix}, v^2 = \frac{u^4}{2au-a^2} \quad \text{(L31.12)}$$

- Clearly, the dielectric and magnetic properties of the material in the spherical shell are both inhomogeneous and anisotropic.

- For example, at points on the x_1 axis $(u, 0, 0)$, we have:

$$\varepsilon_0^{-1}\varepsilon^e = \mu_0^{-1}\mu^e = \frac{b}{b-a}\begin{bmatrix} 1-u^2/v^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(L31.13)}$$



Equivalent anisotropic material with identical trajectories

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

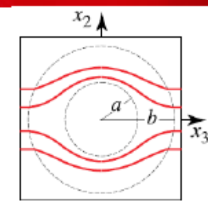
So as you can see this is bit mathematically intensive but this kind of things are possibility. So what is understood here is that the dielectric and the magnetic properties of this kind of materials in this spherical shell are both inhomogeneous and anisotropic. For example, if you take at points on the x_1 axis, okay, that is u00, you will have this kind of a value, okay. And if you, at all these points, the principal axes are anyways aligned to the three coordinate, Cartesian coordinates u_1, x_1, x_2 and x_3. And you can think of the principal value u_1 that varies from 0 to epsilon b minus a over b.

So this is the range over which epsilon 1 will vary because u is varying from a to b. Similarly, the values of, however, you can see here there is no variation. So ε_2 and ε_3 will be simply b over b minus a times epsilon 0. So the values along 2 and 3 direction will be same but along epsilon 1 it will change. The similar result will apply for mu as well. And with that you can see that if you go to the implementation at optical wavelengths, the implementation of invisibility clocks via metamaterials will require the use of advanced nanofabrication technology, something like electron beam or focused ion beam lithography.

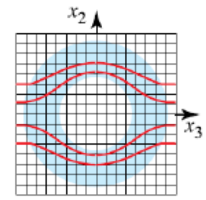
Invisibility Cloaks

$$\varepsilon_0^{-1}\varepsilon^e = \mu_0^{-1}\mu^e = \frac{b}{b-a}\begin{bmatrix} 1-u^2/v^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(L31.13)}$$

- At these points, the principal axes are aligned with the Cartesian coordinates (x_1, x_2, x_3)
- The principal value ε_1 varies from 0 to $\varepsilon_0(b-a)/b$ as u varies from a to b, while the principal values ε_2 and ε_3 are fixed at the value $\varepsilon_0 b/(b-a)$
- Similar results apply for μ.



(a)



(d)

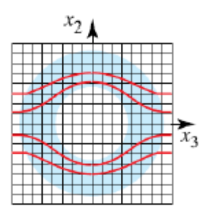
Equivalent anisotropic material with identical trajectories

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

We will look into the details of these different techniques in the subsequent lectures. But you can also see because there is change gradient graded one in the graded values. So you have to think of constituent dielectric and magnetic elements to be of various shapes and dimensions and they must be intricately designed and precisely laid out. So that makes this thing very challenging. So, such elements will be highly resonant and the electromagnetic properties of the material will depend strongly on the wavelength.

Invisibility Cloaks

- At optical wavelengths, the implementation of invisibility cloaks via metamaterials requires the use of advanced nanofabrication technologies such as electron-beam or focused ion-beam lithography.
- The constituent dielectric and magnetic elements, which have various shapes and dimensions, must be intricately designed and precisely laid out.
- Since such elements are highly resonant, the electromagnetic properties of the metamaterial depend strongly on wavelength so that such devices typically operate only over narrow spectral bandwidths.



IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So such devices will typically work over a very very narrow spectral bandwidth. And I hope you are able to understand the challenge here that you require metamaterial to realize this kind of special functionalities like anisotropy in the permittivity and permeability and after that also it will work only over a very narrow bandwidth. So with that we come to an end to this lecture. So we will be discussing carpet clocking and other transformation optics metamaterial devices in the next lecture. So if you have any query you can drop an email to me mentioning MOOC in the subject line. Thank you.